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A Comparison between Galerkin Weighted Residual and Perturbation Techniques for Penetrative Bénard–Marangoni Ferroconvection via Internal Heating

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Abstract

A comparison between Galerkin-type of weighted residual technique (WRT) and regular perturbation technique (RPT) for studied the effects of internal heat generation and magnetic field dependent (MFD) on the onset of Rayleigh–Bénard–Marangoni ferroconvection subjected to constant heat flux conditions. The lower rigid and upper free boundary at which the temperature-dependent surface tension effect is considered is non-deformable. The eigenvalue problem is solved numerically by Galerkin technique and analytically by regular perturbation technique. It is noted that the combined effect of magnetic Rayleigh number and dimensionless internal heat source strength is to reinforce together and to suppress the onset of Bénard–Marangoni ferroconvection. The onset of ferroconvection is augment with an increase in MFD viscosity parameter. In addition, nonlinearity of fluid magnetization is found to have no influence on the onset of ferroconvection.

Keywords: Bénard-Marangoni ferroconvection, Weighted residual technique, Regular perturbation technique, Internal heat generation, MFD viscosity, Insulated boundaries

INTRODUCTION

Ferrofluids are colloidal suspensions of magnetic nanoparticles in a carrier fluid such as water, hydrocarbon (mineral oil or kerosene) or fluorocarbon. The nanoparticles typically have sizes of about 100 Å or 10 nm and they are coated with surfactants in order to prevent the coagulation. Usually, these fluids do not conduct electric current and exhibit a nonlinear paramagnetic behavior. The variety of formulations available for ferrofluids permits a great number of applications, from medical to satellite and vacuum technologies (Rosensweig(1985)), Halbreich et al. (1998), Odenbech ((2002a, 2002b), Kim and Park(2010)). The first macroscopic description of magnetic fluids was given (Neuringer and Rosensweig (1964)). The convective instability of a magnetic fluid layer heated from below in the presence of a uniform vertical magnetic field was discussed later (Finlayson(1970)). Other works in ferrofluid convection can be found in Refs. (Lalas and Carmi (1971), Gotoh and Yamada (1982), Stiles and Kagan (1990), Siddheshwar (1995), Kaloni and Lou (2004), Sunil and Mahajan (2008), Nanjundappa and Shivakumara (2008), Nanjundappa et al. (2009), Singh and Bajaj (2011), Shivakumara et al. (2012)). Recently, Sekhar et al. (2017) have studied the effect of variable viscosity on thermal convection in Newtonian ferromagnetic liquid by different forms of boundary conditions.

On the other hand, if the surface of a ferrofluid layer is free and open to the atmosphere, then ferroconvection can also be induced by temperature dependent surface tension forces at the free surface known as Marangoni ferroconvection. In view of the fact that heat transfer is greatly enhanced due to ferroconvection, Marangoni ferroconvection offers new possibilities for application in cooling of motors in space, loudspeakers, transmission lines and other equipments in micro-gravity environment where magnetic field is already present. In most of the cases, the combined effect of buoyancy and surface tension forces on convective instability in a ferrofluid layer also becomes important. Realizing these aspects, a limited number of studies have addressed the effect of surface tension forces on ferroconvection in a horizontal ferrofluid layer. Linear and non-linear stability of combined buoyancy–surface tension effects in a ferrofluid layer heated from below has been analyzed (Qin and Kaloni (1994)). Odenbech (1995) was carried out experimentally the sounding rockets and at the drop tower on thermomagnetic convection in magneticfluids. The Bénard–Marangoni convection problems of ferrofluid layer heated from below under various assumptions is studied by many authors (Weilepp and Brand (2001), Shivakumara et al. (2002), Hennenberg et al. (2005), Shivakumara and Nanjundappa (2006), Bozhko and Putin (2009), Nanjundappa et al. (2010), Nanjundappa et al. (2013) have investigated the linear stability analysis of penetrative Bénard–Marangoni convection in a horizontal ferromagnetic fluid layer via an internal heating model. The linear stability of the onset of Bénard–Marangoni convection in a horizontal ferromagnetic fluid layer with internal heat source effect and temperature dependent viscosity effect (Nanjundappa et al. (2014)) have been examined. Nanjundappa et al. (2015) have investigated the combined effect of rotation and MFD viscosity on Bénard-Marangoni ferroconvection.

