International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 7 (2018) pp. 5332-5345
© Research India Publications. http://www.ripublication.com

Abundant Solutions with Distinct Physical Structure for Nonlinear Integro and Partial Differential Equations

Taher A. Nofal[1,2]
[1]*Mathematics Department, Faculty of Science, Taif University, Saudi Arabia.*
[2] *Mathematics Department, Faculty of Science, Minia University, Egypt.*

Khaled A Gepreel [1,3]
[1]*Mathematics Department, Faculty of Science, Taif University, Saudi Arabia.*
[3] *Mathematics Department, Faculty of Science, Zagazig University, Egypt.*

Abstract

In this article, we use two direct methods namely the generalized Kudryashov method and the generalized (G'/G)-expansion method to discuss the traveling wave solutions to the nonlinear integro- partial differential equations. In the generalized (G'/G)-expansion method, we suppose the trial equation for G satisfies the nonlinear second order differential equation $AGG''(\xi)-BGG'-EG^2-CG'^2=0$ while Q in the generalized Kudryashov method satisfies Bernoulli first order differential equation $Q'=AQ^2+BQ$ We construct the exact solutions for some nonlinear integro - partial differential equations in mathematical physics via (3+1)- dimensional Gardner type integro- differential equation and (2+1) dimensional Sawada- Kotera nonlinear integro partial differential equation. We obtain the traveling wave solutions as a rational formula in the hyperbolic functions, trigonometric functions and rational function, when G satisfies a nonlinear second order ordinary differential equation and Q satisfies the Bernoulli first order differential equation. When the parameters are taken some special values, the solitary wave are derived from the traveling waves. This method is reliable, simple and gives many new exact solutions.

Keywords: Generalized (G'/G)- expansion method, Generalized Kudryashov method, Traveling wave solutions, Gardner type integro- differential equation, Sawada- Kotera nonlinear integro partial differential equation

INTRODUCTION

The study of partial differential equations has a significant role in identifying some of the physical and natural phenomena surrounding us and through its knowledge of predicting some natural problems that may be induced in the near future. Many natural and physical problems can be visualized in many nonlinear partial differential equations and by analyzing their analytical solutions, physicists and engineers can interpret those. There are many methods for obtaining exact solutions to nonlinear partial differential equations such as the inverse scattering method [1], Hirota's bilinear method [2], Backlund transformation [3], the first integral method [4], Painlevé expansion [5], sine–cosine method [6], homogenous balance method [7], extended trial equation method [8,9], perturbation method [10,11], variation method [12], tanh - function method [13,14], Jacobi elliptic function expansion method [15,16], Exp-function method [17,18] and F-expansion method [19,20] . Wang etal [21] suggested a direct method called the (G'/G) expansion method to find the traveling wave solutions for nonlinear partial differential equations (NPDEs) . Zayed etal [22,23] have used the (G'/G) expansion method and modified (G'/G) expansion method to obtain more than traveling wave solutions for some nonlinear partial differential equations. Shehata [24] have successfully obtained more traveling wave solutions for some important NPDEs when G satisfies a linear differential equations $G''-\mu G=0$. There are many authors have successively applied the (G'/G) Iexpansion method to study the exact solutions for nonlinear evoluation equations see [25-28]. In this paper we use the generalized (G'/G)- expansion function method when G satisfies a nonlinear differential equations $AGG''(\xi)-BGG'-EG^2-CG'^2=0$, where $A,B,C,$ E are real arbitrary constants to find the traveling wave solutions for some nonlinear integro- partial differential equations in mathematical physics . Also we use the generalized Kudryashov method [29,30] to discuss the rational traveling wave solutions for some nonlinear integro- partial differential equations. The solitary wave solutions are deduced form the traveling wave solutions when the parameter are taking some special values.

DESCRIPTION OF THE GENERALIZED (G'/G) EXPANSION FUNCTION METHOD FOR NPDEs

In this part of the manuscript, the generalized (G'/G) expansion method will be given. In order to apply this method to nonlinear partial differential equations we consider the following steps[27,28]

Step 1. We consider the nonlinear partial differential equation, say in two independent variables x and t is given by

$$P(u,u_t,u_x,u_{tt},u_{xx},u_{xt},..)=0, \qquad (1)$$

where $u=u(x,t)$ is an unknown function, P is a polynomial in $u=u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Step 2. We use the following travelling wave transformation:

$$u=U(\xi), \qquad \xi=k_i x_i - wt, \qquad (2)$$

where k_i, w is a nonzero constant. We can rewrite Eq.(1) in the following form:

$$P(U,U',U'',...)=0 \qquad (3)$$

Step 3. We assume that the solutions of Eq. (3) can be expressed in the following form:

$$U(\xi)=\sum_{i=-m}^{m}\frac{a_i\left(G'(\xi_n)/G(\xi_n)\right)^i}{\left[1+\rho G'(\xi_n)/G(\xi_n)\right]^i}, \qquad (4)$$

where $a_i\,(i=0,\pm1,...,\pm m)$ are arbitrary constants , ρ is nonzero constant to be determined later, m is a positive integer and $G(\xi)$ satisfies a nonlinear second order differential equation

$$AGG''(\xi)-BGG'-EG^2-CG'^2=0, \qquad (5)$$

where A,B,C,E are real nonzero constants.

Step 4. Determine the positive integer m by balancing the highest order nonlinear term(s) and the highest order derivative in Eq (3).

Step 5. Substituting Eq. (4) into (3) along with (5), cleaning the denominator and then setting each coefficient of $(G'(\xi)/G(\xi))^i, i=0,\pm1,\pm2,..$ to be zero, yield a set of algebraic equations for $a_i\,(i=0,\pm1,...,\pm m)$, k and ρ .

Step 6. Solving these over-determined system of algebraic equations with the help of Maple software package to determine $a_i\,(i=0,\pm1,...,\pm m)$, k and ρ.

Step 7. The general solution of Eq. (5), takes the following cases :

(i) When $B\neq 0$, $\Omega=B^2+4E(A-C)>0$, $\Gamma=A-C$, we obtain the hyperbolic exact solution of Eq.(5) takes the following form:

$$G(\xi)=\frac{e^{\frac{\xi B}{2\Gamma}}}{[\Omega]^{\frac{A}{2\Gamma}}}\left[C_1\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)+C_2\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)\right]^{\frac{A}{\Gamma}} \qquad (6)$$

where C_1 and C_2 are arbitrary constants. In this case the ratio between G' and G takes the form

$$\frac{G'}{G}=\frac{B}{2\Gamma}+\frac{\sqrt{\Omega}}{2\Gamma}\left[\frac{C_1\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)+C_2\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)}{C_1\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)+C_2\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)}\right] \qquad (7)$$

(ii) When $B\neq 0$, $\Omega=B^2+4E(A-C)<0$, $\Gamma=A-C$, we obtain the trigonometric exact solution of Eq.(5) takes the form

$$\frac{G'}{G}=\frac{B}{2\Gamma}+\frac{\sqrt{-\Omega}}{2\Gamma}\left[\frac{-C_1\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)+C_2\cos(\frac{\sqrt{-\Omega}}{2\Gamma})}{C_1\cos(\frac{\sqrt{-\Omega}}{2\Gamma})+C_2\sin(\frac{\sqrt{-\Omega}}{2\Gamma})}\right] \qquad (8)$$

