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Norm Inequalities for Positive Semidefinite Matrices

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Abstract

In this paper, we introduce and study some inequalities involving symmetric norms and positive semidefinite matrices. Then we will use the properties of symmetric norms to generalize the inequality obtained by Bhatia and Kittaneh[3].

Keywords: Symmetric norms, positive semidefinite matrices, normal matrices, singular values.

INTRODUCTION

The class of Hermitian matrices is a generalization of real numbers and the class of positive semidefinite matrices is a generalization of nonnegative real numbers. This observation often provides insight into the properties and applications of positive semidefinite matrices. In this paper we will use the inequality that said if a, b are positive real number then for any complex number z,

$$|a - |z|b| \leq |a+zb| \leq |a+|z|b| \qquad (1)$$

we discuss several matrix norm inequalities for positive semidefinite matrices. All through this paper $M_{m,n}$ and M_n will stand, respectively, for spaces of all m x n and n x n complex matrices. Let $\|.\|$ denote any symmetric norm (or unitarily invariant norm) on M_n . Therefore, $\|\boldsymbol{UAV}\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$..

PRELIMINARIES

Definition 3.1:

Let $A \in M_n$. Then:

1. A is called Hermitian (or, self adjoint) if $A^* = A$.
2. A is called normal if $A^*A = AA^*$.
3. A is called unitary if $A^*A = AA^* = I_n$, where I_n is the identity matrix of order n.
4. A is called positive semidefinite or nonnegative definite(written as $A \geq 0$) if A is Hermitan and $\langle Ax, x\rangle \geq 0$, for all $x \in \mathbb{C}^n$.

Definition 3.2:

For any matrix $A \in M_n$, we define the absolute value $|A|$ of A to be the positive

semidefinite matrix square root of A^*A. Then the singular values of A, $s_1(A), \ldots, s_n(A)$ are defined to be the eigenvalues of $|A|$ which ordered from largest to smallest

$s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$.

In this paper, we will apply Ky Fan's maximum principle to the absolute value,

$|A|$, of $A \in M_n$, we get for each $k = 1, 2, \ldots, n$ that

$\sum_{j=1}^{k} s_j(A) = \max \sum_{j=1}^{k} |\langle Ax_j, y_j\rangle|$

where the maximum is taken over all choices of orthonormal k-tuples $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$.

Also Fan Dominance Theorem saying that if $A, B \in M_n$,then:

$\|A\| \leq \|B\|$ for all symmetric norms on M_n if and only if

$\sum_{j=1}^{k} s_j(A) \leq \sum_{j=1}^{k} s_j(B)$ for all k = 1,2,…, n.

SEVERAL NORM INEQUALITIES FOR MATRICES

Proposition 4.1

If $A, B \in M_n$ are normal matrices, then , for all symmetric norms,

$\|A + B\| \leq \||A| + |B|\|$.

This satisfies a triangle inequality for normal matrices.

A stronger triangle inequality holds if we use Hermitian matrices:

Proposition 4.2

If X, Y are Hermitian matrices, then for some unitaries U, V

$$|X + Y| \leq \frac{1}{2}(U(|X| + |Y|U^*) + (V(|X| + |Y|V^*))$$

Proposition 4.2 implies Proposition 4.1. by substituting

$$X = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad and \quad Y = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}$$

Now Proposition 4.1. shows that if $A, B \geq 0$, and any complex number z,

$\|A + zB\| \leq \|A + |z|B\|$

Also Bhatia and Kittaneh [3]generalized inequality(1) by using positive semidefinite matrices $A, B \geq 0$, and any complex number z,

$$\sum_{j=1}^{k} s_j(A - |z|B) \leq \sum_{j=1}^{k} s_j(A + zB)$$

$$\leq \sum_{j=1}^{k} s_j(A + |z|B) \quad \text{for all } k = 1,2,\ldots, n.$$

Which equivalent to

$$\|A - |z|B\| \leq \|A + zB\| \leq \|A + |z|B\| \quad (2)$$

By using Fan Dominance Theorem.

Now we will give the generalization of inequality (2)

Theorem 4.1:

If A, $B \in M_n$ are positive semidefinite matrices , and z_1, z_2 any two complex numbers, then

$$\||z_1|A - |z_2|B\| \leq \|z_1 A + z_2 B\| \leq \||z_1|A + |z_2|B\| \quad (3)$$

Proof:

If $z_1 = 0$, its obvious holds.

But if $z_1 \neq 0$, let $z = \frac{z_2}{z_1}$ in inequality (2),then

$$\left\|A - \left|\frac{z_2}{z_1}\right|B\right\| \leq \left\|A + \frac{z_2}{z_1}B\right\| \leq \left\|A + \left|\frac{z_2}{z_1}\right|B\right\|, \text{ so}$$

$$\||z_1|A - |z_2|B\| \leq \|z_1 A + z_2 B\| \leq \||z_1|A + |z_2|B\|. \quad \blacksquare$$

Theorem 4.2:

If A, B_1, …,$B_n \in M_n$ are positive semidefinite matrices , and z_1, z_2 , …, z_n be complex numbers, then

$$\|A + \boldsymbol{z_1} B_1 + \cdots + \boldsymbol{z_n} B_n\| \leq \|A + |\boldsymbol{z_1}|B_1 + \cdots + |\boldsymbol{z_n}|B_n\|$$