In the present study, we have considered the problem of combined buoyancy and surface tension driven convection in a horizontal ferromagnetic fluid layer subjected to the magnetic field dependent viscosity including the additional effect of internal heat generation. Such a study helps in understanding control of Bénard–Marangoni ferroconvection due to non-uniform temperature gradient arising due to internal heating, which is important in the applications of ferrofluid technology. The lower rigid and upper free boundary at which the temperature-dependent surface tension forces are accounted for are considered to be perfectly insulated to temperature perturbations. The resulting eigenvalue problem is solved numerically using the Galerkin technique. Besides, an analytical formula is obtained for the critical Rayleigh/Marangoni number by regular perturbation technique with wave number as a perturbation parameter. The results obtained from both numerical and analytical methods are found to complement with each other suggesting the analytical results obtained are exact.

MATHEMATICAL FORMULATION

We consider a horizontal layer of an electrically non-conducting Boussinesq ferromagnetic fluid of thickness *d* with a uniformly distributed volumetric heat generation. An external uniform magnetic field $\vec{H} = (H_0, 0, 0)$ is applied perpendicular to the ferrofluid layer. The temperatures at the lower-rigid ($z = 0$) and upper-free ($z = d$) boundaries are kept at T_l and $T_u (< T_l)$, respectively. A Cartesian co-ordinate system (*x, y, z*) is used with the origin at the bottom of the surface and the z-axis vertically upward. Gravity acts in the negative z-direction, $\vec{g} = -g\,\hat{k}$, where $\hat{k}$ is the unit vector in the z-direction. At the upper free surface, the surface tension σ is assumed to vary linearly with temperature in the form

$$\sigma = \sigma_0 - \sigma_T\ (T - T_0) \tag{1}$$

where, σ_0 is the unperturbed value and $-\sigma_T$ is the rate of change of surface tension with temperature T. The continuity equation for an incompressible Boussinesq fluid is

$$\nabla \cdot \vec{V} = 0 \tag{2}$$

where, $\vec{V} = (u,\ v,\ w)$ is the velocity vector.

The momentum equation for an incompressible ferrofluid is

$$\rho_0 [\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\ \vec{V}] = -\nabla \mathrm{p} + \rho \vec{g} + 2\nabla \cdot [\eta\ \underline{\underline{D}}] + \mu_0\ (\vec{M} \cdot \nabla) \vec{H} \tag{3}$$

where, p is the pressure, *t* the time, $\rho_0 = \rho(T_a)$ the density of fluid at the average temperature $T_a = (T_u + T_l)/2$, $\vec{B}$ the magnetic induction field, $\vec{H}$ the magnetic field, $\vec{M}$ the magnetization, the coefficient $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Henry}\, m^{-1}$ the magnetic constant and $\underline{\underline{D}} = [\nabla \vec{V} + (\nabla \vec{V})^T]/2$ the rate of strain tensor. The last term in Eq. (3) describes a ponderomotive force which acts on a magnetized fluid in a non-uniform magnetic field (i.e., magnetized fluid tends to move in the direction of increasing magnetic field). The fluid is assumed to be incompressible having variable viscosity. Experimentally, it has been demonstrated that the magnetic viscosity has got exponential variation with respect to magnetic field (Rosenswieg (1966)). As a first approximation for small field variation, linear variation of magnetic viscosity has been used in the form $\eta = \eta_0 (1 + \vec{\delta} \cdot \vec{B})$, where $\vec{\delta}$ is the variation coefficient of magnetic field dependent viscosity and is considered to be isotropic (Vaidayanathan et al. (2000)), η_0 is taken as viscosity of the fluid when the applied magnetic field is absent.

The energy equation for an incompressible fluid which obeys Fourier's law is

$$\left[\rho_0 C_{V,H} - \mu_0 \vec{H} \cdot \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H}\right] \frac{DT}{Dt} + \mu_0 T \left(\frac{\partial \vec{M}}{\partial T}\right)_{V,H} \cdot \frac{D\vec{H}}{Dt} = k_1 \nabla^2 T + Q \tag{4}$$

where, k_1 is the (constant) overall thermal conductivity, Q the overall uniformly distributed effective volumetric internal heat generation, $C_{V,H}$ the specific heat at constant volume and magnetic field, H the magnitude of $\vec{H}$ $\left(H = \left|\vec{H}\right|\right)$, M the magnitude of $\vec{M}$ $\left(M = \left|\vec{M}\right|\right)$ and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ the Laplacian operator. The variation in fluid density ρ with temperature *T* is accounted for only in the buoyancy term and is approximated linearly as

$$\rho = \rho_0 \left[1 - \alpha_t (T - T_a)\right] \tag{5}$$

where, α_t is the coefficient of thermal expansion coefficient.

