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Critical and Stability of Domination in Fuzzy Graphs

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Abstract

In this paper we studied the critical and stability of fuzzy dominating set. We investigate how the removal of a node affects the fuzzy domination number also we studied the stability of fuzzy path, fuzzy trees and fuzzy cycles. Also we obtain sharp bounds and characterizations.

KEYWORDS: Fuzzy dominating set, fuzzy domination number, critical node and fuzzy domination stability.

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INTRODUCTION

Brigham [1] introduced vertex domination critical graphs. Harary et al [2] explained an interesting application in voting situations using the concept of domination. Nagoor Gani and Vijayalakshmi [7] discussed domination critical nodes in fuzzy graph Rosenfeld [8] introduced the notion of fuzzy graph and several fuzzy analogous of graph theoretic concepts such as paths, cycles, connectedness and etc. Somasundaram and Somasundaram [9] discussed domination in fuzzy graphs. Sumner [10] discussed domination critical graphs. Nader Jafari Rad, Elahe Sharifi and Marcin krzywkowski [5] introduced domination stability in graphs. Bauer, Harary, Nieminen and Suffel [2] introduced domination alteration sets in graphs.

PRELIMINARIES:

A *fuzzy graph* G=(σ,μ) is a non-empty set V together with a pair of functions $\sigma: V \to [0,1]$ and μ:VxV→[0,1] such that μ(u,v) ≤ $\sigma(u) \wedge \sigma(v)$ for all u,v $\in V$, where $\sigma(u) \wedge \sigma(v)$is the minimum of $\sigma(u)$ *and* $\sigma(v)$.The *underlying crisp graph* of the fuzzy graph G=(σ,μ) is denoted as $G^*=(\sigma^*, \mu^*)$ where σ^*={ u∈ V/ $\sigma(u)>0$} and μ^*= {(u,v)∈VxV / μ(u,v)>0}.Let G =(σ,μ) be a fuzzy graph and τ be any fuzzy subset of σ,i.e $\tau(u) \le \sigma(u)$for all u,. Then the fuzzy subgraph of G = (σ,μ) induced by τ is the maximal fuzzy sub graph of G = (σ,μ) that has fuzzy node set τ. Evidently this is just the fuzzy graph (τ,ρ), where ρ(u,v) = $\tau(u) \wedge \tau(v)$for all u, v ∈ V.

Two nodes u and v are said to be *neighbours* if μ(u,v) > 0.The *strong neighbourhood* of u is $N_S(u)$ = { v ∈ V : (u,v) is a strong arc}. $N_S[u] = N_S(u) \cup \{u\}$ is the *closed strong neighbourhood* of u. Let G=(σ,μ) be a fuzzy graph. Two nodes u and v of G are *strong adjacent* if (u,v) is strong arc. The *strong degree* of a node v is the minimum number of nodes that are strong adjacent to v. It is denoted by $d_S(v)$. The minimum cardinality of strong neighbourhood δ_S(G) = min{$|N_S(u)|$:u∈ $V(G)$} and the maximum cardinality of strong neighbourhood Δ_S (G) = max {$|N_S(u)|$: u∈ $V(G)$} . Let G be a fuzzy graph. Let S be a set of vertices in G. Let u∈S then the *private neighbourhood* of u is pn(u,S) = { v: $N_S(v) \cap S$ = {u}}. The *external private neighbourhood* epn(v,S) = pn(u,S) \ S. A node u is called a *fuzzy end node* of G = (σ,μ) if it has atmost one strong neighbour in G = (σ,μ).

