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Unravelling Non-Differentiable Manifold Problems based on Lagrange Duality and Wolfe Duality

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Abstract

This paper presented a solution for solving the problems in duality theorems for the category of non-differentiable multi-objective programming issues. Weak and strong duality theorems solve optimization problem based on the formulation of the primal and dual problems. Here the solution concepts of the primal and dual problems are based on the concept of Hybridization of both Lagrange and Wolfe duality theorem. The proof are designed in such a way that the solution for the primal problem must always be greater than or equal to the solution of the dual problem. Consequently the concepts of without duality gap in the weak and strong sense are also introduced, and strong duality theorems in the weak and strong sense are then derived.

Keywords: Second order duality, multi-objective programming issues, Type I function, Lagrange principle, Wolfe duality.

AMS 2002 Subject Classification: 90C29, 90C30, 90C46.

INTRODUCTION

Srivastava and Govil formulated second order Mond-Weir type dual for multi objective nonlinear programming and established duality results used second order (F,ρ,σ) type-I functions and their generalizations. Higher order cone convex, pseudo convex and similar convex functions are studied and better order duality results for a vertex improvement drawback over cones victimization higher order has been introduced [1]. The convexity theory plays an important role in many aspects of mathematical programming. In recent years, in order to relax convexity assumption, various generalized convexity notions have been obtained. One of them is the concept of (F, ρ) convex functions defined by Preda[2], which extended the class of F- convex functions and the classes of ρ convex functions defined by Vial [3,4]. that extended the category of F convex functions and also the categories of ρ convex functions [5].

The study of second order duality is important because of the process advantage over initial order duality because it provides tighter bounds for the worth of the objective function. Mangasarian [6] thought of a nonlinear program and mentioned second order duality using certain inequalities. The conception of second order convex function that was named as bonvex function by Bector and Chandra [7]. Later Bector and Chandra [8] established second order symmetric duality results for a combination of nonlinear programming issues by pseudo bonvexity and pseudo boncavity assumptions [9]. A pair of multi objective second order symmetric dual nonlinear programming problems, second order pseudo-invexity assumptions on the functions are involved in arbitrary cones. Subsequently, weak, strong, converse and self-duality theorems are established under second order pseudo-invexity or second order pseudo-incavity assumptions [10]. A weak second-order sufficient condition holds, and the feasible set is polyhedric, formula for computing the directional derivative of the optimal control with respect to a perturbation. [11] This is a partial extension of the results of [11] to the two norms setting, and is to be compared with [12] and [13], where the derivative of solution is computed under stronger second-order conditions, whereas our theory of second-order necessary conditions guaranties the fact that our sufficient condition is minimal [14].

RELATED RESEARCHES

Xin-Min Yang D, *et al.* [15] have described a technique of second-order symmetric dual models for multi objective nonlinear programming. Where, weak, strong, and converse duality theorems formulated, second-order symmetric dual programs under invexity conditions was used. A pair of second-order nonlinear multi objective programs was formulated the primal problem (MSP) and the dual problem (MSD) were formed. Weak duality was feasible for MSP and strong duality feasible for MSD. There the objective values of MSP and MSD were equal, efficient solution of MSD were proved. The MSD efficient solution was used in converse duality.

TANG LiPing, *et al.* [16] have described the weak duality solution, where objective value of a feasible solution to the primal problem was not less than the corresponding dual one. This solution provides a lower bound for the primal optimal value that was feasible dual solution. The strong duality theorem described, whenever the primal problem had an optimal solution, the dual problem also had one and there was no duality gap. However, the most difficult part of the duality theory, was on the converse duality theorem. It deals with the issues on how to obtain the primal solution from the dual solution and on conditions under which there is no gap between the primal problem and the dual problem.