Maxwell's equations, simplified for a non-conducting fluid with no displacement currents, become

$$\nabla \cdot \vec{B} = 0,\ \nabla \times \vec{\mathrm{H}} = 0 \text{ or } \vec{H} = \nabla \varphi \tag{6a,b}$$

where, φ is the magnetic scalar potential.

The magnetic field $\vec{H}$, magnetization $\vec{M}$, and the magnetic induction $\vec{B}$ are related by

$$\vec{B} = \mu_0 \left(\vec{M} + \vec{H}\right) \tag{7}$$

where

$$\vec{M} = \frac{\vec{H}}{H} M\left(H, T\right). \tag{8}$$

The magnetic equation of state is linearized about the magnetic field, H_0, and the average temperature, T_a to becomes

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$$M = M_0 + \chi(H - H_0) - K(T - T_a) \tag{9}$$

where, $K = -\left(\partial M / \partial T\right)_{H_0, T_a}$ is the pyromagnetic co-efficient, $\chi = (\partial M / \partial H)_{H_0, T_a}$ is the magnetic susceptibility and $M_0 = M(H_0, T_a)$ is the constant mean value of magnetization.

The basic state is quiescent and is given by

$$\vec{V}_b = 0 \tag{10}$$

$$\begin{aligned} p_b(z) = {} & p_0 - \rho_0 g z - \rho_0 \alpha_t g \left[\frac{Q z^3}{6 k_1} - \frac{Q d z^2}{4 k_1} + \frac{\beta z^2}{2} \right] \\ & - \frac{\mu_0 M_0 K}{1+\chi} \left[\frac{Q z^2}{2 k_1} - \frac{Q d z}{2 k_1} + \beta z \right] \\ & - \frac{\mu_0 K^2}{(1+\chi)^2} \left[\frac{Q^2 z^4}{8 k_1^2} + (2\beta - Q d) \frac{Q z^3}{4 k_1} + \left(\beta - \frac{Q d}{2 k_1} \right) \frac{z^2}{2} \right] \end{aligned} \tag{11}$$

$$T_b(z) = -\frac{Q z^2}{2 k_1} + \frac{Q d z}{2 k_1} - \beta z + T_a \tag{12}$$

$$\vec{H}_b(z) = \left[H_0 - \frac{K}{1+\chi} \left(\frac{Q z^2}{2 k_1} - \frac{Q d z}{2 k_1} + \beta z \right) \right] \hat{k} \tag{13}$$

$$\vec{M}_b(z) = \left[M_0 + \frac{K}{1+\chi} \left(\frac{Q z^2}{2 k_1} - \frac{Q d z}{2 k_1} + \beta z \right) \right] \hat{k} \tag{14}$$

where, $\beta = (T_l - T_u)/d = \Delta T / d$ and the subscript b denotes the basic state. It may be noted that $T_b(z)$, $\vec{H}_b(z)$ and $\vec{M}_b(z)$ are distributed parabolically with the fluid layer height due to the presence of internal heat generation. However, for $Q = 0$ (i.e. in the absence of internal heat generation) the basic state distributions are linear in z.

To study the stability of the system, the basic state is perturbed in the form

$$\begin{aligned} & \vec{V} = \vec{V}', \; p = p_b(z) + p', \; \eta = \eta_b(z) + \eta', \\ & T = T_b(z) + T', \vec{H} = \vec{H}_b(z) + \vec{H}', \\ & \vec{M} = \vec{M}_b(z) + \vec{M}', \end{aligned} \tag{15}$$

where the primed quantities represent the perturbed variables. Substituting Eq. (15) into Eq. (7), using Eq. (8), we get (after dropping the primes):

$$\begin{aligned} & H_x + M_x = \left(1 + M_0 / H_0\right) H_x, \\ & H_y + M_y = \left(1 + M_0 / H_0\right) H_y, \\ & H_z + M_z = (1+\chi) H_z - K T, \end{aligned} \tag{16}$$

where, $\left(H_x, H_y, H_z\right)$ and $\left(M_x, M_y, M_z\right)$ are the (x, y, z) components of the magnetic field intensity and magnetization, respectively. In obtaining the above equations it is assumed that $K \beta d << (1+\chi) H_0$ and $K Q d^2 << 2(1+\chi) H_0 k_1$.