(iii) When $B\neq 0$, $\Omega=B^2+4E(A-C)=0$, we obtain the rational exact solution of Eq.(5) takes the form

$$\frac{G'}{G}=\frac{B}{2\Gamma}+\frac{C_2}{C_1+C_2\xi} \qquad (9)$$

(iv) When $B=0$, $\Delta=E\Gamma>0,$, we obtain the hyperbolic exact solution of Eq.(5) takes the following form:

$$\frac{G'}{G}=\frac{\sqrt{\Delta}}{\Gamma}\left[\frac{C_1\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)+C_2\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)}{C_1\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)+C_2\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)}\right] \qquad (10)$$

(v) When $B=0$, $\Delta=E\Gamma<0,$ we obtain the hyperbolic exact solution of Eq.(5) takes the following form:

$$\frac{G'}{G}=\frac{\sqrt{-\Delta}}{\Gamma}\left[\frac{-C_1\sin(\frac{\sqrt{-\Delta}}{\Gamma}\xi)+C_2\cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi)}{C_1\cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi)+C_2\sin(\frac{\sqrt{-\Delta}}{\Gamma}\xi)}\right] \qquad (11)$$

Step 8. Substituting the constants $\alpha_i\,(i=0,\pm1,...,\pm m)$, k and ρ which obtained by solving the algebraic equations in Step 5, and the general solutions of Eq.(5) in step 6 into

Eq.(4) , we obtain more new exact solutions of Eq. (1) immediately.

DESCRIPTION OF THE GENERALIZED KUDRYASHOV METHOD FOR NPDE

The basic steps in the application of the GKM detailed in the following [29,30]:

Step 1. We suppose the exact solution of Eq. (3) to be in the following rational form:-

$$V(\xi)=\frac{\sum_{i=0}^{N} a_i Q^i(\xi)}{\sum_{j=0}^{M} a_j Q^j(\xi)} \tag{12}$$

Where a_i, b_j are constants to be determined later such that $a_N \neq 0,\ b_M \neq 0$. We suppose the trial equation for Q satisfies the first order Bernoulli differential equation:

$$Q' = AQ^2 + BQ \tag{13}$$

Step 3. Determine the positive integer numbers N and M in Eq. (12) by balancing the highest order derivatives and the nonlinear terms in Eq. (3).

Step 4. Substituting Eqs. (12) and (13) into Eq. (3), we obtain a polynomial in Q^{i-j}, $(i, j = 0,1,2, \ldots)$. Setting all coefficients of this polynomial to be zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica software package to get the unknown parameters $a_i(i = 0,1,2, \ldots, N)$ and $b_j(j = 0,1,2, \ldots, M)$. Consequently, we obtain the exact solutions of Eq. (1).

TRAVELING WAVE SOLUTIONS FOR THE FIRST EQUATION TO THE GARDNER TYPE INTEGRO-DIFFERENTIAL EQUATION

In this section, we use two different methods namely the generalized (G'/G) expansion method and the generalized kuderyshov method to discuss the exact solutions for the nonlinear evolution equations in mathematical physics via the (3+1) dimensional Gardner type intergro- differential equations which are very important in the mathematical science and have been paid attention by many researchers in physics and engineering. The (3+1) dimensional Gardner type integro- differential equation takes the following form:

$$\begin{aligned} &u_t + 6\beta u\, u_x + u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x + 3\sigma^2 \int_{-\infty}^{x} u_{yy} dx' \\ &- 3\alpha\sigma u_x \int_{-\infty}^{x} u_y\, dx' + 3\sigma^2 \int_{-\infty}^{x} u_{zz} dx' = 0 \end{aligned} \tag{14}$$

where α, β, δ and σ are arbitrary constants. Gardner type intergro- differential equations have many applications in different branches of physics such as plasma physics, fluid physics, and quantum field theory [1–7]. We take the transformation:

$$u = v_x, \tag{15}$$

to convert the Gardner type intergro- differential equations to the nonlinear partial differential equation:

$$\begin{aligned} &v_{xt} + 6\beta v_x v_{xx} + v_{xxxx} - \frac{3}{2}\alpha^2 v_x^2 v_{xx} + 3\sigma^2 v_{yy} \\ &- 3\alpha\sigma v_{xx} v_y + 3\sigma^2 v_{zz} = 0. \end{aligned} \tag{16}$$

Traveling wave transformation

$$v = \phi(\xi), \qquad \xi = k_1 x + k_2 y + k_3 z - wt, \tag{17}$$

permits us to convert the nonlinear partial differential equation (16) to the following ordinary differential equation

$$\begin{aligned} &- k_1 w\phi'' + 6\beta k_1^3 \phi'\phi'' + k_1^4 \phi^{(4)} - \frac{3}{2}\alpha^2 k_1^4 \phi'^2 \phi'' \\ &+ 3\sigma^2 k_2^2 \phi'' - 3\alpha\sigma k_1^2 k_2 \phi''\phi' + 3\sigma^2 k_3^2 \phi'' = 0. \end{aligned} \tag{18}$$

By using the integration equation (18) can be written in the following form:

$$\begin{aligned} &\frac{1}{2}(3\sigma^2 k_2^2 + 3\sigma^2 k_3^2 - k_1 w)\phi'^2 + (\beta k_1^3 - \frac{1}{2}\alpha\sigma k_1^2 k_2)\phi'^3 \\ &+ \frac{1}{2}k_1^4 \phi''^2 - \frac{1}{8}\alpha^2 k_1^4 \phi'^4 + c_1\phi' + c_2 = 0. \end{aligned} \tag{19}$$

where c_1 and c_2 are the integration constants. If , we take $\psi(\xi) = \phi'(\xi)$ equation (19) can be reduced to the following ODE's:

$$\begin{aligned} &\frac{1}{2}(3\sigma^2 k_2^2 + 3\sigma^2 k_3^2 - k_1 w)\, \psi^2 + (\beta k_1^3 - \frac{1}{2}\alpha\sigma k_1^2 k_2)\psi^3 \\ &+ \frac{1}{2}k_1^4 \psi'^2 - \frac{1}{8}\alpha^2 k_1^4 \psi^4 + c_1\psi + c_2 = 0. \end{aligned} \tag{20}$$

Generalized (G'/G) expansion method to the (3+1) dimensional Gardner type integro- differential equation :

In this subsection we discuss the solution of Eq.(20) by using generalized (G'/G) expansion method. Balancing the highest order derivative ψ'^2 with the nonlinear term ψ^4, we get the solution formula of Eq.(20) has the following form:

$$\begin{aligned} \psi(\xi) &= a_0 + \frac{a_1 G'(\xi)/G(\xi)}{[1 + \rho\, G'(\xi)/G(\xi)]} \\ &+ \frac{b_1[1 + \rho\, G'(\xi)/G(\xi)]}{G'(\xi)/G(\xi)} \end{aligned} \tag{21}$$

where a_0, a_1, b_1 and ρ are constants to be determined later. Substituting Eq. (21) along with (5) into Eq. (20) and cleaning the denominator and collecting all terms with the same order of $(G'(\xi)/G(\xi))$ together, the left hand side of Eq. (20) are converted into polynomial in $(G'(\xi)/G(\xi))$. Setting each coefficient of these polynomials to be zero , we derive a set of algebraic equations for $a_0, a_1, b_1, k_1, k_2, k_3, w$, σ. Solving the set of algebraic equations by using Maple or Mathematica , software package to get the following results:

Case 1.

