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# Numerical Ways for Solving Fuzzy Differential Equations

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**Abstract**

Fuzzy differential equations are suggested as a way of modeling uncertain and incompletely specified systems.

Runge-Kutta algorithms for solving fuzzy ordinary differential equations are considered. A theorem of convergence for the solution is stated and proved.

Keywords: fuzzy ordinary differential equation, fuzzy numbers, Runge-Kutta.

## INTRODUCTION

In this paper we consider the first – order initial value problem

$$\acute{x}(t) \;=\; g(t,x), \quad t \in [0,T]$$

$$x(t_0) \;= x_0, \quad t_0 \in [0,T] \;, for\ some\ T > 0, \qquad (1)$$

Where, $x_0$ is a fuzzy number, $x$ is a fuzzy function of t, g $(t,x)$ is a fuzzy function of the crisp variable $t$ and fuzzy variable$x$, and $\acute{x}$ is the fuzzy derivative of [2].

Sufficient conditions for the existence of a unique solution to *Equation* (1) are that $g$ is continuous and satisfy Lipschitz condition [2]

$$|g(t,x) - g(t,z)| \le L|x-z|, \qquad L > 0 \qquad (2)$$

Kaleva's definition for fuzzy numbers [1] will be adopted here.

**Definition 1** *A fuzzy number u is a pair of functions* $(u_1,u_2)$ *of functions* $u_1(\alpha), u_2(\alpha);\ \alpha \in [0,1], which\ satisfy$:

1. $u_1(\alpha)$ *is a bounded monotonic increasing left continuous function.*
2. $u_2(\alpha)$ *is a bounded monotonic decreasing left continuous function.*
3. $u_1(\alpha) \le u_2(\alpha), \alpha \in [0,1]$

The set of all fuzzy numbers is denoted by $E^1$. The fuzzy number space $E^1$ as in [7] can be embedded into the Banach space $B = \bar{C}[0,1] \times \bar{C}[0,1]$ where the metric is usually defined as,

$$\|u,v\| = max\left\{\sup_{0\le\alpha\le1} |u(\alpha)|\,,\ \sup_{0\le\alpha\le1} |v(\alpha)|\right\} \qquad (3)$$

By [2] we may replace *Equation* (1) by the equivalent system

$$\acute{x}_1(t) \;= \min\{g(t,w), w \in [x_1,x_2]\}$$
$$= G_1(t,x_1,x_2), \qquad x_1(t_0) = x_{01}$$
$$\acute{x}_2(t) \;= \max\{g(t,w), w \in [x_1,x_2]\}$$
$$= G_2(t,x_1,x_2), \qquad x_2(t_0) = x_{02} \qquad (4)$$

*which has a unique solution* $(x_1,x_2) \in B$,

*which is a fuzzy function, where* $[x_1(t;\alpha), x_2(t;\alpha)] \in E^1$.

The parametric form of *Equation* (4) following [3] is given by

$$\acute{x}_1(t;\alpha) \;=\; G_1\big(t, x_1(t;\alpha), x_2(t;\alpha)\big), \quad x_1(t_0;\,\alpha) = x_{01}(\alpha)$$

$$\acute{x}_2(t;\alpha) \;=\; G_2\big(t, x_1(t;\alpha), x_2(t;\alpha)\big), \quad x_2(t_0;\,\alpha) = x_{02}(\alpha) \qquad (5)$$

For$\alpha \in [0,1]$. A solution to *Equation* (5) must solve *Equation* (4), since equality between two fuzzy numbers in B yields a pointwise equality because we use the sup *norm*.

In very few cases fuzzy initial value problems are solved analytically; however in general, numerical algorithms are needed and some of these algorithms have been developed by using the standard Euler method [4, 5] and the Taylor method [6]. In the following we will develop an algorithm based on Runge-Kutta Methods.

## RUNGE-KUTTA METHOD

The Rung-Kutta class of numerical solutions is one-step method which can be constructed of any order of accuracy and without the need of evaluating higher order derivatives. We will approximate the exact solution$(Y_1(t;\alpha), Y_2(t;\alpha))$ by $[y_1(t;\alpha), y_2(t;\alpha)]$. The *t-axis* is discretized over the finite interval $[t_0, T]$ **.** The subdivision points $t_n$, $n=0,\dots,N$, are often chosen equally spaced; that is $t_n=t_0+nh$, where the step size $h$ is $h = \frac{T-t_0}{N}$. The exact and approximate solutions at $t_n, 0 \le n \le N$, are denoted by $[Y_{1,n}(\alpha), Y_{2,n}(\alpha)]$ and$[y_{1,n}(\alpha), y_{2,n}(\alpha)]$, respectively.