S.J. Lia, *et al*. [17] have described, Weir and Mond method that explain weak, strong and converse duality for weak minima of multiple objective optimization problems under different pseudo-convexity and quasi convexity assumptions. Mond–Weir type of duality results was differentials of generalized d-type-I functions that involved in the multi objective programming problem. In that Mond–Weir duality scheme for optimization problems involving set functions, that defined on a measure space with the variables being measurable sets .In Mond–Weir type of duality results under generalized pseudo convexity and generalized quasi convexity assumptions were investigated on duality properties of optimization problems with set-valued mappings that satisfy an invex property and by virtue of tangent derivative of set-valued mapping.

Anurag Jayswal, *et al*. [18] Duality is one of the most prominent in operation research that helps generating useful insights about the optimization problem. Second order duality has even greater significance over first order duality, since it provides tighter bounds for the value of the objective function when approximations were used, because it involves more parameters. Another advantage of second order duality have been defined feasible point in the primal, where, first order duality conditions do not apply, then second order duality used to provide a lower bound to the value of the primal problem.

Mishra SK, [19] have described second order symmetric duality, under second order F-convexity F-concavity/second order F-pseudo convexity F-pseudo concavity for second order Wolfe and Mond-Weir type models, respectively. These second order duality results are then used to formulate Wolfe type, and Mond-Weir type second order minimax mixed integer dual programs and a symmetric duality theorem is established under separability and second order F-convexity F-concavity of the kernel function. Fractional symmetric duality results and self-duality have also been discussed in second order duality.

HYBRID LAGRANGE-WOLFE DUALITY FOR MULTI-OBJECTIVE OPTIMIZATION PROBLEM

Duality is an important feature of optimization problem. There are several types of problems such as convex, linear, constrained, equality etc. By the employment of lagrangian theorem, convex, inequality, concave problems can be solved and by utilizing Wolfe theorem invexity problems can be solved. Therefore by the hybridization of lagrange and wolfe duality theorems all these problems can be solved. As this hybrid duality theorem solves all these problems that directs to the solution for multi-objective optimization problem.

LAGRANGE DUALITY THEOREM

Generally the term "dual problem" refers to Lagrangian dual problem but other dual problems like Wolfe dual problem and Fenchel dual problem can also be used. The Lagrangian dual problem is acquired by creating the Lagrangian, using nonnegative Lagrange multipliers to add the constraints to the objective function, and then solving for some primal variable values that minimize the Lagrangian. This solution obtains primal variables as functions of Lagrange multipliers, which is called as dual variables, so that the new problem is to maximize the objective function with respect to the dual variables under the derived constraints on the dual variables.

Primal and dual problem

Let us consider the non-linear programming problem,

Minimize $f(a^*)$ (1)

Subject to: $g_x(a^*) \le 0$ for $x = 1,2,\ldots\ldots i$

$h_x(a^*) = 0$ for $x = 1,2,\ldots\ldots j$

$a^* \in A$

The Lagrangian dual problem for non-linear programming problem is defined as,

Maximize $\phi_L(m,n)$ (2)

Subject to: $m \ge 0$

Where,

$$\phi_L(m,n) = \inf\{ f(a^*) + \sum_{x=1}^{i} m_x g_x(a^*) + \sum_{x=1}^{j} n_x h_x(a^*) : a^* \in A \quad (3)$$

Theorem 1: Weak duality

Consider the primal problem P given by eqn. (1) and its Lagrangian dual problem D given by eqn. (2). Let a^* be a feasible solution to P that is, $a^* \in A$ $g(a^*) \le 0$ and $h(a^*) = 0$. Also let (m,n) be a feasible solution to D; that is $m \ge 0$. Then,

$$f(a^*) \ge \phi_L(m,n)$$

Proof:

We use the definition of ϕ_L given in eqn.(3) and the facts that $a^* \in A$, $m \ge 0$ $g(a^*) \le 0$ and $h(a^*) = 0$

To prove:

$$\phi_L(m,n) = f(a^*) + \sum_{x=1}^{i} m_x g(a^*) + \sum_{x=1}^{j} n_x h(a^*) \le f(a^*)$$

Corollary:

$$\inf\{ f(a^*) : a^* \in A, g(a^*) \le 0, h(a^*) = 0\} \ge Sup\{\phi_L(m,n) : m \ge 0\}$$

System 1:

$g(a^*) \le 0, h(a^*) = 0$, for some $a^* \in A$, has no solution.