$$a_0 = \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1^2}, \qquad a_1 = \frac{B^2 - 4E(C-A)}{2k_1^2 EA\alpha},$$

$$b_1 = -\frac{2E}{A\alpha k_1^2}, \qquad \rho = \frac{B}{2E},$$

$$w = \frac{1}{2\alpha^2 k_1 A^2}[-4\alpha^2(B^2 - 4E(C-A)) + 12A^2 k_1^2 \beta^2 - 12A^2 k_1 k_2 \beta\alpha\sigma + 3A^2\sigma^2\alpha^2 k_2^2 + 6A^2\sigma^2\alpha^2(k_2^2 + k_3^2)]$$

$$c_1 = \frac{1}{2A^2 k_1^2 \alpha^4}\{4\sigma k_2 \alpha^3 [B^2 - 4E(C-A)] - 8\alpha^2 k_1 \beta [B^2 - 4E(C-A)] - A^2\sigma^3\alpha^3 k_2^3 - 12A^2\sigma k_2 \alpha k_1^2 \beta^2 + 6A^2\sigma^2 k_2^2 k_1 \alpha^2 \beta + 8A^2 k_1^3 \beta^3\}$$

$$c_2 = \frac{1}{8\alpha^6 k_1^4 A^4}\{-32A^2 k_1^2 \beta^2 \alpha^2 [B^2 - 4E(C-A)] - 8A^2\sigma^2 k_2^2 \alpha^4 [B^2 - 4E(C-A)] + 32A^2\alpha^3 k_1 k_2 B\sigma[B^2 - 4E(C-A)] + 16\alpha^4 [B^2 - 4E(C-A)]^2 + A^4(\sigma k_2 \alpha - 2k_1\beta)^4\} \tag{22}$$

where $C, B, E, A, \sigma, \alpha, \beta$, k_1, k_2, k_3 are arbitrary constants. There are many other cases which are omitted her for convenience to the reader. In this case the traveling wave solution of Eq.(20) takes the following form:

$$\psi(\xi) = \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1^2} + \frac{(B^2 - 4E(C-A))G'(\xi)/G(\xi)}{k_1^2 A\alpha[2E + B\, G'(\xi)/G(\xi)]} - \frac{[2E + BG'(\xi)/G(\xi)]}{A\alpha k_1^2 G'(\xi)/G(\xi)} \tag{23}$$

There are many families to discuss the types of the traveling wave solutions of Eq.(20) as follows:

Family 1. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) > 0$, we obtain the hyperbolic exact solution of Eq.(20) takes the following:

$$\psi(\xi) = \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1^2} + \frac{\Omega[(BC_1 + C_2\sqrt{\Omega})\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + (BC_2 + C_1\sqrt{\Omega})\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]}{k_1^2 A\alpha[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{\Omega}\}\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 + BC_1\sqrt{\Omega}\}\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]}$$
$$- \frac{[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{\Omega}\}\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 + BC_1\sqrt{\Omega}\}\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]}{A\alpha k_1^2[(BC_1 + C_2\sqrt{\Omega})\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + (BC_2 + C_1\sqrt{\Omega})\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]}.$$

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 7 (2018) pp. 5332-5345
© Research India Publications. http://www.ripublication.com

Consequently the hyperbolic traveling wave solution of Eq.(14) has the following form:

$$\begin{aligned} u_1(x,y,z,t) &= \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1} + \\ &+ \frac{\Omega[(BC_1 + C_2\sqrt{\Omega})\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + (BC_2 + C_1\sqrt{\Omega})\ \sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]}{k_1 A\alpha[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{\Omega}\}\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 + BC_1\sqrt{\Omega}\}\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]} \\ &- \frac{[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{\Omega}\}\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 + BC_1\sqrt{\Omega}\}\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]}{A\alpha k_1[(BC_1 + C_2\sqrt{\Omega})\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + (BC_2 + C_1\sqrt{\Omega})\ \sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]}. \end{aligned} \tag{24}$$

where

$$\begin{aligned} \xi &= k_1 x + k_2 y + k_3 z - \frac{t}{2\alpha^2 k_1 A^2}[-4\alpha^2(B^2 - 4E(C-A)) + 12A^2 k_1^2\beta^2 - 12A^2 k_1 k_2\beta\alpha\sigma \\ &+ 3A^2\sigma^2\alpha^2 k_2^2 + 6A^2\sigma^2\alpha^2(k_2^2 + k_3^2)] \end{aligned}$$

Family 2. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) < 0$, we obtain the trigonometric exact solution of Eq.(20) takes the following form

$$\begin{aligned} \psi(\xi) &= \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1^2} + \\ &+ \frac{\Omega[(BC_1 + C_2\sqrt{-\Omega})\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + (BC_2 - C_1\sqrt{-\Omega})\ \sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]}{k_1^2 A\alpha[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{-\Omega}\}\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 - BC_1\sqrt{-\Omega}\}\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]} \\ &- \frac{1}{A\alpha k_1^2}\frac{[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{-\Omega}\}\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 - BC_1\sqrt{-\Omega}\}\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]}{[(BC_1 + C_2\sqrt{-\Omega})\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + (BC_2 - C_1\sqrt{-\Omega})\ \sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]}. \end{aligned} \tag{25}$$

Consequently the periodic trigonometric traveling wave solution of Eq.(14) has the following form:

$$\begin{aligned} u_2(x,y,z,t) &= \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1} + \\ &+ \frac{\Omega[(BC_1 + C_2\sqrt{-\Omega})\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + (BC_2 - C_1\sqrt{-\Omega})\ \sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]}{k_1 A\alpha[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{-\Omega}\}\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 - BC_1\sqrt{-\Omega}\}\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]} \\ &- \frac{1}{A\alpha k_1}\frac{[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{-\Omega}\}\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 - BC_1\sqrt{-\Omega}\}\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]}{[(BC_1 + C_2\sqrt{-\Omega})\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + (BC_2 - C_1\sqrt{-\Omega})\ \sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]}, \end{aligned} \tag{26}$$

Family 3. When $B \neq 0$, $\Omega = 0$, we obtain the rational exact solution of Eq.(20) takes the following form:

$$\psi(\xi) = \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1^2} - \frac{1}{A\alpha k_1^2} \frac{[\{(4E\Gamma + B^2)(C_1 + C_2\xi) + 2BC_2\Gamma]}{B(C_1 + C_2\xi) + 2\Gamma C_2}. \quad (27)$$

Consequently the rational traveling wave solution of Eq.(14) has the following form:

$$u_3(x,y,z,t) = \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1} - \frac{1}{A\alpha k_1} \frac{[\{(4E\Gamma + B^2)(C_1 + C_2\xi) + 2BC_2\Gamma]}{B(C_1 + C_2\xi) + 2\Gamma C_2}, \quad (28)$$

Family 4. When $B = 0$, $\Delta = E\Gamma > 0$, we obtain the hyperbolic exact solution of Eq.(15) takes the following form:

$$\psi(\xi) = \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1^2} - \frac{2\sqrt{\Delta}}{A\alpha k_1^2} \frac{[C_1 \sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]}{[C_1 \cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]} - \frac{2E\Gamma}{A\alpha k_1^2 \sqrt{\Delta}} \frac{[C_1 \cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]}{[C_1 \sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]}. \quad (29)$$