To obtain a $p$ – stage Runge-Kutta Method ($p$ function evaluation per step) for the fuzzy initial problem (1) we let

$$y_{1,n+1}(\alpha) \;\approx\; y_{1,n}(\alpha) + h\emptyset_n(\alpha,h),$$
$$y_{2,n+1}(\alpha) \;\approx\; y_{2,n}(\alpha) + h\psi_n(\alpha,h), \qquad (6)$$

Where,

$$\emptyset(\alpha,h) = \emptyset\big(t_n y_{1,n}, y_{2,n;}\alpha; h\big) = \sum_{i=1}^{p} w_i k_i(\alpha) \qquad (7)$$

$$\psi(\alpha,h) = \psi\left(t_n, y_{1,n}, y_{2,n;}\alpha; h\right) = \sum_{i=1}^{p} v_i l_i(\alpha) \qquad (8)$$

$$k_i(\alpha) \;=\; G_1(t_n + ha_i, y_{1,n} + h\sum_{j=1}^{i-1} \eta_{ij} k_j, y_{2,n} + h\sum_{j=1}^{i-1} \eta_{ij} k_j) \qquad (9)$$

$$l_i(\alpha) = G_2(t_n + hb_i, y_{1,n} + h\sum_{j=1}^{i-1}\xi_{ij}l_j, y_{2,n} + h\sum_{j=1}^{i-1}\xi_{ij}l_j), \quad (10)$$

With $\sum_{i=1}^{p} w_i = 1, \sum_{i=1}^{p} v_i = 1, a_1 = b_1 = 0$ *and* $a_i, b_i, \eta_{ij}, \xi_{ij}, i = 1, \dots, p, j = 1, \dots, i-1$ *, are real numbersto be chosen,*

*where the initial conditions are*

$y_{1,0}(\alpha) = x_{01}(\alpha), \quad y_{2,0}(\alpha) = x_{02}(\alpha).$

The polygon Curves

$$y_1(t; h; \alpha) \approxeq \{[t_0, y_{1,0}(\alpha)], [t_1, y_{1,1}(\alpha)], \dots, [t_N, y_{1,N}, (\alpha)]\},$$

$$y_2(t; h; \alpha) \approxeq \{[t_0, y_{2,0}(\alpha)], [t_1, y_{2,1}(\alpha)], \dots [t_N, y_{2,N}, (\alpha)]\} \quad (11)$$

are the Runge-Kutta approximates to $Y_1$ (t; ) *and* $Y_2(t; \alpha)$, respectively, over the interval $t_0 \le t \le t_N$.

Note that when $p = 1$, we have Euler method. But the most powerful method is the Classical 4th Order Runge-Kutta Method, where the approximate fuzzy solutions are defined by

$$y_{1,n+1}(\alpha) = y_{1,n}(\alpha) + \frac{1}{6}(k_1(\alpha) + 2k_2(\alpha) + 2k_3(\alpha) + k_4(\alpha))h,$$

$$y_{2,n+1}(\alpha) = y_{2,n}(\alpha) + \frac{1}{6}(l_1(\alpha) + 2l_2(\alpha) + 2l_3(\alpha) + l_4(\alpha))h, \quad (12)$$

Where,

$k_1(\alpha) = G_1(t_n, y_{1,n}(\alpha), y_{2,n}(\alpha)),$

$k_2(\alpha) = G_1(t_n + 0.5h, y_{1,n}(\alpha) + 0.5k_1(\alpha)h, y_{2,n}(\alpha) + 0.5k_1(\alpha)h),$

$k_3(\alpha) = G_1(t_n + 0.5h, y_{1,n}(\alpha) + 0.5k_2(\alpha)h, y_{2,n}(\alpha) + 0.5k_2(\alpha)h),$

$k_4(\alpha) = G_1(t_n + h, y_{1,n}(\alpha) + k_3(\alpha)h, y_{2,n}(\alpha) + k_3(\alpha)h),$

$l_1(\alpha) = G_2(t_n, y_{1,n}(\alpha), y_{2,n}(\alpha)),$

$l_2(\alpha) = G_2(t_n + 0.5h, y_{1,n}(\alpha) + 0.5l_1(\alpha)h, y_{2,n}(\alpha) + 0.5l_1(\alpha)h),$

$l_3(\alpha) = G_2(t_n + 0.5h, y_{1,n}(\alpha) + 0.5l_2(\alpha)h, y_{2,n}(\alpha) + 0.5l_2(\alpha)h),$