Define the set

$S = \{(p,q,r) : p > \alpha(a^*), q \ge g(a^*), r = h(a^*)$, for some $a^* \in A$

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The set S is convex, since A,α and g are convex and h is affine. Since System 1 has no solution, we have that $(0,0,0) \notin S$.

Recall the following corollary of the Supporting Hyper plane Theorem:

Corollary

Let S be a nonempty convex set in R^n and $\bar{a}^* \notin \operatorname{int} S$. Then there is a nonzero vector p such that $p^U(a^* - \bar{a}^*) \le 0$ for each $a^* \in \operatorname{cl} S$.

We then have, from the above corollary, that there exists a nonzero vector $(m_0, m\ \ , n\ \ )$ such that

$$(m_0, m\ \ , n\ \ )^U[(p,q,r)-(0,0,0)] \\ = m_0 p + m^U q + n^U r \ge 0 \tag{4}$$

For each $(p,q,r) \in \operatorname{cl} S$.

Now, fix an $a^* \in A$. Noticing, from the definition of S , that p and q can be made arbitrarily large, we have that in order to satisfy (4),

We must have $m_0 \ge 0$ and $m \ge 0$.

We have that there exists a nonzero vector $(m_0, m\ \ , n\ \ )$ with $(m_0, m\ \ ) \ge (0,0)$ such that

$(m_0, m\ \ , n\ \ )^U[(p,q,r)-(0,0,0)] = m_0 p + m^U q + n^U r \ge 0$, for each $(p,q,r) \in \operatorname{cl} S$.

Also, note that $[\alpha(a^*), g(a^*), h(a^*)] \in \operatorname{cl} S$ and we have from the above inequality that

$m_0 \alpha(a^*) + m^U g(a^*) + n^U h(a^*) \ge 0$.

Since the above inequality is true for each $a^* \in A$, we conclude that

System:1 $m_0 \alpha(a^*) + m^U g(a^*) + n^U h(a^*) \ge 0$ for some $(m_0, m, n) \ne (0,0,0), (m_0, m) \ge (0,0)$ and for all $a^* \in A$ has a solution.

To prove the converse, assume that $m_0 > 0$.

Suppose that $a^* \in A$ is such that $g(a^*) \le 0$. But since $m_0 > 0$ we must then have that $\alpha(a^*) \ge 0$.

System:2 $\alpha(a^*) < 0$, $g(a^*) \le 0$, $h(a^*) = 0$ for some $a^* \in A$.

Has no solution and this completes the proof.

Theorem 2:Strong duality

Let A be a nonempty convex set in R^n .Let $f : R^n \to R$ and $g : R^n \to R^m$ be convex and $h : R^n \to R^l$ be affine. Suppose that the following constraint qualification is satisfied. There exists an $\hat{a}^* \in A$ such that $g(\hat{a}^*) < 0$ and $h(\hat{a}^*) = 0$ and $0 \in \operatorname{int} h(a^*)$ where $h(A) = \{h(a^*) : a^* \in A\}$. Then,

$$\inf\{ f(a^*) : a^* \in A, g(a^*) \le 0, h(a^*) \\ = 0\} = \sup\{\varphi_L(m,n) : m \ge 0\}, \tag{5}$$

Where $\phi_L(m,n) = \inf\{ f(a^*) + m^U g(a^*) + n^U h(a^*) : a^* \in A\}$.
Furthermore, if the $\inf$ is finite, then $\sup\{\phi_L(m,n) : m \ge 0$, is achieved at $\bar{m}, \bar{n}$ with $\bar{m} \ge 0$ if the $\inf$ is achieved at $\bar{a}^*$ then $\bar{m}^U g(\bar{a}^*) = 0$

Let $\gamma = \inf\{ f(a^*) : a^* \in A, g(a^*) \le 0, h(a^*) = 0$.

By assumption there exists a feasible solution $\hat{a}^*$ for the primal problem and hence $\gamma < \infty$.