Consequently the rational traveling wave solution of Eq.(14) has the following form:

$$u_4(x,y,z,t) = \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1} - \frac{2\sqrt{\Delta}}{A\alpha k_1} \frac{[C_1 \sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]}{[C_1 \cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]} - \frac{2E\Gamma}{A\alpha k_1 \sqrt{\Delta}} \frac{[C_1 \cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]}{[C_1 \sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]}. \quad (30)$$

where

$$\xi = k_1 x + k_2 y + k_3 z - \frac{t}{2\alpha^2 k_1 A^2}[16\alpha^2 E(C - A)) + 12A^2 k_1^2 \beta^2 - 12A^2 k_1 k_2 \beta\alpha\sigma + 3A^2\sigma^2\alpha^2 k_2^2 + 6A^2\sigma^2\alpha^2(k_2^2 + k_3^2)]. \quad (31)$$

Family 5. When $B = 0$, $\Delta = E\Gamma < 0$, we obtain the hyperbolic exact solution of Eq.(20) takes the following form:

$$\psi(\xi) = \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1^2} - \frac{2\sqrt{-\Delta}}{A\alpha k_1^2} \frac{[-C_1 \sin(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]}{[C_1 \cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi) + C_2 \sin(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]} - \frac{2E\Gamma}{A\alpha k_1^2 \sqrt{-\Delta}} \frac{[C_1 \cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi) + C_2 \sin(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]}{[-C_1 \sin(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]}. \quad (32)$$

Consequently the rational traveling wave solution of Eq.(14) has the following form:

$$u_5(x,y,z,t) = \frac{2\beta k_1 - \alpha\sigma k_2}{\alpha^2 k_1} - \frac{2\sqrt{-\Delta}}{A\alpha k_1} \frac{[-C_1 \sin(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]}{[C_1 \cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi) + C_2 \sin(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]} - \frac{2E\Gamma}{A\alpha k_1 \sqrt{-\Delta}} \frac{[C_1 \cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi) + C_2 \sin(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]}{[-C_1 \sin(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2 \cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]}, \quad (33)$$

where ξ is defined as Eq.(31).

Generalized Kudryashov method to the (3+1) dimensional Gardner type integro- differential equation :

In this subsection we discuss the solution of Eq.(20) by using generalized Kudryashov method. Balancing the highest order derivative ψ'^2 with the nonlinear term ψ^4, we have

$$N - M = 1. \quad (34)$$

Equation (34) has infinitely solutions, in the special if $M = 1$ then $N = 2$. Consequently the solution formula of

Eq.(20) has the following form:

$$\psi(\xi)=\frac{a_0+a_1Q(\xi)+a_1Q^2(\xi)}{b_0+b_1Q(\xi)} \quad (35)$$

where a_0,a_1,a_2,b_0,b_1 are constants to be determined later and

$$Q'(\xi)=AQ^2(\xi)+BQ(\xi). \quad (36)$$

Substituting Eqs. (35) and (36) into Eq. (20), we obtain a polynomial in $Q^{i-j},(i,j=0,1,2,...)$, Setting all coefficients of this polynomial to be zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica software package to get the unknown parameters $a_0,a_1,a_2,b_0,b_1,k_1,k_2,c_1$ and w .

$$a_0=-\frac{b_0(-2\beta k_1+2B\alpha k_1^2+\alpha\sigma k_2)}{\alpha^2k_1^2},$$

$$a_1=-\frac{2b_0A(-2\beta k_1+2B\alpha k_1^2+\alpha\sigma k_2)}{B\alpha^2k_1^2},$$

$$a_2=-\frac{4b_0A^2}{B\alpha}, \qquad b_1=\frac{2b_0A}{B},$$

$$c_1=-\frac{1}{2\alpha^4k_1^2}\{8k_1^5\beta\alpha^2B^2-8\beta^3k_1^3+\sigma^3k_2^3\alpha^3+12\alpha\beta^2k_1^2k_2\sigma-6\alpha^2\beta k_1k_2^2\sigma^2-4\alpha^3k_2\sigma k_1^4B^2\},$$

$$c_2=-\frac{1}{8\alpha^6k_1^4}\{-32\alpha\beta^3k_1^3k_2\sigma+24\alpha^2\beta^2k_1^2k_2^2\sigma^2-8\alpha^3\beta k_1k_2^3\sigma^3-8\sigma^2k_2^2\alpha^4k_1^4B^2+16\beta^4k_1^4+32\alpha^3\beta k_1^5B^2k_2\sigma+\sigma^4k_2^4\alpha^4+16\alpha^4k_1^8B^4-32\alpha^2\beta^2k_1^6B^2\}$$

$$w=-\frac{1}{2\alpha^2k_1}\{-12\beta^2k_1^2+12\sigma\alpha\beta k_1k_2-9\sigma^2\alpha^2k_2^2+4k_1^4\alpha^2B^2-6\sigma^2\alpha^2k_3^2\} \quad (37)$$

where b_0, $k_1,k_2,k_3,\alpha,\beta,\sigma$, and A,B are arbitrary constants . In this case the traveling wave solution takes the form:

$$\psi(\xi)=-\frac{B(-2\beta k_1+2B\alpha k_1^2+\alpha\sigma k_2)+2A(-2\beta k_1+2B\alpha k_1^2+\alpha\sigma k_2)Q(\xi)+4A^2\alpha k_1^2Q^2(\xi)}{\alpha^2k_1^2b_0B+2b_0A\alpha^2k_1^2Q(\xi)} \quad (38)$$

Substituting by the general solutions of Eq.(13) into (38) we have the rational traveling wave solution:

$$\psi(\xi)=-\frac{1}{\alpha^2k_1^2b_0(1-A\,Ce^{B\xi})^2+2b_0A\alpha^2k_1^2\,Ce^{B\xi}\,(1-A\,Ce^{B\xi})]}\{4A^2\alpha k_1^2BC^2e^{2B\xi}+(-2\beta k_1+2B\alpha k_1^2+\alpha\sigma k_2)(1-A\,Ce^{B\xi_n})^2+2AC(-2\beta k_1+2B\alpha k_1^2+\alpha\sigma k_2)e^{B\xi_n}(1-A\,Ce^{B\xi_n})\} \quad (39)$$

where

$$\xi=k_1x+k_2y+k_3z+\frac{t}{2\alpha^2k_1}\{-12\beta^2k_1^2+12\sigma\alpha\beta k_1k_2-9\sigma^2\alpha^2k_2^2+4k_1^4\alpha^2B^2-6\sigma^2\alpha^2k_3^2\}$$

There are many other cases which omit for convenience to the reader.

Remark 1: The general G'/G expansion method is more effective than the generalized Kuderyshov method. The general G'/G expansion is complicated than the generalized Kuderyshov method but determine many types of exact solutions such as the hyperbolic functions, trigonometric functions and rational function but the generalized Kuderyshov method determine only one type of solution.