Substituting Eq. (15) in Eq. (3), linearizing, eliminating the pressure by operating curl twice, and retaining the z-component of the resulting equation, we obtain (after dropping the primes):

$$\begin{aligned} & \left[\rho_0 \frac{\partial}{\partial t} - \eta_0 \left\{ 1 + \delta \mu_0 (M_0 + H_0) \right\} \nabla^2 \right] \nabla^2 w = \\ & -\mu_0 K \left(-\frac{Q z}{k_1} + \frac{Q d}{2 k_1} - \beta \right) \frac{\partial}{\partial t} \left(\nabla_h^2 \varphi \right) \\ & + \rho_0 \alpha_t g \nabla_h^2 T + \frac{\mu_0 K^2}{1+\chi} \left(-\frac{Q z}{k_1} + \frac{Q d}{2 k_1} - \beta \right) \nabla_h^2 T \end{aligned} \tag{17}$$

where, $\nabla_h^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the horizontal Laplacian operator. Equation (4), after using Eq. (15), and linearizing, takes the form (after dropping the primes):

$$\begin{aligned} & \rho_0 C_0 \frac{\partial T}{\partial t} - \mu_0 T_0 K \frac{\partial}{\partial t} \left(\frac{\partial \varphi}{\partial z} \right) = k_1 \nabla^2 T \\ & + \left[1 - \frac{\mu_0 T_0 K^2}{1+\chi} \right] \left(\frac{Q z}{k_1} - \frac{Q d}{2 k_1} + \beta \right) w \end{aligned} \tag{18}$$

where, $\rho_0 C_0 = \rho_0 C_{V,H} + \mu_0 H_0 K$ and we have assumed $\beta d << T_0$. Equation (6a,b), after substituting Eq. (15) and using Eq. (16), may be written as (after dropping the primes)

$$\left(1 + \frac{M_0}{H_0} \right) \nabla_h^2 \varphi + (1+\chi) \frac{\partial^2 \varphi}{\partial z^2} - K \frac{\partial T}{\partial z} = 0 \tag{19}$$

The normal mode expansion of the dependent variables is assumed in the form

$$\{w, T, \varphi\} = \{W(z), \Theta(z), \Phi(z)\} \exp\left[i(\ell x + m y) + \sigma t \right] \tag{20}$$

where, ℓ and m are wave numbers in the x and y directions, respectively and σ is the growth rate with is complex. On substituting Eq. (20) into Eqs.(17)-(19) and non-dimensionalizing the variables by setting

$$\begin{aligned} & (x^*, y^*, z^*) = \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d} \right), \quad W^* = \frac{d}{\nu} W, \\ & \Theta^* = \frac{\kappa}{\beta \nu d} \Theta, \quad \Phi^* = \frac{(1+\chi)\,\kappa}{\mathrm{K} \beta \nu d^2} \Phi, \\ & t^* = \frac{\nu}{d^2} t, \quad \sigma^* = \frac{d^2}{\nu} \sigma, \quad \Lambda^* = \mu_0 H_0 (1+\chi) \delta \end{aligned} \tag{21}$$

where, $\nu = \eta_0 / \rho_0$ is the kinematic viscosity and $\kappa = k_1 / \rho_0 C_0$ is the thermal diffusivity, we obtain (after ignoring the asterisks)

$$\left[(1+\Lambda)(D^2-a^2)-\sigma\right](D^2-a^2)W = -R_m a^2 \left[N_S(1-2z)-1\right](D\Phi-\Theta)+R_t a^2 \Theta \tag{22}$$

$$\left(D^2-a^2-\Pr\sigma\right)\Theta-\Pr M_2\sigma\Phi = [N_S(1-2z)-1](1-M_2)W \tag{23}$$

$$\left(D^2-M_3 a^2\right)\Phi = D\Theta \tag{24}$$

Here, $D = d/dz$ is the differential operator, $a=\sqrt{\ell^2+m^2}$ the horizontal wave number, $R_m = R_t M_1 = \mu_0 K^2 \beta^2 d^4 /(1+\chi)\mu\kappa$ the magnetic Rayleigh number, $M_1 = \mu_0 K^2 \beta /(1+\chi)\rho_0 \alpha_t g$ the magnetic number, $R_t = g\alpha_t \beta d^4/\nu\kappa$ the gravity thermal Rayleigh number, $N_s = Qd/2\kappa\beta$ the dimensionless heat source strength, $\Pr = \nu/\kappa$ the Prandtl number, $M_2 = \mu_0 T_0 K^2/(1+\chi)(\rho_0 C)_1$ the magnetic parameter, $M_3 = (1+M_0/H_0)/(1+\chi)$ the measure of nonlinearity of the magnetization, $\Lambda = \delta\mu_0(M_0+H_0)$ the non dimensional magnetic filed dependent viscosity parameter and σ is the growth rate. Moreover, the typical value of M_2 for magnetic fluids with different carrier liquids is of the order of 10^{-6} and hence its effect is neglected as compared to unity. Equations (22)-(24), after noting the above facts, now become