$l_4(\alpha) = G_2(t_n + h, y_{1,n}(\alpha) + l_3(\alpha)h, y_{2,n}(\alpha) + l_3(\alpha)h).$

**Lemma 1.** *Let a sequence of numbers* $\{W_n\}_{n=0}^{N}$ *satisfies,*

$|W_{n+1}| \le A|W_n| + B, \quad 0 \le n \le N-1,$

*for some given positive constants A and B, then*

$|W_n| \le A^n|W_0| + B(A^n - 1)/A - 1, \quad 0 \le n \le N-1.$

**Proof.** Straightforward.

**Lemma 2.** *Let the sequences of numbers* $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ *satisfy*

$$|W_{n+1}| \le |W_n| + A\,max\{|W_n|, |V_n|\} + B,$$
$$|V_{n+1}| \le |V_n| + A\,max\{|W_n|, |V_n|\} + B,$$

*for some given positive constants A and B, and denote*

$$U_n = |W_n| + |V_n|, \quad 0 \le n \le N.$$

*Then*

$U_n \le A^{*n}U_o + B^*(A^{*n} - 1)/A^* - 1 \quad 0 \le n \le N$

*where* $A^* = 1 + 2A$ *and* $B^* = 2B$

**Proof.** We have,

$|W_{n+1}| + |V_{n+1}| \le |W_n| + |V_n| + 2A(|W_n| + |V_n|) + 2B =$

$(1 + 2A)(|W_n| + |V_n|) + 2B$

By applying **lemma1** for $U_n$, $0 \le n \le N$. We conclude the result.

Now we will introduce the convergence theorem for the Runge-Kutta method.

*The domain of* $\emptyset(t,u,v)$, $\psi(t,u,v)$, $G_1(t,u,v)$ *and* $G_2(t,u,v)$ *is*

$K = \{(t,u,v) : t_0 \le t \le T, -\infty < v < \infty, -\infty < u < v\}.$

**Theorem 1** *Let* $\emptyset(t,u,v)$ *and* $\psi(t,u,v)$ *belong to* $C^1(K)$ *and each satisfies a Liptchiz condition in u and v. Assume* $\emptyset(\alpha,h)$, $\psi(\alpha,h)$ *satisfy the consistency condition*

$$G_1(t_n, y_{1,n}(\alpha), y_{2,n}(\alpha)) = \emptyset(\alpha, 0),$$
$$G_2(t_n, y_{1,n}(\alpha), y_{2,n}(\alpha)) = \psi(\alpha, 0),$$

*Then the Runge-Kutta approximates of (11) converge to the exact solutions*

$Y_1(t;\alpha)$, $Y_2(t;\alpha)$.

**Proof.** It is sufficient to prove that

$$\lim_{h\to 0} y_{1,N}(t; h; \alpha) = Y_1(T, \alpha), and \quad (13)$$

$$\lim_{h\to 0} y_{2,N}(t; h; \alpha) = Y_2(T, \alpha).$$

*Let*

$$W_{n+1} = Y_{1,n+1}(t; \alpha) - y_{1,n+1}(t; \alpha),$$
$$V_{n+1} = Y_{2,n+1}(t; \alpha) - y_{2,n+1}(t; \alpha),$$

$$W_{n+1} = Y_{1,n+1}(t; \alpha) - y_{1,n+1}(t; \alpha)$$
$$= Y_{1,n}(t; \alpha) + h\left(G_1\begin{pmatrix} t_n + \theta h, Y_{1,n}(t_n + \theta h), \\ Y_{2,n}((t_n + \theta h); \alpha; h) \end{pmatrix}\right)$$
$$- y_{1,n}(t; \alpha) - h\emptyset(t_n, y_{1,n}, y_{2,n}; \alpha; h).$$

Then

$$|W_{n+1}| \le |W_n| +$$
$$h(|\,G_1\begin{pmatrix} t_n + \theta h, Y_{1,n}(t_n + \theta h), \\ Y_{2,n}((t_n + \theta h); \alpha; h) \end{pmatrix} - \emptyset(t_n, y_{1,n}, y_{2,n}; \alpha; h)\,|)$$

We can write the part in the parentheses as:

$$|\,G_1\begin{pmatrix} t_n + \theta h, Y_{1,n}(t_n + \theta h), \\ Y_{2,n}((t_n + \theta h); \alpha; h) \end{pmatrix} - \emptyset(t_n, y_{1,n}, y_{2,n}; \alpha; h)\,| =$$

$$|\,G_1(t_n + \theta h, Y_{1,n}(t_n + \theta h), Y_{2,n}(t_n + \theta h); \alpha; h) - G_1(t, Y_{1,n}, Y_{2,n}; \alpha) + \emptyset(t_n, Y_{1,n}, Y_{2,n}; \alpha; 0) - \emptyset(t_n, Y_{1,n}; Y_{2,n}; \alpha; h) + \emptyset(t_n, Y_{1,n}, Y_{2,n}; \alpha; h) - \emptyset(t_n, y_{1,n}, y_{2,n}; \alpha; h)\,|$$