TRAVELING WAVE SOLUTIONS FOR (2+1) DIMENSIONAL SAWADA- KOTERA NONLINEAR INTEGRO PARTIAL DIFFERENTIAL EQUATION

In this section, we use genealized (G'/G) expansion method to discuss the exact solutions for the nonlinear evolution equations in mathematical physics via (2+1) dimensional Sawada- Kotera nonlinear integro partial differential equation which are very important in the mathematical science and have been paid attention by many researchers in physics and engineering. The (2+1) dimensional Sawada- Kotera nonlinear integro partial differential equation takes the following form:

$$u_t = (u_{xxxx} + 5u\,u_{xx} + \frac{5}{3}u^3 + u_{xy})_x - 5\int_{-\infty}^{x} u_{yy}dx' + 5uu_y + 5u_x \int_{-\infty}^{x} u_y\, dx' \quad (40)$$

The transformation (15) convert the (2+1) dimensional Sawada- Kotera nonlinear integro partial differential equation (40) to the following partial differential equation

$$v_{xt} = (v_{5x} + 5v_x v_{xxx} + \frac{5}{3}v_x^3 + v_{xxy})_x - 5v_{yy} + 5v_x v_{xy} + 3v_{xx}v_y \quad (41)$$

Traveling wave transformation (17) permits us to convert the nonlinear partial differential equation (41) to the following ordinary differential equation

$$k_1\, w\phi'' + k_1(k_1^5\phi^{(5)} + 5k_1^4\phi'\phi''' + \frac{5}{3}k_1^3\phi'^3 + k_1^2k_2\phi''')' - 5k_2^2\phi'' + 10k_1^2k_2\phi''\phi' = 0. \quad (42)$$

By using the integration equation (42) can be written in the following form:

$$(k_1\, w - 5k_2^2)\phi' + k_1(k_1^5\phi^{(5)} + 5k_1^4\phi'\phi''' + \frac{5}{3}k_1^3\phi'^3 + k_1^2k_2\phi''') + 5k_1^2k_2\phi'^2 + c_1 = 0, \quad (43)$$

where c_1 is the integration constant. If , we take $\psi(\xi) = \phi'(\xi)$ equation (43) can be reduced to the following ODE's:

$$(k_1\, w - 5k_2^2)\psi + k_1^6\psi^{(4)} + 5k_1^5\psi\psi'' + \frac{5}{3}k_1^4\psi^3 + k_1^3k_2\psi'' + 5k_1^2k_2\psi^2 + c_1 = 0, \quad (44)$$

We discuss the solution of Eq.(44) by using generalized rational (G'/G) expansion method. Balancing the highest order derivative $\psi^{(4)}$ with the nonlinear term ψ^3, we get the solution formula of Eq.(44) has the following form:

$$\psi(\xi) = a_0 + \frac{a_1 G'(\xi)/G(\xi)}{[1+\rho\, G'(\xi)/G(\xi)]} + \frac{b_1[1+\rho\, G'(\xi)/G(\xi)]}{G'(\xi)/G(\xi)} + \frac{a_2[G'(\xi)/G(\xi)]^2}{[1+\rho\, G'(\xi)/G(\xi)]^2} + \frac{b_2[1+\rho\, G'(\xi)/G(\xi)]^2}{[G'(\xi)/G(\xi)]^2} \quad (45)$$

where a_0, a_1, b_1, a_2, b_2 and ρ are constants to be determined later. Substituting Eq. (45) along with (5) into Eq. (44) and cleaning the denominator and collecting all terms with the same order of $(G'(\xi)/G(\xi))$ together, the left hand side of Eq. (44) are converted into polynomial in $(G'(\xi)/G(\xi))$. Setting each coefficient of these polynomials to be zero , we derive a set of algebraic equations for $a_0, a_1, b_1, a_2, b_2, k_1, k_2, k_3, w, c_1$ and ρ. Solving the set of algebraic equations by using Maple or Mathematica , software package to get the following results:

Case 1.

$$a_0 = -\frac{1}{5A^2k_1^2}[9k_2A^2 - 10k_1^3(B^2 - 4E(C-A))],$$

$$b_2 = -\frac{12E^2k_1}{A^2},$$

$$a_2 = -\frac{3k_1[B^2 - 4E(C-A)]^2}{4A^2E^2}, \qquad \rho = \frac{B}{2E},$$

$$a_1 = b_1 = 0,$$

$$w = \frac{2}{5k_1A^4}\{-40k_1^6(B^2 - 4E(C-A))^2 + 17k_2^2A^4\}$$

$$\begin{aligned} c_1 = \frac{-1}{75A^6k_1^2}\{&-42240k_1^6k_2E^2C^2A^2 + 38400k_1^9B^4EC \\ &-153600k_1^9B^2E^2C^2 - 153600k_1^9B^2E^2A^2 - 38400k_1^9B^4EA \\ &+307200k_1^9B^2CAE^2 - 3200k_1^9B^6 + 243k_2^3A^6 \\ &-204800k_1^9E^3A^3 + 204800k_1^9E^3C^3 + 21120k_1^6B^2A^2ECk_2 \\ &-61440k_1^9E^3C^2A + 61440k_1^9E^3A^2C - 2640k_1^6B^4k_2A^2 \\ &-42240k_1^6E^2A^4k_2 - 21120k_1^6B^2EA^3k_2 \\ &+84480k_1^6E^2CA^3k_2\} \end{aligned} \quad (46)$$

where $C, B, E, A, \sigma, k_1, k_2$ are arbitrary constants. There are many other cases which are omitted her for convenience to the reader. In this case the traveling wave solutions of Eq.(44) take the following form:

$$\psi(\xi) = -\frac{1}{5A^2k_1^2}[9k_2A^2 - 10k_1^3(B^2 - 4E(C-A))] - \frac{3k_1[B^2 - 4E(C-A)]^2[G'(\xi)/G(\xi)]^2}{4A^2E^2[1+\rho\, G'(\xi)/G(\xi)]^2} - \frac{12E^2k_1[1+\rho\, G'(\xi)/G(\xi)]^2}{A^2[G'(\xi)/G(\xi)]^2} \quad (47)$$

There are many families to discuss the types of the traveling wave solutions of Eq.(44) as follows:

Family 1. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) > 0$, $\Gamma = A - C$ we obtain the hyperbolic exact solution of Eq.(44) takes the following:

$$\psi(\xi) = -\frac{1}{5A^2k_1^2}[9k_2A^2 - 10k_1^3(B^2 - 4E(C-A))] - \frac{3k_1\Omega^2[(BC_1 + C_2\sqrt{\Omega})\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + (BC_2 + C_1\sqrt{\Omega})\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]^2}{A^2[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{\Omega}\}\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 + BC_1\sqrt{\Omega}\}\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]^2} - \frac{3k_1[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{\Omega}\}\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 + BC_1\sqrt{\Omega}\}\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]^2}{A^2[(BC_1 + C_2\sqrt{\Omega})\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + (BC_2 + C_1\sqrt{\Omega})\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]^2} \quad (48)$$

Consequently the hyperbolic traveling wave solution of Eq.(40) has the following form:

$$u_1(x,y,t) = -\frac{1}{5A^2}[9k_2A^2 - 10k_1^3(B^2 - 4E(C-A))] - \frac{3k_1^2\Omega^2[(BC_1 + C_2\sqrt{\Omega})\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + (BC_2 + C_1\sqrt{\Omega})\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]^2}{A^2[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{\Omega}\}\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 + BC_1\sqrt{\Omega}\}\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]^2} - \frac{3k_1^2[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{\Omega}\}\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 + BC_1\sqrt{\Omega}\}\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]^2}{A^2[(BC_1 + C_2\sqrt{\Omega})\cosh(\frac{\sqrt{\Omega}}{2\Gamma}\xi) + (BC_2 + C_1\sqrt{\Omega})\sinh(\frac{\sqrt{\Omega}}{2\Gamma}\xi)]^2} \quad (49)$$

where

$$\xi = k_1x + k_2y - \frac{2t}{5k_1A^4}\{-40k_1^6(B^2 - 4E(C-A))^2 + 17k_2^2A^4\} \quad (50)$$