A *path* ρ in a fuzzy graph is a sequence of distinct nodes $u_0,u_1,u_2,...u_n$ such that $\mu(u_{i-1},u_i)>0$; $1\le i \le n$ here n≥ 0 is called the *length* of the path ρ .The consecutive pairs (u_{i-1},u_i) are called the *arcs* of the path. A path ρ is called a *cycle* if $u_o=u_n$ and n≥3. An arc (u,v) is said to be a *strong arc* if μ(u,v) ≥ μ^∞ (u,v) and the node v is said to be a *strong neighbour* of u. If μ(u,v) = 0 for every v∈V then u is called *isolated node*. Two nodes that are joined by a path are said to be *connected*. Let G= (σ,μ) be a fuzzy graph and u be a node in G then there exist a node v such that (u,v) is a strong arc then we say that *u dominates v*. Let G=(σ,μ) be a fuzzy graph. A set D of V is said to be *fuzzy dominating set* of G if every v∈V-D there exist u∈D such that u dominates v. A fuzzy dominating set D of G is called a *minimal fuzzy dominating set* of G if no proper subset of D is a fuzzy dominating set. The *fuzzy domination number* γ_f(G) of the fuzzy graph G is the smallest number of nodes in any fuzzy dominating set of G. A fuzzy dominating set D of a fuzzy graph G such that $|D| = \gamma_f$(G) is called minimum fuzzy dominating set.

FUZZY DOMINATING CRITICAL NODES

Definition 3.1:

Let G=(σ,μ) be a fuzzy graph. A node v of G is said to be *fuzzy dominating critical node* if its removal either increases (or) decreases the fuzzy domination number.

We partition the nodes of G into three disjoint sets according to how their removal affects γ_f (G). Let V= $V_f^0 \cup V_f^+ \cup V_f^-$ for

V_f^0= { v∈V : γ_f (G-v) = γ_f (G)}

V_f^+ = { v∈V : γ_f (G-v) > γ_f (G)}

V_f^- = { v∈V : γ_f (G-v) < γ_f(G)}

STABILITY OF FUZZY DOMINATING SET

Definition 4.1:

The *domination stability (or)* $\gamma_f -$ *stability* of a fuzzy graph is the minimum number of nodes whose removal changes the fuzzy domination number.

γ_f^+- *Stability* of a fuzzy graph G denoted by γ_f^+ (G) is defined as the minimum number of nodes whose removal increases γ_f (G).

γ_f^- - *Stability* of a fuzzy graph G denoted by γ_f^- (G) is defined as the minimum number of nodes whose removal decreases γ_f (G).

We denote the $\gamma_f -$ stability of G by stγ_f(G). Thus the domination stability of a fuzzy graph G is stγ_f(G) = min {γ_f^- (G) , γ_f^+ (G)}. If G is a disconnected fuzzy graph with components $G_1, G_2, \ldots G_k$ then stγ_f(G) = min { stγ_f(G_1), stγ_f(G_2),..... stγ_f(G_k)}.

Theorem 4.2:

The removal of a node v from G increases γ_f (G) if and only if (i) v is not isolated and v is in every minimum dominating set for G, and (ii) there is no dominating set for G-N_S[v] with fuzzy domination number γ_f (G) which also dominates N_S(v).

STABILITY OF FUZZY TREES

Theorem 5.1:

For any fuzzy tree T with at least three points γ_f (T-v) > γ_f (T) if and only if v is in every minimum fuzzy dominating set for T.

Proof:

By theorem 1 the necessity of v being in every minimum fuzzy dominating set for T is immediate.

Suppose v is in every minimum fuzzy dominating set of T. Note that γ_f (T-v) ≥ γ_f (T), for otherwise a minimum fuzzy dominating set of T-v could be extended to a fuzzy dominating set of T which avoids v and has cardinality at most γ_f (T). Let N_S(v) = { $v_1, v_2, \ldots v_m$} and T_i be the component of T-v containing v_i.

If γ_f (T-v) = γ_f (T), then for each i, v_i is in no minimum fuzzy dominating set of T_i, for otherwise such a fuzzy dominating set could be extended to a fuzzy dominating set of T which avoids v and has cardinality at most γ_f (T). Thus, for each i, γ_f (T-$\cup_{j\neq i} T_j$) = γ_f (T_i)+1, and so for any fuzzy dominating set D of T, $|D \cap V(T_i)| \geq \gamma_f$ (T_i).It follows that γ_f (T) ≥ $\sum_{i=1}^{n} \gamma_f$ (Ti)+1 = γ_f (T) +1, a contradiction.