If $\gamma < -\infty$ we then conclude from the corollary of the Weak Duality Theorem that $\sup\{\phi_L(m,n) : m \ge 0\} = -\infty$ and hence, (5) is satisfied.

Thus, suppose that γ is finite, and consider the following system:

$f(a^*) - \gamma < 0, g(a^*) \le 0, h(a^*) = 0$, for some $\hat{a}^* \in A$.

By the definition of γ this system has no solution. Hence, from the previous lemma, there exists a nonzero vector (m_0, m, n) with $(m_0, m) \ge (0,0)$ such that

$$m_0[f(a^*) - \gamma] + m^U g(a^*) + n^U h(a^*) \ge 0 \text{ , for all } \hat{a}^* \in A \tag{6}$$

We will next show that $m_0 > 0$ Suppose, by contradiction that $m_0 = 0$

By assumption, there exists an $\hat{a}^* \in A$ such that $g(\hat{a}^*) < 0$ and $h(\hat{a}^*) = 0$. Substituting in (6) we obtain $m^U g(\hat{a}^*) \ge 0$. But since $g(\hat{a}^*) < 0$ and $m \ge 0, m^U g(\hat{a}^*) \ge 0$ is only possible if $m = 0$.

From (6) $m_0 = 0$ and $m = 0$ imply that $n^U h(a^*) \ge 0$ for all $\hat{a}^* \in A$ .but since $0 \in \operatorname{int} h(a^*)$ we can choose an $\hat{a}^* \in A$ such that $h(a^*) = -\lambda n$, where $\lambda > 0$.
Therefore $0 \le n^U h(a^*) = \lambda \|v\|^2$ which implies that $n = 0$.

Thus, it has been shown that $m_0 = 0$ implies that $(m_0, m, n) = (0,0,0)$ which is a contradiction. We conclude, then, that $m_0 > 0$.

Dividing (6) by m_0 and denoting $\overline{m} = m/m_0$ and $\overline{n} = n/m_0$ we obtain

$$f(a^*) + \overline{m}^U g(a^*) + \overline{n}^U h(a^*) \geq \gamma \text{ for all } \hat{a}^* \in A \quad (7)$$

To prove:

$$\varphi_L(\overline{m}, \overline{n}) = \inf\left\{ f(a^*) + \overline{m}^U g(a^*) + \overline{n}^U h(a^*) : a^* \in A \right\} \geq \gamma$$

We then conclude, from the Weak Duality Theorem, that $(\overline{m}, \overline{n}) = \gamma$ and, from the corollary of the Weak Duality Theorem, we conclude that $(\overline{m}, \overline{n})$ solves the dual problem.

Finally, to complete the proof, assume that $\overline{a}^*$ is an optimal solution to the primal problem; that is, $\overline{a}^* \in A$, $g(\overline{a}^*) \leq 0$, $h(\overline{a}^*) = 0$ and $f(\overline{a}^*) = \gamma$.

From (7), assuming $a^* = \overline{a}^*$, we get $\overline{m}^U g(\overline{a}^*) \geq 0$ .since $\overline{m} \geq 0$ and $g(\overline{a}^*) \leq 0$ we get $\overline{m}^U g(\overline{a}^*) = 0$

This completes the proof.

Example 1: Weak duality

Weak duality equation is represented as,

$$\phi_L(m,n) = f(a^*) + m^u g(a^*) + n^u h(a^*) \leq f(a^*) \quad (8)$$

Where, $\phi_L(m,n)$ is the solution for dual problem and $f(a^*)$ is the primal problem.

The value for dual problem must always be less than the value of primal problem.