Family 2. When $B \neq 0$, $\Omega = B^2 + 4E(A-C) < 0$, we obtain the trigonometric exact solution of Eq.(44) takes the following form

$$\psi(\xi) = -\frac{1}{5A^2k_1}[9k_2A^2 - 10k_1^3(B^2 - 4E(C-A))] + - \frac{3k_1\Omega^2[(BC_1 + C_2\sqrt{-\Omega})\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + (BC_2 - C_1\sqrt{-\Omega})\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]^2}{A^2[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{-\Omega}\}\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 - BC_1\sqrt{-\Omega}\}\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]^2} - \frac{3k_1}{A^2}\frac{[\{(4E\Gamma + B^2)C_1 + BC_2\sqrt{-\Omega}\}\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + \{(4E\Gamma + B^2)C_2 - BC_1\sqrt{-\Omega}\}\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]^2}{[(BC_1 + C_2\sqrt{-\Omega})\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi) + (BC_2 - C_1\sqrt{-\Omega})\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]^2}. \quad (51)$$

Consequently the periodic trigonometric traveling wave solution of Eq.(40) has the following form:

$$u_2(x,y,t)=-\frac{1}{5A^2}[9k_2A^2-10k_1^3(B^2-4E(C-A))]+$$
$$-\frac{3k_1^2\Omega^2[(BC_1+C_2\sqrt{-\Omega})\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)+(BC_2-C_1\sqrt{-\Omega})\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]^2}{A^2[\{(4E\Gamma+B^2)C_1+BC_2\sqrt{-\Omega}\}\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)+\{(4E\Gamma+B^2)C_2-BC_1\sqrt{-\Omega}\}\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]^2} \quad (52)$$
$$-\frac{3k_1^2}{A^2}\frac{[\{(4E\Gamma+B^2)C_1+BC_2\sqrt{-\Omega}\}\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)+\{(4E\Gamma+B^2)C_2-BC_1\sqrt{-\Omega}\}\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]^2}{[(BC_1+C_2\sqrt{-\Omega})\cos(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)+(BC_2-C_1\sqrt{-\Omega})\sin(\frac{\sqrt{-\Omega}}{2\Gamma}\xi)]^2}.$$

where ξ is defined as Eq.(50).

Family 3. When $B\neq 0$, $\Omega=0$, we obtain the rational exact solution of Eq.(44) takes the following form:

$$\psi(\xi)=-\frac{9k_2}{5k_1^2}-\frac{3k_1}{A^2}\frac{[\{(4E\Gamma+B^2)(C_1+C_2\xi)+2BC_2\Gamma]^2}{[B(C_1+C_2\xi)+2\Gamma C_2]^2} \quad (53)$$

Consequently the rational traveling wave solution of Eq.(40) has the following form:

$$u_3(x,y,z,t)=-\frac{9k_2}{5k_1}-\frac{3k_1^2}{A^2}\frac{[\{(4E\Gamma+B^2)(C_1+C_2\xi)+2BC_2\Gamma]^2}{[B(C_1+C_2\xi)+2\Gamma C_2]^2}, \quad (54)$$

where ξ is defined as Eq.(50).

Family 4. When $B=0$, $\Delta=E\Gamma>0$, we obtain the hyperbolic exact solution of Eq.(44) takes the following form:

$$\psi(\xi)=-\frac{1}{5A^2k_1^2}[9k_2A^2-40k_1^3\Delta]-\frac{48k_1\Delta}{4A^2}\frac{[C_1\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)+C_2\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2}{[C_1\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)+C_2\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2}-\frac{12\Delta k_1[C_1\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)+C_2\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2}{A^2[C_1\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)+C_2\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2} \quad (55)$$

Consequently the rational traveling wave solution of Eq.(40) has the following form:

$$u_4(x,y,t)=-\frac{1}{5A^2k_1}[9k_2A^2-40k_1^3\Delta]-\frac{48k_1^2\Delta}{4A^2}\frac{[C_1\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)+C_2\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2}{[C_1\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)+C_2\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2}-\frac{12\Delta k_1^2[C_1\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)+C_2\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2}{A^2[C_1\sinh(\frac{\sqrt{\Delta}}{\Gamma}\xi)+C_2\cosh(\frac{\sqrt{\Delta}}{\Gamma}\xi)]^2} \quad (56)$$

where

$$\xi=k_1x+k_2y-\frac{2t}{5k_1A^4}\{-640k_1^6\Delta^2+17k_2^2A^4\} \quad (57)$$

Family 5. When $B=0$, $\Delta=E\Gamma<0$, we obtain the hyperbolic exact solution of Eq.(44) takes the following form:

$$\psi(\xi) = -\frac{1}{5A^2k_1^2}[9k_2A^2 - 40k_1^3\Delta] + \frac{48k_1\Delta}{4A^2}\frac{[-C_1\sin(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2\cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]^2}{[C_1\cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi) + C_2\sin(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]^2} + \frac{12\Delta k_1[C_1\cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi) + C_2\sin(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]^2}{A^2[-C_1\sin(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2\cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]^2}. \quad (58)$$

Consequently the rational traveling wave solution of Eq.(40) has the following form:

$$u_5(x,y,t) = -\frac{1}{5A^2k_1}[9k_2A^2 - 40k_1^3\Delta] + \frac{48k_1^2\Delta}{4A^2}\frac{[-C_1\sin(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2\cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]^2}{[C_1\cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi) + C_2\sin(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]^2} + \frac{12\Delta k_1^2[C_1\cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi) + C_2\sin(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]^2}{A^2[-C_1\sin(\frac{\sqrt{\Delta}}{\Gamma}\xi) + C_2\cos(\frac{\sqrt{-\Delta}}{\Gamma}\xi)]^2}. \quad (59)$$

where ξ is defined as Eq.(51).

3.2. Numerical solutions for KdV equation

In this section we give some figures to illustrate some of our results which obtained in this section. To this end , we select some special values of the parameters to show the behavior of the extended rational (G'/G) expansion method for the KdV equation.

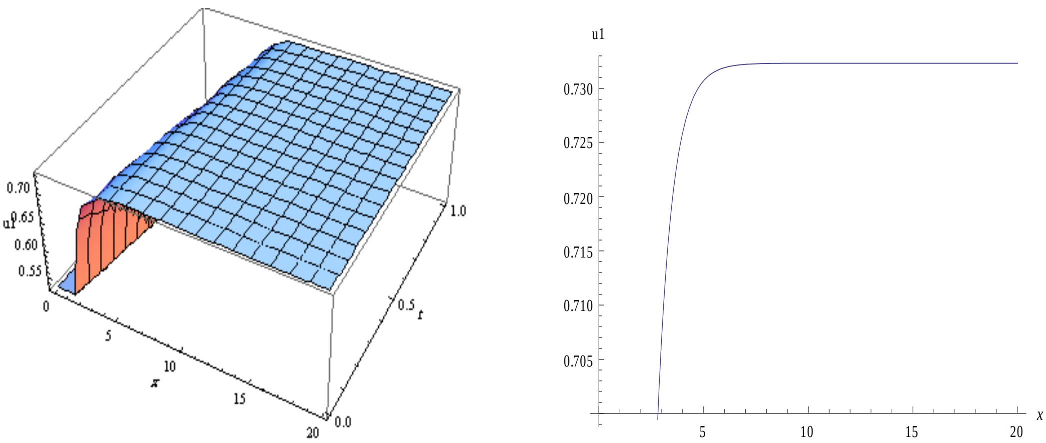

Figure 1. The exact extended (G'/G) expansion solution U_1 in Eq. (24) and its projection at $t=0$ when the parameters take special values $E=1, C1=2, A=2$, B=5, $C=1, k_1=5, k_2=3, k_3=7, \beta=11, \sigma=3, \alpha=1, z=3, y=5$

