International Journal of Applied Engineering Research, ISSN 0973-4562 Volume 13, Number 5 (2018) pp. 2688–2693

Effect of Gravity Modulation and Internal Heat Generation on Rayleigh-Bénard Convection in Couple Stress Fluid with Maxwell-Cattaneo Law

Maria Thomas
Research scholar, Department of Mathematics, CHRIST (Deemed to be University), Bengaluru, Karnataka-560029, India.

Sangeetha George K.
Department of Mathematics, CHRIST (Deemed to be University) Bengaluru, Karnataka-560029, India.

Abstract: The linear stability of a horizontal layer of couple stress fluid heated from below in the presence of time periodic body force and internal heat generation is considered. The classical Fourier heat law is replaced by the Maxwell-Cattaneo law to assimilate inertial effects. The regular perturbation method based on the small amplitude of modulation is employed to compute the critical Rayleigh number and the corresponding wave number. It is shown that the onset of convection can be advanced or delayed by proper regulation of various governing parameters.

Keywords: Couple Stress Fluid, Maxwell-Cattaneo Law, Gravity Modulation, Internal Heat Generation.

INTRODUCTION

Fourier's law describes the phenomenon of heat conduction in most practical engineering applications. However, some systems can exhibit time lag in response or wave-like behaviour. Since Fourier's law fails to account such mechanisms, there were several theoretical developments to understand the nature of the deviations. Maxwell-Cattaneo law is given by

$$\vec{Q} + \tau \frac{\partial \vec{Q}}{\partial t} = -\kappa \nabla T,$$

where τ represents the time lag that is required for the heat flux to reach a steady state following a perturbation to the temperature gradient. Delayed response implies that the system has some thermodynamic inertia and the parameter τ is a measure of this inertia. Bissell [1, 2] studied Rayleigh-Bénard convection problem using the hyperbolic heat-flow model and developed a linear theory for thermal instability by oscillatory modes. Stranges et al. [3] examined the thermal convection for fluids possessing significant thermal relaxation time and found that non-Fourier effects are important when the Cattaneo number is significant. Pranesh and Ravi [4] and Shivakumara [5] have respectively studied temperature modulation in a Newtonian fluid and fluid-saturated Brinkman porous medium with Cattaneo effect.

Understanding non-Newtonian fluid behaviour is very important for the better explanation of the behaviour of rheologically complex fluids such as liquid crystals, polymeric suspensions that have long-chain molecules and lubrication. Couple stress fluid theory introduced by Stokes [6] takes into account the presence of couple stresses, body couples and non-symmetric stress tensor. This model results in equations that are similar to the Navier Stokes equations and thus facilitating a comparison with the results for the classical non polar case. Pranesh and Sangeetha [7] studied the effect of temperature modulation on the onset of convection in a dielectric couple stress fluid. Bhadauria et al. [8] investigated the effect of temperature and gravity modulation on double diffusive convection in a couple stress liquid. Shankar et al. [9] analyzed the combined effect of couple stresses and a uniform horizontal AC electric field on the stability of buoyancy-driven parallel shear flow of a vertical dielectric fluid.

Many authors have studied the effect of periodic modulation on the onset of convection on a heated fluid layer. The gravitational modulation, which can be realized by vertically oscillating horizontal liquid layer, acts on the entire volume of the liquid and has a stabilizing or destabilizing effect depending on the amplitude and frequency of the modulation. Natural convection driven by internal heat generation is observed in many physical phenomena in nature such as in the earth's mantle. Understanding the effects of internal heat generation is also significant for reactor safety analysis, metal waste, spent nuclear fuel, fire and combustion studies and strength of radioactive materials. Bhadauria and Kiran [10] analyzed double diffusive convection in an electrically conducting viscoelastic fluid layer heated from below and found that modulated gravity field can be used either to delay or enhance the heat and mass transfer in the system. Sameena and Pranesh [11] studied the effects of fluctuating gravity in a weak electrically conducting couple stress fluid with a saturated porous layer. Vasudha et al. [12] studied the effect of gravity modulation on the onset of convection in a micropolar fluid with internal heat generation using linear stability analysis.