Given condition: $h(a^*) = 0$, $m \geq 0$ and $g(a^*) \leq 0$ for all $h(a^*) = 0$

Possible conditions: $m = 1 \,\&\, g(a^*) = -1$;

$m = 0 \,\&\, g(a^*) = -1$

Condition 1: $m = 1, g(a^*) = -1 \,\&\, h(a^*) = 0$

Substitute the above condition in eqn. (8) we get,

$f(a^*) + 1(-1) + 0 \leq f(a^*)$

$f(a^*) - 1 \leq f(a^*)$

Substitute $f(a^*) = 5$ we get, $4 \leq 5$

Condition 2: $m = 0, h(a^*) = 0 \,\&\, g(a^*) = -1$

Substitute the above condition in eqn. (8) we get,

$f(a^*) + 0 + 0 \leq f(a^*)$

$f(a^*) \leq f(a^*)$

Substitute $f(a^*) = 5$ we get, $5 \leq 5$

Hence proved.

Example 2: Strong duality

Strong duality equation is represented as,

$$f(a^*) + \overline{m}^U g(a^*) + \overline{n}^U h(a^*) \geq \gamma \quad (9)$$

Given condition: $f(a^*) = \gamma, \overline{m}^u g(a^*) \geq 0, \overline{n}^u h(a^*) \geq 0$

Possible conditions: $\overline{m}^u g(a^*) = 1, \overline{n}^u h(a^*) = 1$

$\overline{m}^u g(a^*) = 0, \overline{n}^u h(a^*) = 0$

$\overline{m}^u g(a^*) = 1, \overline{n}^u h(a^*) = 0$

$\overline{m}^u g(a^*) = 0, \overline{n}^u h(a^*) = 1$

Condition 1: $\overline{m}^u g(a^*) = 1, \overline{n}^u h(a^*) = 1, \; f(a^*) = \gamma$

Substitute the above condition in eqn. (9) we get,

$\gamma + 1 + 1 \geq \gamma$

$\gamma + 2 \geq \gamma$

Substitute $\gamma = 5$

$5 + 2 \geq 5$

$7 \geq 5$

Condition 2: $\overline{m}^u g(a^*) = 0, \overline{n}^u h(a^*) = 0, \; f(a^*) = \gamma$

Substitute the above condition in eqn. (9) we get,

$\gamma + 0 + 0 \geq \gamma$

$\gamma \geq \gamma$

Substitute $\gamma = 5$

$5 \geq 5$

Condition 3: $\overline{m}^u g(a^*) = 1, \overline{n}^u h(a^*) = 0, \; f(a^*) = \gamma$

Substitute the above condition in eqn. (9) we get,

$\gamma + 1 + 0 \geq \gamma$

$\gamma + 1 \geq \gamma$

Substitute $\gamma = 5$

$5+1 \geq 5$

$6 \geq 5$

Condition 4: $\overline{m}^u g(a^*) = 0, \overline{n}^u h(a^*) = 1,\ f(a^*) = \gamma$

Substitute the above condition in eqn. (9) we get,

$\gamma + 0 + 1 \geq \gamma$

$\gamma + 1 \geq \gamma$

Substitute $\gamma = 5$

$5+1 \geq 5$

$6 \geq 5$

Hence proved.

FRANK-WOLFE DUALITY THEOREM

Strong and weak duality theorems are presented below.

First of all, an ordering is introduced. Let A and B be two sets of closed intervals. In conventional optimization problems, the weak duality theorem says that theobjective value of the primal problem is always greater than or equal to the objectivevalue of the dual problem. In some sense, $\min(f, A)$ and $\max(\phi_W, Y)$ can be regardedas kinds of optimal objective values of the primal problem (IVP) and dual problem(IVD), respectively. Therefore, we are going to present the weak duality theorem forproblems (IVP) and (IVD) by showing that $\min(f, A)$and $\max(\phi_W, Y)$.

Theorem 1: Weak duality theorem

Let A_0 be an open subset of R^n. Let f and $G_x, x = 1,2,.....k$ be convex and H-differentiable on X_0 Then, we have $\min(f, A), \max(\phi_W, Y)$.

Proof:

The result follows immediately from the equation below,

$$F^T(a) \geq (\phi_W(a,u))^T$$

In the sequel, we are going to present the strong duality theorem by considering

$$\min(f, A) \cap \max(\phi_W, Y) \neq \emptyset$$

In other words, there exist $f(a^*) \in \min(f, A)$ and $\phi_W(a_0, u_0) \in \max(\phi_W, Y)$ such that

$$f(a^*) = \phi_W(a_0, u_0)$$

Therefore, we provide the following definition.