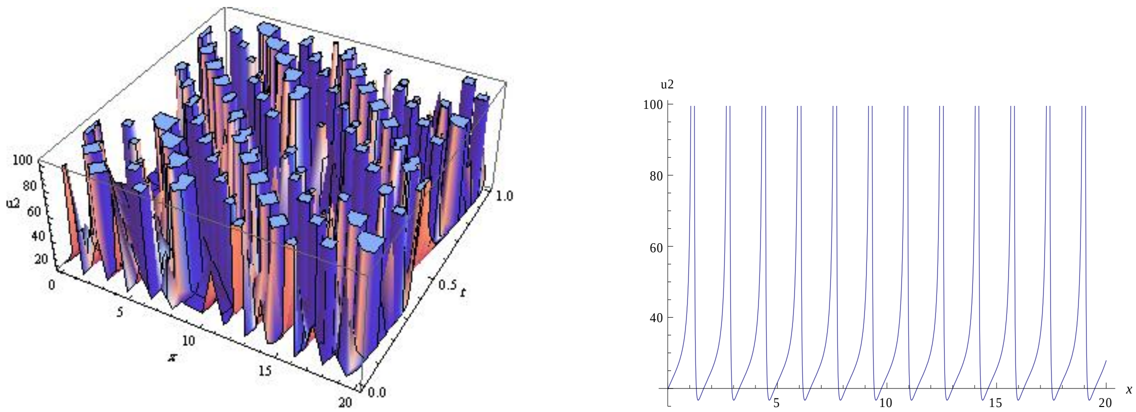

Figure 2. The exact extended (G'/G) expansion solution U_2 in Eq. (26) and its projection at $t=0$ when the parameters take special values $E=1,\ C1=2,\ C2=3,\ A=1,$ B=1, $C=10,\ k_1=5, k_2=3, k_3=7, \beta=11, \sigma=3, \alpha=1, z=3, y=5$

.

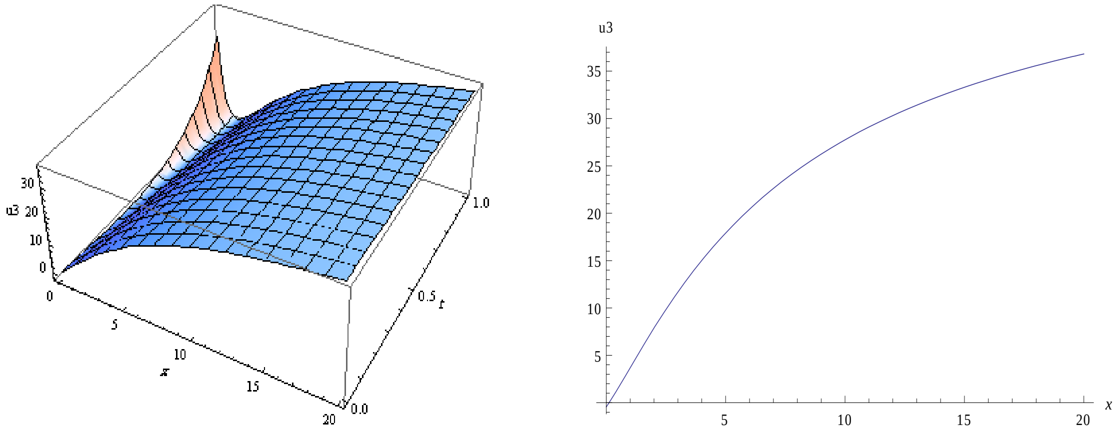

Figure 3. The exact extended (G'/G) expansion solution U_3 in Eq. (29) and its projection at $t=0$ when the parameters take special values $E=1,\ C1=2,\ C2=3,\ A=1,$ B=2, $C=2,\ k_1=5, k_2=3, k_3=7, \beta=11, \sigma=3, \alpha=1, z=3, y=5$

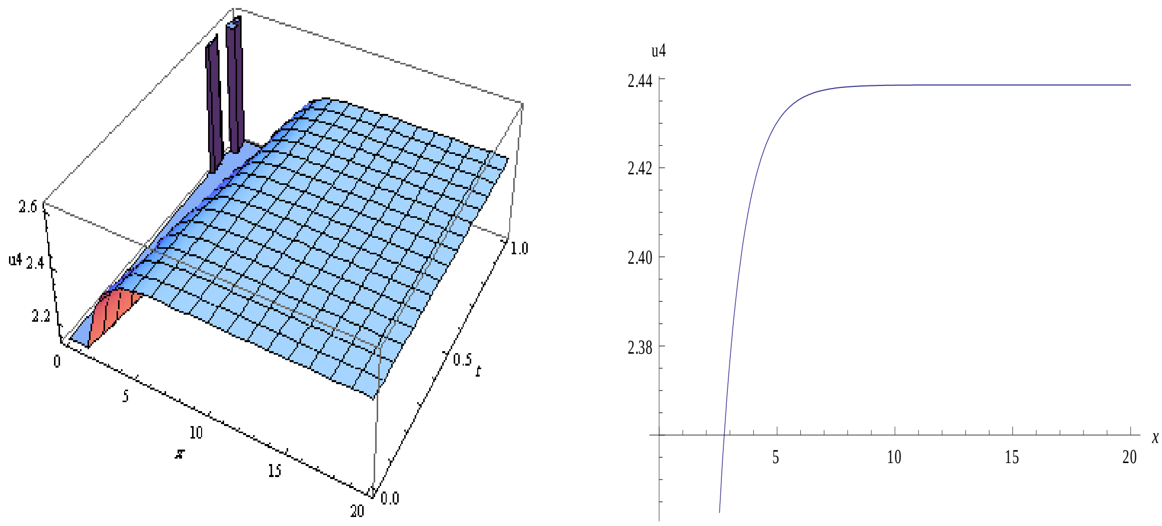

Figure 4. The exact extended (G'/G) expansion solution U_4 in Eq. (30) and its projection at $t=0$ when the parameters take special values $E=1,\ C1=2,\ C2=3,\ A=1,$ B=0, $C=2,\ k_1=5, k_2=3, k_3=7, \beta=11, \sigma=3, \alpha=1, z=3, y=5$

.

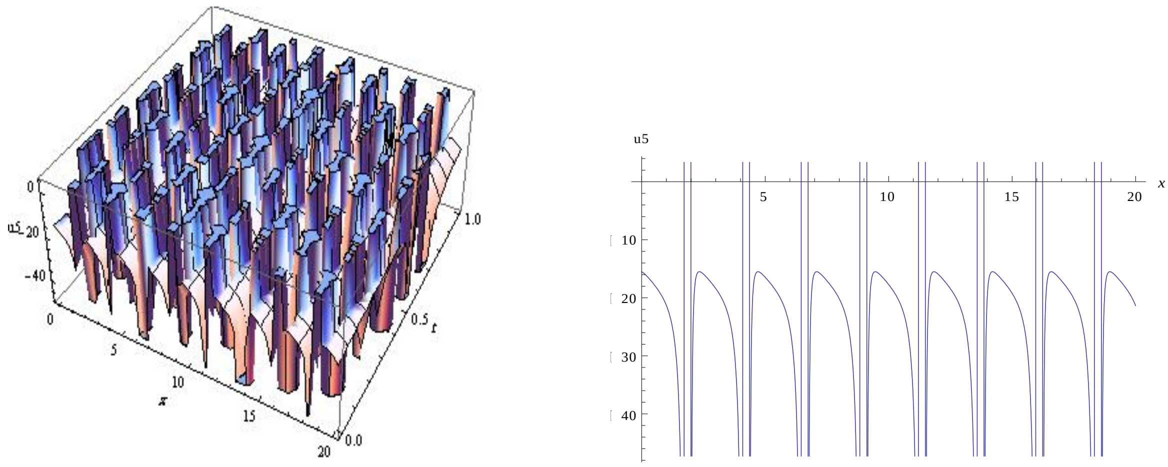

Figure 5. The exact extended (G'/G) expansion solution U_5 in Eq. (33) and its projection at $t=0$ when the parameters take special values $E=1,\ C1=2,\ C2=3,\ A=2,$ B=0, $C=1,\ k_1=5, k_2=3, k_3=7, \beta=11, \sigma=3, \alpha=1, z=3, y=5$