Proposition 5.2:

If a cut node v of G is in every minimum fuzzy dominating set for G then γ_f (G-v) > γ_f (G)

Theorem 5.3:

Let T be a tree. Then γ_f^+(T) = 2 if and only if there are points v and u such that (i) every minimum fuzzy dominating set contains either v or u, (ii) v is in every minimum fuzzy dominating set for T-u and u is in every minimum fuzzy dominating set for T-v and (iii) no node is in every minimum fuzzy dominating set for T.

Proof:

The necessity of the conditions is clear. Furthermore sufficiency is easily established if we can prove that γ_f (T-v) = γ_f (T), for then condition (ii) will serve as the hypothesis for theorem 2 applied to T-v. The fact that γ_f (T-v) ≤ γ_f (T) follows from condition (ii) and theorem 2.

Suppose γ_f (T-v) < γ_f (T), and let S be a minimum fuzzy dominating set for T which contains v but not u. Let $v_1, v_2, \ldots v_m$ be the points strong adjacent to v. Then S= {v} ∪ $\bigcup_{i=1}^{m} S_i$ where S_i is a minimum collection of points from T_i – v_i. Note that if there are two or more values of i for which γ_f (T_i) = |Si|+1then γ_f^+(T) = 1, which contradicts condition (iii). Suppose there exists one value of i such that γ_f (T_i) = |Si|+1.

Then γ_f (T-v) = $\sum_{i=1}^{m} \gamma$(Ti) = 1+ $\sum_{i=1}^{m}$|Si| = γ_f (T), a contradiction. If γ_f (T_i) = |Si| for all i, then $\bigcup_{i=1}^{m}$Si is a minimum fuzzy dominating set for T-v which does not contain u, and we are done.

Proposition 5.4:

For all fuzzy graphs G, min { γ_f^+(G), γ_f^-(G)} ≤ δ_S(G) +1.

Theorem 5.5:

If G is a fuzzy graph with a fuzzy end node, then γ_f^+(G) ≥ 2 implies γ_f^-(G) ≤ 2. In particular this is true for fuzzy trees.

Proof:

Let v be a node of T which is adjacent to a fuzzy end node u of T. If γ_f (T-v) < γ_f (T) we are done. If not, since we know γ_f (T-v) ≤ γ_f (T), it follows that γ_f (T-v) = γ_f (T).

However T-v = {u}∪ T', where T' is a sub tree of T, and hence γ_f (T-v) = 1+ γ_f(T').But then γ_f (T-u-v) = γ_f(T') < γ_f (T-v) = γ_f (T) and so γ_f^-(T) ≤ 2.

Theorem 5.6:

For every fuzzy tree T there exists a node v∈T such that γ_f (T-v) = γ_f (T).

Proof:

We first note that if there is a node v∈T which is strong adjacent to two (or more) fuzzy end nodes u_1 and u_2 of T then v is in every minimum fuzzy dominating set for T and γ_f (T- u_1) = γ_f (T). If not, then T contains a node w of strong degree two which is strong adjacent to an end node u.

Let T' = T-w-u. Now for any fuzzy graph G, if d_s(v) = 1, then γ_f (G-v) ≤ γ_f (G). Hence γ_f (T') ≤ γ_f (T-v) ≤ γ_f (T). However clearly γ_f (T') ≥ γ_f (T) -1.Now if γ_f (T') = γ_f (T) -1, then γ_f (T) = γ_f (T-w). Otherwise γ_f (T') = γ_f (T) = γ_f (T-u).

STABILITY OF FUZZY PATHS AND CYCLES:

Theorem 6.1:

For n≥7, γ_f^+(P_n) + γ_f^-(P_n) = 4.

Proof:

Let path P_n = v_1,v_2,...v_n. We show that γ_f^+(P_n) + γ_f^-(P_n) = 4 by proving this separately for n≡ 0,1 and 2(mod3).