Definition 1: Two kinds of concepts for no duality gap are presented below:

(i) We say that the primal problem (IVP) and the dual problem (IVD) have no duality gap in the weak sense if and only if $\min(f, A) \cap \max(\phi_W, Y) \neq \emptyset$.

(ii) We say that the primal problem (IVP) and the dual problem (IVD) have no duality gap in the strong sense if and only if there exist $f(a^*) \in \min(f, A)$ and $\phi_W(a^*, u^*) \in \max(\phi_W, Y)$ such that $f(a^*) = \phi_W(a^*, u^*)$.

Where, a^* is a feasible solution to primal problem. a^*, u^* is a feasible solution to dual problem

The fact that the primal problem (IVP) and the dual problem (IVD) have no duality gap in the strong sense implies that the primal problem (IVP) and the dual problem(IVD) have no duality gap in the weak sense. We also see that, if $a^* = a_0$, then the fact that the primal problem (IVP) and the dual problem (IVD) have no dualitygap in the weak sense implies that the primal problem (IVP) and the dual problem(IVD) have no duality gap in the strong sense.

Theorem 2:Strong duality theorem in the weak sense

Let A_0 be an open subset of R^n. Let f and $G_x, x = 1,2,.....k$ be convex and H-differentiable on A_0. Suppose if one of the following conditions is satisfied, then

(i) There exist a feasible solution a^* of the primal problem (IVP) such that $f(a^*) \in obj_D(\phi_W, Y)$.

(ii) There exist a feasible solution a^*, u^* of the dual problem (IVD) such that $\phi_W(a^*, u^*) \in obj_P(f, A)$.

Then, the primal problem (IVP) and dual problem (IVD) have no duality gap in the weak sense.

Proof The result follows immediately from the equation below,

$$\phi_W(a_0, u_0) = f(a^*) \geq \phi_W(a, u)$$

Theorem 3: Strong duality theorem in the strong sense

Let A_0 be an open subset of R^n. Let f and $G_x, x = 1,2,.....k$ be convex and H-differentiable on A_0. Suppose that a^* is a feasible solution of the primal problem (IVP) and that (a^*, u^*) is a feasible solution of the dual problem (IVD) with the property that

$$\sum_{x=1}^{k} u_x^* G_x\left(a^*\right) = [0,0]$$

Where $u^* = u_1, u_2 \ldots\ldots u_k$. Then a^* solves the primal problem (IVP), (a^*, u^*) solves the dual problem (IVD) and the dual problem (IVD) has no duality gap in the strong sense.

Condition:

$$\sum_{x=1}^{k} u_x^* G_x\left(a^*\right) \geq 0; G_x^T\left(a^*\right) \leq 0; u_x, u_x^* \geq 0$$

To prove:

$$\sum_{x=1}^{k} u_x G_x\left(a^*\right) = [0,0]$$

From the above equation we have that,

$$\phi_W\left(a^*, u^*\right) = f\left(a^*\right) + \sum_{x=1}^{k} u_x^* G_x\left(a^*\right) = f\left(a^*\right)$$

Let (a^*, u^*) be a feasible solution of the dual problem (IVD). Suppose that $\arg-\max\left(\phi_W\left(a^*, u^*\right), R_+^m\right)$ contains $u = 0$ and u infinitely large in magnitude.Then, a^* is a feasible solution of primal problem (IVP) and in the sequel, we are going to provide sufficient conditions to guarantee that

$$\sum_{x=1}^{k} u_x^* G_x\left(a^*\right) = [0,0]$$

$$\phi_W\left(a^*, u^*\right) = f\left(a^*\right) + \sum_{x=1}^{k} u_x^* G_x\left(a^*\right) = f\left(a^*\right)$$

Proof:

We have that $\phi\left(a^*, u^*\right) \geq \phi\left(a^*, u\right)$ for all $u \in \arg-\max\left(\phi\left(a^*, u^*\right), R_+^k\right)$.