Proof:

$$\|A + \boldsymbol{z_1} B_1 + \cdots + \boldsymbol{z_n} B_n\| = \sum_{j=1}^{k} s_j(A + \boldsymbol{z_1} B_1 + \cdots + \boldsymbol{z_n} B_n)$$

$$= \max \sum_{j=1}^{k} |\langle (A + \boldsymbol{z_1} B_1 + \cdots + \boldsymbol{z_n} B_n)\, x_j, y_j \rangle|$$

(using Ky Fan´s maximum principle over all choices of orthonormal k-tuples x_1, …, x_k and y_1, …, y_k**).**

$$= \max \sum_{j=1}^{k} |\langle Ax_j, y_j\rangle + \boldsymbol{z_1}\langle B_1 x_j, y_j\rangle + \cdots + \boldsymbol{z_n}\langle B_n x_j, y_j\rangle|$$

$$\leq \max \sum_{j=1}^{k} (|\langle Ax_j, y_j\rangle| + |\boldsymbol{z_1}||\langle B_1 x_j, y_j\rangle| + \cdots + |\boldsymbol{z_n}||\langle B_n x_j, y_j\rangle|)$$

$$\leq \max \sum_{\boldsymbol{j=1}}^{\boldsymbol{k}} \left(\sqrt{\langle Ax_j, x_j\rangle \cdot \langle Ay_j, y_j\rangle} + |\boldsymbol{z_1}| \sqrt{\langle B_1 x_j, x_j\rangle \cdot \langle B_1 y_j, y_j\rangle} + \ldots + |\boldsymbol{z_n}| \sqrt{\langle B_n x_j, x_j\rangle \cdot \langle B_n y_j, y_j\rangle}\right)$$

(using Cauchy-Schwars inequality)

$$\leq \frac{1}{2} \sum_{\boldsymbol{j=1}}^{\boldsymbol{k}} \langle (A + |\boldsymbol{z_1}|B_1 + \cdots + |\boldsymbol{z_n}|B_n)x_j, x_j\rangle + \langle (A + |\boldsymbol{z_1}|B_1 + \cdots + |\boldsymbol{z_n}|B_n)y_j, y_j\rangle$$

(using arithmetic-geometric mean inequality).

Now using Ky Fan´s maximum principle ,we get

$$\sum_{j=1}^{k} s_j(A + z_1 B_1 + \cdots + z_n B_n) \leq \sum_{j=1}^{k} s_j(A + |\boldsymbol{z_1}|B_1 + \cdots + |\boldsymbol{z_n}|B_n)$$

Finally, by Fan Dominance Theorem

$$\|A + \boldsymbol{z_1} B_1 + \cdots + \boldsymbol{z_n} B_n\| \leq \|A + |\boldsymbol{z_1}|B_1 + \cdots + |\boldsymbol{z_n}|B_n\|. \quad \blacksquare$$

But the inequality

$\|A - |\boldsymbol{z_1}|B_1 - \cdots - |\boldsymbol{z_n}|B_n\| \leq \|A + \boldsymbol{z_1} B_1 + \cdots + \boldsymbol{z_n} B_n\|$ turns out to be false**.**

If n=2 and $A = B_1 = B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\boldsymbol{z_1 = i}$ and $\boldsymbol{z_2 = -1 - i}$.

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REFERENCES

[1] Horn, R. A. and Johnson, C. R. (1985). **Matrix Analysis,** Cambridge University Press, Cambridge.

[2] Horn, R. A. and Johnson, C. R. (1990). **Topics inMatrix Analysis,** Cambridge University Press, Cambridge.

[3] R. Bhatia, F. Kittaneh, Norm inequalities for positive operators, Lett. Math. Phys. 43 (1998) 225–231.

[4] Bhatia, R. and Kittaneh, F. (2009). The singular values of $A + B$ and $A + iB$, **Linear Algebra Appl.** 431: 1502-1508.

[5] Kalton N.and Sukochev F.(2008). Symmetric norms and spaces of operators, J. Reine Angew. Math. 621: 81-121.

[6] Bhatia, R. (1997). **Matrix Analysis,** Springer-Verlag, New York.

[7] Bourin, J . C. (2005), Hermitian operators and convex functions, J, of Inequal. In Pure and appl. Math. 6 no5, Article 139

[8] Zhan, X. (2000). Singular values of of differences of positive semidefinite matrices, **SIAM J. Matrix. Anal. Appl.** 22: 819-823.

[9] Bhatia, R. and Kittaneh, F. (2008). The matrix arithmetic-geometric mean inequality revisited**, Linear Algebra Appl.** 428: 2177-2191.

[10] Tao, Y. (2006). More results on singular value inequalities**, Linear Algebra Appl.** 416:724-729.

[11] Kittaneh, F. (2007). Inequalities for commutators of positive operators, **J. Funct. Anal.** 250: 132-143.

[12] Kittaneh, F. (2002). Norm inequalities for sums of positive operators, **J. Operator Theory.** 48: 95-103.

[13] Kittaneh, F. (2004). Norm inequalities for sums and differences of positive operators, **Linear Algebra Appl.** 383: 85-91.