Case (i) n≡ 0 (mod3). Clearly v_2 is in every minimum fuzzy dominating set, hence by theorem 2 γ_f^+(P_n) = 1. To see that γ_f^-(P_n) = 3 first note that γ_f(P_n-3) = γ_f (P_n)-1; hence γ_f^-(P_n) ≤ 3. Since γ_f(P_n-1) = γ_f(P_n-2) = γ_f (P_n) the only way to lower the fuzzy domination number of P_n by removing either one or two nodes is to disconnect P_n.

Suppose we create two components, A and B, containing a and b nodes respectively, by removing either one or two nodes from P_n. Let k=1/3n. Then γ_f (A) + γ_f (B) = $\left\lceil\frac{1}{3}a\right\rceil$ + $\left\lceil\frac{1}{3}b\right\rceil$ ≥ $\frac{1}{3}$a + $\frac{1}{3}$b ≥k-$\frac{2}{3}$ and so γ_f (A) + γ_f (B) ≥ k. The last possibility, namely two nodes from P_n and creating three components, is immediate and we omit the details.

Case (ii) n≡ 1 (mod3). Now γ_f(P_n-1) = γ_f (P_n) – 1 and hence γ_f^-(P_n) = 1. If we remove {v_2,v_4,v_6} from P_n we obtain three isolated nodes and P_{n-6}. Since γ_f(P_n-6) = γ_f (P_n)-2 we conclude that γ_f^+(P_n) ≤ 3. Now note that no node of P_n is in every minimum fuzzy dominating set of P_n.

In fact the only pairs of nodes satisfying condition (i) of theorem 3 are {v_1,v_2} and {v_{n-1},v_n}. However in either case condition (ii) is not satisfied. Hence by theorem 2 and 3, γ_f^+(P_n) = 3.

Case (iii) n≡ 2(mod3).Here $v_2$and v_{n-1} satisfy the hypothesis of theorem 3 and thus γ_f^+(P_n) =2. Now by theorem 4 γ_f^-(P_n) ≤ 2. To see that γ_f^-(P_n) ≠1 we appeal to an argument similar to that used in case (i).

Theorem 6.2:

For n≥8, γ_f^+(C_n) + γ_f^-(C_n) = 6.

Proof:

It suffices to show that for n≡ 0,1 and 2(mod3), we have respectively γ_f^+(C_n) = γ_f^-(C_n) = 3, γ_f^+(C_n) = 5 and γ_f^-(C_n) = 1, and γ_f^+(C_n) = 4, γ_f^-(C_n) = 2. We indicate how to prove that γ_f^+(C_n) = 5 when n≡ 1 (mod3). The remaining cases follow easily from the proof of theorem 6.

Suppose n≡ 1 (mod3) and let k=$\left\lceil\frac{1}{3}n\right\rceil$.If we denote C_n by v_0,v_1,...v_n = v_0, then removal of the set of nodes {v_0,v_2,v_4,v_6,v_8} leaves four isolated nodes and P_{n-9}. However γ_f(P_n-9) = γ_f (P_n) – 3 = γ_f (C_n)-3 and thus γ_f^+(C_n)≤5.If we remove only a single node from C_n, we obtain P_{n-1} and since γ_f(P_n-1) = k-1, we know γ_f^+(C_n) ≥ 2. It remains to show that removal of fewer than four nodes from P_{n-1} will not cause the fuzzy domination number to exceed k.

Suppose three nodes are removed from P_{n-1} leaving four components A_i, 1≤i≤4, containing a_i points respectively, and that $\sum_{i=1}^{4} \gamma_f(A_i)$ ≥k+1. Since a_i ≥ 3 γ_f(A_i) -2 we have $\sum_{i=1}^{4} a_i$ ≥ [3$\sum_{i=1}^{4} \gamma_f(A_i)$] -8 ≥ 3(k+1) -8 = 3k-5.

However, $\sum_{i=1}^{4} a_i$ = 3(k-1) – 3 = 3k-6, a contradiction. Analogous arguments will show that if less than four components are formed as a result of removing fewer than four nodes from P_{n-1} the fuzzy domination number will never exceed k.

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