By definition, we see that $\left(\phi\left(a^*, u^*\right)\right)^T \geq \left(\phi\left(a^*, u\right)\right)^T$ for all $u \in \arg-\max\left(\phi\left(a^*, u^*\right), R_+^k\right)$.
Since $u, u^* \geq 0$, $\left(\phi\left(a^*, u^*\right)\right)^T \geq \left(\phi\left(a^*, u\right)\right)^T$ implies,

$$f^T\left(a^*\right) + \sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right) \geq f^T\left(a^*\right) + \sum_{x=1}^{k} u_x G_x^T\left(a^*\right)$$

The above equation can be written as,

$$f^T\left(a^*\right) + \sum_{x=1}^{k} u_x G_x^T\left(a^*\right) \leq f^T\left(a^*\right) + \sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right)$$

$$\sum_{x=1}^{k} u_x G_x^T\left(a^*\right) \leq \sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right) \qquad (10)$$

Example 3:

Given condition: $\sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right) \geq 0$

Where, $u^* = u$

Possible conditions:

$$\sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right) = 1$$

$$\sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right) = 0$$

Condition 1: $\sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right) = 1$

Apply the above conditions in eqn. (10) we get,

$$\sum_{x=1}^{k} u_x G_x^T\left(a^*\right) \leq \sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right)$$

$1 \leq 1$

Condition 2: $\sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right) = 0$

Apply the above conditions in eqn. (10) we get,

$$\sum_{x=1}^{k} u_x G_x^T\left(a^*\right) \leq \sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right)$$

$0 \leq 0$

Hence proved.

HYBRID LAGRANGE-WOLFE DUALITY THEOREM

Common Lagrange equation is represented as,

$$\phi_L(m,n) = \inf\{ f\left(a^*\right) + \sum_{x=1}^{i} m_x g_x\left(a^*\right) + \sum_{x=1}^{j} n_x h_x\left(a^*\right) \leq f\left(a^*\right)$$

Summation operation is removed in the above equation can be rewritten as,

$$\phi_L\left(m^*, n^*\right) = f\left(a^*\right) + m^* g\left(a^*\right) + n^* h\left(a^*\right) \leq f\left(a^*\right)$$

Where $m = m^*$

Common Wolfe equation is represented as,

$$\phi_W\left(a^*, u^*\right) = \sum_{x=1}^{k} u_x G_x^T\left(a^*\right) \leq \sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right)$$

Summation operation is removed in the above equation and can be rewritten as,

$$\phi_W\left(a^*,u^*\right)=uG^T\left(a^*\right)\le u^*G^T\left(a^*\right)$$

Given conditions for lagrange theorem:
$h\left(a^*\right)=0; m, m^*=1{,}0, g\left(a^*\right)=-1{,}0$

Given conditions for Wolfe theorem: $u_x^*=1{,}0; G_x^T\left(a^*\right)=-1{,}0$

The equation for hybrid lagrange and wolfe can be represented as,

$$\varphi_{LW}\left(a^*,m^*,n^*,u^*\right)=f\left(a^*\right)+m^*g\left(a^*\right)+n^*h\left(a^*\right)+\sum_{x=1}^{k}u_xG_x^T\left(a^*\right)\le f\left(a^*\right)+\sum_{x=1}^{k}u_x^*G_x^T\left(a^*\right) \quad (11)$$

Given condition:

Condition for both lagrange and wolfe duality theorem is represented below,

$$h\left(a^*\right)=0; m^*, m=1{,}0; g\left(a^*\right)=-1{,}0; \sum_{x=1}^{k}u_x^*G_x^T\left(a^*\right)=0{,}1$$

Condition 1: $m=1; g\left(a^*\right)=-1; h\left(a^*\right)=0; \sum_{x=1}^{k}u_x^*G_x^T\left(a^*\right)=1$