$$(1+\Lambda)(D^2-a^2)^2 W = -R_m a^2 \left[N_S(1-2z)-1\right](D\Phi-\Theta)+R_t a^2\Theta \tag{25}$$

$$(D^2-a^2)\Theta = [N_S(1-2z)-1]W \tag{26}$$

$$(D^2-M_3 a^2)\Phi = D\Theta \tag{27}$$

It is considered that the lower boundary is rigid-ferromagnetic, while the upper free boundary at which the surface tension effects are accounted for is taken to be non-deformable and flat. In addition, both the boundaries are assumed to be perfectly insulated to temperature perturbations. The boundary conditions are then given by

$$W = DW = D\Theta = \Phi = 0 \;\; at \;\; z=0 \tag{28}$$

$$W = (1+\Lambda)D^2W + Ma\; a^2\Theta = D\Theta = D\Phi = 0 \quad \text{at} \quad z=1 \tag{29}$$

where, $Ma = \sigma_T \Delta T\, d/\mu\kappa$ is the Marangoni number.

METHOD OF SOLUTION

Equations (25)–(27) together with the boundary conditions constitute an eigenvalue problem with R_m or R_t or Ma as an eigenvalue. The eigenvalue problem is solved both numerically using the Galerkin-type of WRT as well as analytically using a RPT with wave number as a perturbation parameter.

1. Numerical Solution by Galerkin-type of WRT

The Galerkin method is used to solve the eigenvalue problem. In this method, the test (weighted) functions are the same as the base (trial) functions. Accordingly, W, Θ and Φ are written as

$$W(z)=\sum_{i=1}^{n} A_i W_i(z),\; \Theta(z)=\sum_{i=1}^{n} B_i\,\Theta_i(z), \quad \Phi(z)=\sum_{i=1}^{n} C_i\,\Phi_i(z) \tag{30}$$

where, A_i, B_i and C_i are unknown constants to be determined. The base functions $W_i(z)$, $\Theta_i(z)$ and $\Phi_i(z)$ are generally chosen such that they satisfy the corresponding boundary conditions but not the differential equations. We select the trial functions as

$$W_i = (z^4 - 5z^3/2 + 3z^2/2)T^*_{i-1},\quad \Theta_i = z(1-z/2)T^*_{i-1},\quad \Phi_i = z^2(1-2z/3)T^*_{i-1} \tag{31}$$

where, T_i^*'s are the modified Chebyshev polynomials. The above trial functions satisfy all the boundary conditions except the natural one, namely $(1+\Lambda)D^2W + Ma\; a^2\Theta = 0$ at $z=1$ but the residual from this condition is included as residual from the differential equation. Substituting Eq. (30) into Eqs.(25)-(27), multiplying momentum Eq. (25) by $W_j(z)$, energy Eq. (26) by $\Theta_j(z)$ and magnetization Eq. (27) by $\Phi_j(z)$; performing the integration by parts with respect to z between z = 0 and z = 1 and using the boundary conditions, we obtain a system of linear homogeneous algebraic equations in A_i, B_i and C_i. A nontrivial solution to the system requires the characteristic determinant of the coefficient matrix must vanish and this leads to a relation involving the parameters R_t, Ma, R_m, M_1, M_3, Λ, N_s and a in the form

$$f(R_t,\, Ma,\, R_m,\, M_1,\, M_3,\, \Lambda,\, N_s,\, a) = 0 \tag{32}$$

The critical values of R_{tc} or Ma_c are found as a function of wave number a for various values of physical parameters. It is observed that the convergence is achieved with six terms in the series expansion of Eq. (30).

2. Analytical solution by RPT

Since the critical wave number is negligibly small when the boundaries are perfectly insulated to temperature perturbations (i.e., $D\Theta = 0$ at $z = 0, 1$), the eigenvalue problem is also solved analytically using regular perturbation technique with wave number a as a perturbation parameter. Accordingly, the variables W, Θ and Φ are expanded in powers of a^2 as

$$(W, \Theta, \Phi) = (W_0, \Theta_0, \Phi_0) + a^2 (W_1, \Theta_1, \Phi_1) + \cdots \quad (33)$$