CONCLUSION

In this paper we use the generalized (G'/G) expansion method and generalized Kudryashov method to construct a series of some new traveling wave solutions for some nonlinear integro- partial differential equations in the mathematical physics. We constructed the rational exact solutions in many different functions such as hyperbolic function solutions, trigonometric function solutions and rational exact solution. The performance of this method reliable, effective and powerful for solving the nonlinear partial differential equations.

Conflict of Interests The authors declare that there is no conflict of interests regarding the publication of this paper.

REFERENCES

[1] M.J. Ablowitz and P.A. Clarkson, Solitons, nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge Univ. Press, Cambridge, 1991.

[2] R.Hirota, Exact solution of the KdV equation for multiple collisions of solutions, Phys. Rev. Letters 27 (1971) 1192-1194.

[3] M.R.Miura, Backlund Transformation,Springer-Verlag, Berlin,1978.

[4] A. Bekir , F. Tascan and O. Unsal, Exact solutions of the Zoomeron and Klein Gordon Zahkharov equations, J. Associat. Arab Univ. Basic Appl. Sci. 17 (2015) 1-5.

[5] J.Weiss, M.Tabor and G.Garnevalle, The Paineleve property for partial differential equations, J.Math.Phys. 24 (1983) 522-526

[6] D.S.Wang, Y.J.Ren and H.Q.Zhang, Further extended sinh-cosh and sin-cos methods and new non traveling wave solutions of the (2+1)-dimensional dispersive long wave equations, Appl. Math.E-Notes, 5 (2005) 157-163.

[7] M.L. Wang, Exact solutions for a compound KdV-Burgers equation , Phys. Lett. A 213 (1996) 279-287.

[8] K.A. Gepreel and T.A. Nofal, Extended trial equation method for nonlinear partial differential equations, Z. Naturforsch A70(2015) 269-279.

[9] F. Belgacem, H Bulut, H. Baskonus and T. Aktuk, Mathematical analysis of generalized Benjamin and Burger- KdV equation via extended trial equation method, J. Associat. Arab Univ. Basic Appl. Sci. 16 (2014) 91-100.

[10] J.H. He, Homotopy perturbation method for bifurcation of nonlinear wave equations, Int. J. Nonlinear Sci. Numer. Simul., 6 (2005) 207-208 .

[11] E.M.E.Zayed, T.A. Nofal and K.A.Gepreel, The homotopy perturbation method for solving nonlinear Burgers and new coupled MKdV equations, Zeitschrift fur Naturforschung Vol. 63a (2008) 627 -633.

[12] H.M. Liu, Generalized variational principles for ion acoustic plasma waves by He's semi-inverse method, Chaos, Solitons & Fractals, 23 (2005) 573 -576.

[13] H.A. Abdusalam, On an improved complex tanh -function method, Int. J. Nonlinear Sci.Numer. Simul., 6 (2005) 99-106.

[14] E.M.E.Zayed ,Hassan A.Zedan and Khaled A. Gepreel, Group analysis and modified extended Tanh- function to find the invariant solutions and soliton solutions for nonlinear Euler equations , Int. J. Nonlinear Sci. Numer. Simul., 5 (2004) 221-234.

[15] Y. Chen and Q. Wang, Extended Jacobi elliptic function rational expansion method and abundant families of Jacobi elliptic functions solutions to (1+1) dimensional dispersive long wave equation, Chaos, Solitons and Fractals, 24 (2005) 745-757 .

[16] S.Liu, Z. Fu, S.D. Liu and Q. Zhao, Jacobi elliptic

function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Letters A, 289 (2001) 69-74.

[17] S. Zhang and T.C. Xia, A generalized F-expansion method and new exact solutions of Konopelchenko-Dubrovsky equations, Appl. Math. Comput., 183 (2006) 1190-1200 .

[18] J.H. He and X.H. Wu, Exp-function method for nonlinear wave equations, Chaos, Solitons and Fractals, 30 (2006) 700-708.

[19] M.A. Abdou, The extended F-expansion method and its applications for a class of nonlinear evolution equation, Chaos, Solitons and Fractals 31(2007) 95 - 104 .

[20] M .Wang and X. Li, Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation, Chaos, Solitons and Fractals 24 (2005) 1257- 1268.

[21] M.Wang, X.Li and J.Zhang, The (G'/G)- expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics, Phys.Letters A, 372 (2008) 417-423.

[22] E.M.E.Zayed and K.A.Gepreel, The (G'/G)-expansion method for finding traveling wave solutions of nonlinear PDEs in mathematical physics, J. Math. Phys., 50 (2009) 013502-013514.

[23] E.M.E.Zayed and Khaled A.Gepreel, Some applications of the (G'/G) expansion method to non-linear partial differential equations, Appl. Math. and Comput., 212 (2009) 1-13.

[24] A. R. Shehata, E.M.E.Zayed and K.A Gepreel, Eaxct solutions for some nonlinear partial differential equations in mathematical physics , J. Information and computing Science 6 (2011) 129-142.

[25] MA Akbar, N. Ali, E. Zayed , A generalized and improved *(G′/G)*-expansion method for nonlinear evolution equations. Mathematical Problems in Engineering (2012) Article ID: 459879, 22 pages, doi:10.1155/2012/459879.

[26] Khaled A. Gepreel, Taher A. Nofaland Khulood O. Alweail, Extended rational (*G′/G)* - expansion method for nonlinear partial differential equations, Journal of Information and Computing Science, 11 (2016) 030-057.

[27] H. Naher and F. Abdullah, New generalized (G′/G)-expansion method to the Zhiber- Shabat Equation and Liouville Equations , IOP Conf. Series: Journal of Physics: Conf. Series 890 (2017) 012018.

[28] A. A. Al-Shawba, K. A. Gepreel, F. A. Abdullah, A. Azmi,Abundant Closed Form Solutions of the Conformable Time Fractional Sawada-Kotera-Ito Equation Using (G'/G) - Expansion Method, Accepted for publication in J. Results in Physics (2018) 9 (2018) 337–343.

[29] Khaled A Gepreel, Taher A Nofal and AMERA O.ALASMARI, Exact Solutions for Nonlinear Integro-Partial Differential Equations Using the Generalized Kudryashov Method, Journal of the Egyptian Mathematical Society 25 (2017), 438-444.

[30] Khaled A Gepreel, Taher A Nofal , Explicit traveling wave solutions for nonlinear differential difference equations in mathematical physics, International Journal of Applied Engineering Research 13(5) (2018) 3034-3042.