Apply the above conditions in eqn. (11), we get

$$f\left(a^*\right)+1(-1)+0+1\le f\left(a^*\right)+1$$

$$f\left(a^*\right)\le f\left(a^*\right)+1$$

Substitute $f\left(a^*\right)=5$

$$5\le 6$$

Condition 2:

$$m=1; g\left(a^*\right)=0; h\left(a^*\right)=0; \sum_{x=1}^{k}u_x^*G_x^T\left(a^*\right)=1$$

Apply the above conditions in eqn. (11), we get

$$f\left(a^*\right)+1\times 0+0+1\le f\left(a^*\right)+1$$

$$f\left(a^*\right)+1\le f\left(a^*\right)+1$$

Substitute $f\left(a^*\right)=5$

$$6\le 6$$

Condition 3:

$$m=1; g\left(a^*\right)=0; h\left(a^*\right)=0; \sum_{x=1}^{k}u_x^*G_x^T\left(a^*\right)=0$$

Apply the above conditions in eqn. (11), we get

$$f\left(a^*\right)+1\times 0+0+0\le f\left(a^*\right)+0$$

$$f\left(a^*\right)\le f\left(a^*\right)$$

Substitute $f\left(a^*\right)=5$

$$5\le 5$$

Condition 4:

$$m=1; g\left(a^*\right)=-1; h\left(a^*\right)=0; \sum_{x=1}^{k}u_x^*G_x^T\left(a^*\right)=0$$

Apply the above conditions in eqn. (11), we get

$$f\left(a^*\right)+1(-1)+0+0\le f\left(a^*\right)+0$$

$$f\left(a^*\right)-1\le f\left(a^*\right)$$

Substitute $f\left(a^*\right)=5$

$$4\le 5$$

Condition 5:

$$m=0; g\left(a^*\right)=-1; h\left(a^*\right)=0; \sum_{x=1}^{k}u_x^*G_x^T\left(a^*\right)=1$$

Apply the above conditions in eqn. (11), we get

$$f\left(a^*\right)+0(-1)+0+1\le f\left(a^*\right)+1$$

$$f\left(a^*\right)+1\le f\left(a^*\right)+1$$

Substitute $f\left(a^*\right)=5$

$$6\le 6$$

Condition 6:

$$m=0; g\left(a^*\right)=0; h\left(a^*\right)=0; \sum_{x=1}^{k}u_x^*G_x^T\left(a^*\right)=1$$

Apply the above conditions in eqn. (11), we get

$$f\left(a^*\right)+0+0+1\le f\left(a^*\right)+1$$

$$f\left(a^*\right)+1\le f\left(a^*\right)+1$$

Substitute $f\left(a^*\right)=5$

$$6\le 6$$

Condition 7:

$$m=0; g\left(a^*\right)=0; h\left(a^*\right)=0; \sum_{x=1}^{k}u_x^*G_x^T\left(a^*\right)=0$$

Apply the above conditions in eqn. (11), we get

$$f\left(a^*\right)+0+0+0\le f\left(a^*\right)+0$$

$$f\left(a^*\right)\le f\left(a^*\right)$$

Substitute $f\left(a^*\right)=5$

$$5\le 5$$

Condition 8:

$$m = 0; g\left(a^*\right) = -1; h\left(a^*\right) = 0; \sum_{x=1}^{k} u_x^* G_x^T\left(a^*\right) = 0$$

Apply the above conditions in eqn. (11), we get

$$f\left(a^*\right) + 0(-1) + 0 + 0 \leq f\left(a^*\right) + 0$$

$$f\left(a^*\right) \leq f\left(a^*\right)$$

Substitute $f\left(a^*\right) = 5$

$$5 \leq 5$$

Hence proved.

CONCLUSION

In this paper, nonlinear Lagrangian functions and Frank-wolfe duality theorem was introduced for constrained multi-objective optimization issues. This approach results in weak and strong duality and Sufficiency results based on both nonlinear Lagrangian functions and wolfe optimization. The constrained multi-objective optimization problem was optimized using the obtained weak duality.This paper proposed a theorem based on the combination of both lagrangian and wolfe theorem. As a result, the proposed hybrid theorem solved multi-objective problems like lagrangian theorem, convex, inequality, concave problems and invexity issues.

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