Substituting Eq.(33) into Eqs.(25)-(27) and also in the boundary conditions, and collecting the terms of zero-th order, we obtain

$$(1+\Lambda) D^4 W_0 = 0 \quad (34a)$$

$$D^2 \Theta_0 = -W_0 \quad (34b)$$

$$D^2 \Phi_0 = -D\Theta_0 \quad (34c)$$

with the boundary conditions

$$W_0 = DW_0 = 0 = D\Theta_0 = \Phi_0 \ \text{ at } z = 0 \quad (35a)$$

$$W_0 = (1+\Lambda) D^2 W_0 = 0 = D\Theta_0 = D\Phi_0 \text{ at } z = 1 \quad (35b)$$

The solution to the zero-th order equations is found to be

$$W_0 = 0,\ \Theta_0 = 1 \text{ and } \Phi_0 = 0 \quad (36)$$

The first order equations are then

$$(1+\Lambda) D^4 W_1 = R_t - R_m \left[N_s (1-2z) - 1 \right] \quad (37a)$$

$$D^2 \Theta_1 = 1 + W_1 \left[N_s (1-2z) - 1 \right] \quad (37b)$$

$$D^2 \Phi_1 = D\Theta_1 \quad (37c)$$

with the boundary conditions

$$W_1 = DW_1 = D\Theta_1 = \Phi_1 = 0 \ \text{ at } z = 0 \quad (38a)$$

$$W_1 = (1+\Lambda) D^2 W_1 + Ma = D\Theta_1 = D\Phi_1 = 0 \ \text{ at } z = 1. \quad (38b)$$

The general solution of Eq. (36a) is given by

$$W_1 = c_1 + c_2 z + c_3 z^2 + c_4 z^3 + \left[\frac{5 R_t - R_m [N_s (5-2z) - 5]}{120(1+\Lambda)} \right] z^4 \quad (39)$$

where,

$$c_1 = 0 = c_2,$$

$$c_3 = \frac{15 R_t + R_m (40 - N_s) + 60\, Ma}{240 (1+\Lambda)},$$

$$c_3 = \frac{-25 R_t + 2 R_m (7 N_s - 25) - 60\, Ma}{240 (1+\Lambda)}.$$

From Eq. (37b), after using the condition that $D\Theta_1 = 0$ at z = 0 and z = 1, it follows that

$$1 = \int_0^1 [1 - N_s (1-2z)] W_1 \, dz. \quad (40)$$

Substituting for W_1 from Eq.(39) into Eq.(40) and carrying out the integration leads to an expression of the form

$$\frac{R_{tc} + R_m}{320} + \frac{Ma_c}{48} + \frac{R_{tc} N_s}{2880} + \frac{Ma_c\, N_s}{240} + \frac{R_m N_s}{1440} + \frac{11 R_m\, N_s^{\,2}}{100800} = 1 + \Lambda. \quad (41)$$

From Eq.(41) it is interesting to note that the parameter M_3 is not appearing in the expression and hence the nonlinearity of magnetization has no effect on the onset of Bénard-Marangoni ferroconvection. Since at the onset of convection $a_c = 0$ (very large wave length), one would expect that M_3 has no effect on the stability of the system. Besides, it can be seen that the parameters $R_m (= R_t M_1)$ and M_3 have no influence on the onset of pure Marangoni ferroconvection ($R_t = 0$) in the absence of internal heat generation ($N_s = 0$). The numerical calculations carried out in the previous section also reflected the same behavior.

It is interesting to check Eq.(41) under the limiting conditions. In the absence of magnetic field (i.e., $R_m = 0$), internal heat source strength (i.e., $N_s = 0$) and when $\Lambda = 0$; Eq. (41) gives

$$\frac{R_{tc}}{320} + \frac{Ma_c}{48} = 1 \quad (42)$$

known result for ordinary viscous fluid layer (Garcia-Ybarra et al. (1987), Yang and Yang (1990)). When we set $R_{tc} = 0$, $R_m = 0$ and $\Lambda = 0$, Eq. (41) reduces to

$$Ma_c = \frac{240}{5 + N_s} \quad (43)$$

which corresponds to the result obtained (Wilson (1997)). When $N_s = 0$, Eq. (43) simply reduces to $Ma_c = 48$ and this is the known exact value for the clear viscous fluid layer (Pearson (1958)).