Let

$$\chi_1(h) = \max_{\substack{t\in[t_0,T]\\ \theta\in[0,1]}} \left| G_1\begin{pmatrix} t+\theta h, Y_1(t_n+\theta h), \\ Y_2(t+\theta h);\alpha;h\end{pmatrix} - G_1(t, Y_1, Y_2; \alpha) \right|$$

And

$$\xi_1(h) = \max_{t\in[t_0,T]} |\emptyset(t, Y_1 Y_2; \alpha; 0) - \emptyset(t, Y_1, Y_2;\ \alpha; h)|.$$

Since $\emptyset(t, Y_{1,n}, Y_{2,n}\ ; \alpha; h)$ is continuous and satisfies a Lipchitz condition, then

$$\left|\emptyset(t_n, Y_{1,n}, Y_{2,n}\ ;\ \alpha;\mathrm{h}) - \emptyset(t_n, y_{1,n}, y_{2,n}\ ;\ \alpha;\mathrm{h})\right| \le 2L \max\{|W_n|, |V_n|\},$$

Thus

$$|W_{n+1}| \le |W_n| + h(\chi_1(h) + \xi_1(h)) + 2hL \max\{|W_n|, |V_n|\}. \quad (14)$$

Similarly

$$V_{n+1} = Y_{2,n+1}(t;\alpha) - y_{2,n+1}(t;\alpha)$$
$$= Y_{2,n}(t;\alpha) + h\left(G_2\begin{pmatrix} t_n+\theta h, Y_{1,n}(t_n+\theta h), \\ Y_{2,n}((t_n+\theta h);\alpha;h)\end{pmatrix}\right)$$
$$- y_{2,n}(t;\alpha) - h\psi(t_n, y_{1,n}, y_{2,n};\alpha;h). \quad (15)$$

So,

$$|V_{n+1}| \le |V_n| + h\left(\left|G_2\begin{pmatrix} t_n+\theta h, Y_{1,n}(t_n+\theta h), \\ Y_{2,n}((t_n+\theta h);\alpha;h)\end{pmatrix} - \psi(t_n, y_{1,n}, y_{2,n};\alpha;h)\right|\right) \quad (16)$$

We can write the part in the parentheses as:

$$\left|(G_2\begin{pmatrix} t_n+\theta h, Y_{1,n}(t_n+\theta h), \\ Y_{2,n}((t_n+\theta h);\alpha;h)\end{pmatrix} - \psi(t_n, y_{1,n}, y_{2,n};\alpha;h)\right| =$$
$$\left| G_2\begin{pmatrix} t_n+\theta h, Y_{1,n}(t_n+\theta h), \\ Y_{2,n}((t_n+\theta h);\alpha;h)\end{pmatrix} - G_2(t, Y_{1,n}, Y_{2,n}\ ;\ \alpha) + \right.$$
$$\psi(t_n, Y_{1,n}, Y_{2,n};\alpha;0) - \psi(t_n, Y_{1,n}, Y_{2,n}\ ;\alpha;h)$$
$$\left. +\psi(t_n, Y_{1,n}, \alpha; h) - \psi(t_n, y_{1,n}, y_{2,n};\ \alpha; h)\right| \quad (17)$$

Let

$$\chi_2(h) = \max_{\substack{t\in[t_0,T]\\ \theta\in[0,1]}} \left| G_2\begin{pmatrix} t+\theta h, Y_1(t_n+\theta h), \\ Y_2(t+\theta h);\alpha;h\end{pmatrix} - G_2(t, Y_1, Y_2;\alpha) \right|$$

And

$$\xi_2(h) = \max_{t\in[t_0,T]} |\psi(t, Y_1, Y_2;\alpha;0) - \psi(t, Y_1, Y_2;\ \alpha;h)|.$$

Since $\psi(t, Y_{1,n}, Y_{2,n};\ \alpha; h)$ is continuous and satisfies a Lipchitz condition, then

$$\left|\psi(t_n, Y_{1,n}, Y_{2,n};\ \alpha;h) - \psi(t_n, y_{1,n}, y_{2,n};\ \alpha;h)\right| \le 2L\, max\{|W_n|, |V_n|\}.$$

Thus

$$|V_{n+1}| \le |V_n| + h(\chi_2(h) + \xi_2(h)) + 2hL \max\{|W_n|, |V_n|\}. \quad (18)$$

Applying **Lemma 2** on (14) and (18) we have

$$|W_n| \le (1+4hL)^n |U_0| + 2h\left[\frac{(1+4hL)^n - 1}{4hL}\right](x_1(h) + \xi_1(h)),$$

$$|V_n| \le (1+4hL)^n |U_0| + 2h\left[\frac{(1+4hL)^n - 1}{4hL}\right](x_2(h) + \xi_2(h)),$$

where$|U_0| = |W_0| + |V_0|$.

In particular

$$|W_N| \le (1+4hL)^N |U_0| + 2h\left[\frac{(1+4hL)^{\frac{T-t_0}{h}} - 1}{4hL}\right](x_1(h) + \xi_1(h)),$$

$$|V_N| \le (1+4hL)^N |U_0| + 2h\left[\frac{(1+4hL)^{\frac{T-t_0}{h}} - 1}{4hL}\right](x_2(h) + \xi_2(h)),$$

Since $W_0 = V_0 = 0$, we have

$$|W_N| \le 2\left[\frac{e^{4L(T-t_0)} - 1}{4L}\right](x_1(h) + \xi_1(h)),$$

$$|V_N| \le 2\left[\frac{e^{4L(T-t_0)} - 1}{4L}\right](x_2(h) + \xi_2(h)),$$

so taking the limit as h →0, we have $W_N \to 0$, $V_N \to 0$, and the proof is completed.

**Illustration**: Consider the fuzzy differential equation

$\acute{x}(t) \ = x(t),$

$x\ (0) = (0.75 + 0.25\alpha, 1.125 - 0.125\alpha).$ (19)

Using *Equation* (5) this is equivalent to the system

$\acute{x}_1(t;\alpha) = \ x_1\ (t;\alpha),\ \ x_1(0;\alpha) = 0.75 + 0.25\alpha$

$\acute{x}_2(t;\alpha) = \ x_2\ (t;\alpha),\ \ x_2(0;\alpha) = 1.125 - 0.125\alpha$ (20)

where the exact solution at $t$ =1, is given by

$$Y = e[0.75 + 0.25\alpha, 1.125 - 0.125\alpha]$$

The approximate solution at $t$ =1 for the Classical 4th Order Runge-Kutta Method using $h$ = 0.10 is

$$y_1 \ = \ (0.75 + 0.25\alpha)\left[1 + 0.10 + \frac{0.10^2}{2} + \frac{0.10^3}{6} + \frac{0.10^4}{24}\right]^{10},$$

$$y_2 \ = \ (1.125 - 0.125\alpha)\left[1 + 0.10 + \frac{0.10^2}{2} + \frac{0.10^3}{6} + \frac{0.10^4}{24}\right]^{10},$$

That is

$$y = 2.71818746\ [0.75 + 0.25\alpha, 1.125 - \ 0.125\ \alpha],$$

which is more accurate than the Euler solution given by

$$y = 2.59374246\ [0.75 + 0.25\ \alpha, 1.125\ \ - 0.125\ \alpha].$$

## REFERENCES

[1] Osmo Kaleva, (1990),The Cauchy Problem For Fuzzy Differential Equations, *Fuzzy Sets and Systems*, 35 (3), 389 – 396.

[2] Seppo Seikkala, (1987), On The Fuzzy Initial Value Problem, *Fuzzy Sets and Systms,* 24 (3), 319 – 330.

[3] Shiji Song and Congxin Wu, (2000) ,Existence and Uniqueness Of Solutions To Cauchy Problem Of Fuzzy Differential Equations, *Fuzzy Sets and Systems*, 110 (1) , 55-67.

[4] Menahem Friedman, Ming Ma and Abraham Kandel, (1999), Numerical Solutions Of Fuzzy differential Equations and Integral Equations, *Fuzzy Sets and Systems*, 106 (1), 35-48.

[5] Ming Ma, Menahem Friedman and Abraham Kandel, (1999), Numerical Solutions Of Fuzzy Differential Equations, *Fuzzy Sets and Systems*, 105 (1), 133-138.

[6] S. Abbasbandy and T. Allah Viranloo , (2004), Numerical Solutions Of Fuzzy Differential Equations By Runge-Kutta Method, *Nonlinear studies*, 11 (1), 117-129.

[7] Wu Congxin and Ming Ma,(1991), On Embedding Problem Of Fuzzy Sets and Spaces, *Fuzzy Sets and Systems*, 44 (1), 33-38.