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RESULTS AND DISCUSSION

The linear stability analysis is carried out to investigate the combined effect of internal heat generation and MFD viscosity on the onset of coupled Bénard-Marangoni convection in a horizontal ferrofluid layer in the presence of a uniform vertical magnetic field. The lower rigid and the upper horizontal free boundaries are assumed to be perfectly insulated to temperature perturbations. The presence of internal heating makes the basic temperature, magnetic field and magnetization distributions to deviate from linear to nonlinear which in turn have significant influence on the stability of the system. To assess the impact of internal heat source strength N_s on the criterion for the onset of thermomagnetic convection, the distributions of dimensionless basic temperature, $\tilde{T}_b(z)$, magnetic field intensity, $\tilde{H}_b(z)$ and magnetization, $\tilde{M}_b(z)$ are exhibited graphically in Fig.1 for different values of N_s. From the figure it is observed that increase in the internal heat source strength amounts to large deviations in these distributions which in turn enhance the disturbances in the ferrofluid layer and thus reinforce instability on the system.

The critical stability parameters, Ma_c or R_{tc} and the corresponding critical wave number a_c are computed numerically by the Galerkin technique as well as analytically by employing a regular perturbation technique for different values of M_1, R_m, Λ and N_s. The salient characteristics of these parameters on the stability of the system are exhibited graphically in Figures 2-6. In these figures, the results obtained from the above two techniques are compared. In general, it is seen that the results obtained by regular perturbation technique coincide exactly with those obtained numerically and thus provides a justification for the use of regular perturbation technique in solving the eigenvalue problem when the boundaries are insulating to temperature perturbations. In the present context, we affirm that the analytical results obtained for the present case are exact.

The presence of internal heating makes the basic temperature, magnetic field and magnetization distributions to deviate from linear to nonlinear which in turn have significant influence on the stability of the system. To assess the impact of internal heat source strength N_s on the criterion for the onset of thermomagnetic convection, the distributions of dimensionless basic temperature, $\tilde{T}_b(z)$, magnetic field intensity, $\tilde{H}_b(z)$ and magnetization, $\tilde{M}_b(z)$ are exhibited graphically in Figure 1 for different values of N_s. From the figure it is observed that increase in the internal heat source strength amounts to large deviations in these distributions which in turn enhance the disturbances in the ferrofluid layer and thus reinforce instability on the system.

We note that three different types of forces are influencing the stability characteristics of the system namely, the buoyancy, the surface tension and the magnetic forces. To know the impact of these forces on the onset when they are acting alone and simultaneously, the results are discussed separately for the following cases:

(i) Bénard ferroconvection (buoyancy and magnetic forces)

(ii) Marangoni ferroconvection (surface tension and magnetic forces), and

(iii) Bénard-Marangoni ferroconvection (buoyancy, surface tension and magnetic forces).

1. Bénard ferroconvection

Here, we consider the absence of surface tension forces (i.e., $Ma = 0$) and this case corresponds to pure Bénard ferroconvection. Figure 2 shows the variation of critical thermal Rayleigh number R_{tc} as a function of dimensionless internal heat source strength N_s for different values of magnetic Rayleigh number R_m when MFD parameter $\Lambda = 0.2$. This case corresponds to convective instability only due to buoyancy forces. The figure clearly indicates that R_{tc} decreases monotonically with N_s indicating the effect of increasing internal heating is to destabilize the system. This is because increasing N_s amounts to large deviation in the basic temperature distribution of the parabolic type which in turn enhances the thermal disturbances in the fluid layer. The curve of $R_m = 0$ (i.e. absence of magnetic force) corresponds to the case of ordinary viscous fluid and it lies above all other curves of different R_m $(\neq 0)$. It is observed that R_{tc} decreases with an increase in the value of magnetic Rayleigh number R_m and thus the effect of magnetic forces to advance the onset of ferroconvection. In other words, ferromagnetic fluids carry heat more efficiently than ordinary viscous fluids. Besides, it may be noted that the difference in the critical Rayleigh numbers among different values of R_m diminishes as the value of R_m increases and also at lower values of N_s. Moreover, with increasing N_s, R_{tc} decreases slowly in the absence of magnetic force (i.e., $R_m = 0$) but quite rapidly with an increase in the strength of magnetic forces (i.e., $R_m \neq 0$).

Figure 3 represents the variation of critical thermal Rayleigh number R_{tc} as a function of dimensionless internal heat source strength N_s for different values of magnetic field dependent (MFD) viscosity parameter Λ when $R_m = 10$. The critical thermal Rayleigh number R_{tc} increase with an increase in MFD viscosity parameter Λ and thus it has a stabilizing effect on the system. That is, the effect of increasing Λ is to delay the onset of Bénard ferroconvection.

2. Marangoni ferroconvection

In the absence of buoyancy forces (i.e., $R_t = 0$) and this case corresponds to pure Marangoni ferroconvection. This type of convection offers new possibilities for applications in microgravity environments. As in the previous case, effect of various physical parameters on the onset of Marangoni ferroconvection is analyzed and the results are presented in Figures 4 and 5. The variation of Ma_c shown as a function of N_s for different values of R_m on the stability of the Marangoni ferroconvection in the case of $\Lambda = 0.2$ is shown in Figure 4. The results for $R_m = 0$ (i.e. absence of magnetic force) correspond to the case of ordinary viscous fluids and it is observed that higher heating is required to have instability in that case. However, Thus, magnetic Rayleigh number R_m increases, the critical Marangoni number Ma_c decreases and this is due to an increase in the destabilizing magnetic force which favors the ferrofluid to flow more easily. Figure 5 shows the plot of Ma_c as a function of N_s for different values of MFD viscosity parameter Λ when $R_m = 10$. As MFD viscosity parameter Λ increases, Ma_c increases and hence its effect is to delay the onset of Marangoni ferroconvection.

3. Bénard–Marangoni ferroconvection

We look into the simultaneous presence of thermal buoyancy and surface tension forces on the stability of the system. A plot of critical thermal Rayleigh number R_{tc} as a function critical Marangoni number Ma_c is shown in Figure 6 for different values of M_1 with $N_s = 2$ and $\Lambda = 0.02$. From the figures it is obvious that there is a strong coupling between the critical Rayleigh and the Marangoni numbers. That is, when the buoyancy force is predominant the surface tension force becomes negligible and vice-versa. From Figure.6 it is seen that an increase in the value of M_1 is to decrease the value of R_{tc} and thus its effect is to hasten the onset of ferroconvection. This is attributed to the increase in destabilizing magnetic force. Nonetheless, the curves of different M_1 converge to the same value $Ma_c = 34$ when $R_{tc} = 0$ indicating that it has no effect on Marangoni ferroconvection. Figure 7 shows that increase in the value of N_s is to decrease both Ma_c and R_{tc}. Thus the non-linear temperature distributions arising due to volumetric distribution of heat sources is to advance the onset of ferroconvection. It is further observed that the critical Rayleigh/Marangoni numbers are independent of the nonlinearity of fluid magnetization parameter M_3 and the analytically obtained results also confirm this finding (see Eq. 39).

Figure 8 shows the locus of the critical Marangoni number Ma_c and the thermal Rayleigh number R_{tc} for different values of MFD viscosity parameter Λ for $N_s = 2$ and $\Lambda = 0.02$. From the Figure 8, it is seen that the critical Rayleigh (R_{tc}) and Marangoni numbers (Ma_c) increase with an increase in the MFD viscosity parameter (Λ) and thus it has a stabilizing effect on the system. That is, the effect of increasing Λ is to delay the onset of Bénard–Marangoni ferroconvection.

The perturbed vertical velocity eigenfunction $W(z)$ for different boundary combinations are presented in Figures 9–11 for different values of Λ, N_s and R_m, respectively. As can be seen, increase in the value of MFD viscosity Λ, (see Figure 9), decrease in dimensionless internal heat source strength N_s (see Figure 10) and magnetic Rayleigh number R_m (see Figure 11) are to decrease the velocity and hence their effect is to delay the onset of ferroconvection in a ferrofluid layer.

CONCLUSIONS

The onset of penetrative Bénard–Marangoni ferroconvection in a ferrofluid layer is investigated theoretically via internal heating with magnetic field dependent viscosity. The lower rigid and the upper horizontal free boundaries are considered to be perfectly insulated to temperature perturbations. The combined effect of internal heat source strength measured through the parameter N_s and MFD viscosity parameter Λ on the stability characteristics of the system is analyzed in detail and the following conclusions can be drawn from the present study:

(i) The effect of increase in the value of magnetic field dependent viscosity parameter Λ is to delay the onset of Bénard-Marangoni ferroconvection, while increase in the value of magnetic Rayleigh number R_m and dimensionless internal heat source strength N_s is to reinforce together and to hasten the onset of coupled Bénard-Marangoni ferroconvection. Thus magnetic field dependent viscosity plays a crucial role in controlling Bénard-Marangoni ferroconvection.

(ii) The nonlinearity of fluid magnetization parameter M_3 has no effect on the onset of ferroonvection.

(iii) The buoyancy and surface tension forces complement with each other and it is always found that $Ma_c < R_{tc}$; a result in accordance with ordinary viscous fluids.

(iv) The critical eigenvalues obtained analytically by RPT and numerically by the Galerkin-type of WRT complement with each other indicating the analytical solutions obtained are exact.

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