We study in the present paper the combined effect of gravity modulation and internal heat generation on the onset of convection in a couple-stress fluid layer. Since the amplitude and frequency of the modulation are externally controlled parameters, the onset of convection can be delayed or advanced by the proper tuning of these parameters.

MATHEMATICAL FORMULATION

We consider a layer of couple stress fluid confined between two horizontal wall as shown in Figure 1. A Cartesian coordinate system (x, y, z) is chosen such that the origin is at the lower wall and z-axis is vertically upward. The system is under the influence of a periodically varying vertical gravity field given by

$$\vec{g}(t) = -g_0\,(1 + \varepsilon\cos\omega t)\hat{k} \tag{1}$$

where ε is the amplitude of the modulation, g_0 is the mean gravity and ω is the frequency of the modulation.

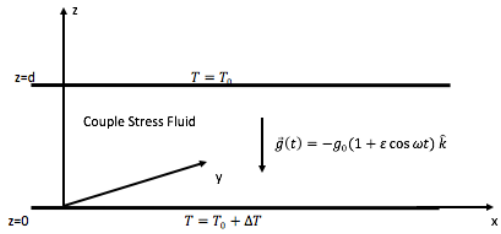

FIGURE 1. Physical configuration of the problem

Under the Boussinesq approximation, the governing equations are given by ([13, 4])

$$\nabla.\vec{q} = 0, \tag{2}$$

$$\rho_0\Big[\frac{\partial\vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q}\Big] = -\nabla p + \rho\vec{g}(t) + \mu\nabla^2\vec{q} - \mu'\nabla^4\vec{q}, \tag{3}$$

$$\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T = -\nabla.\vec{Q} + Q_1(T - T_0), \tag{4}$$

$$\tau\Big[\frac{\partial\vec{Q}}{\partial t} + \vec{\omega_1}\times\vec{Q}\Big] = -\vec{Q} - \kappa\nabla T, \tag{5}$$

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{6}$$

where $\vec{q}$ is the velocity, T is the temperature, p is the pressure, ρ is the density, ρ_0 is the density at $T = T_0$, $\vec{Q}$ is the heat flux vector, μ is the dynamic viscosity, μ' is the couple stress viscosity, τ is the relaxation time, κ is the thermal conductivity, α is the coefficient of thermal expansion, Q_1 is the internal heat source and $\vec{\omega_1} = \frac{1}{2}\nabla\times\vec{q}$.

The thermal boundary conditions considered are

$$T = T_0 + \Delta T \text{ at } z = 0 \text{ and } T = T_0 \text{ at } z = d. \tag{7}$$

The basic state of the fluid is quiescent and is given by

$$\vec{q_b}(z) = \vec{0},\ \rho = \rho_b(z),\ p = p_b(z),\ T = T_b(z),\ \vec{Q} = \vec{Q}_b(z). \tag{8}$$

Substituting (8) in (2)-(6), the pressure p_b, heat flux $\vec{Q}_b$, temperature T_b, and density ρ_b satisfy the following equations:

$$\frac{\partial p_b}{\partial z} = -\rho_b g_0\,(1 + \varepsilon\ \cos\omega t), \tag{9}$$

$$\frac{d\vec{Q}_b}{dz} = Q_1(T_b - T_0), \tag{10}$$

$$\vec{Q}_b = -\kappa \frac{\partial T_b}{\partial z}, \tag{11}$$

$$\frac{d\vec{Q}_b}{dz} = -\kappa \frac{d^2 T_b}{dz^2}, \tag{12}$$

$$\rho_b = \rho_0[1 - \alpha(T_b - T_0)]. \tag{13}$$

(12) is solved for $T_b(z)$ subject to the boundary conditions (7) and we obtain

$$T_b = T_0 + \frac{\Delta T}{\sin(\sqrt{Ri})} \sin\left(\sqrt{Ri}(1 - \frac{z}{d})\right) \tag{14}$$

where $Ri = \dfrac{Q_1\, d^2}{\kappa}$ is the internal Rayleigh number.

Let the basic state be disturbed by an infinitesimal thermal perturbation. We assume a solution for $\vec{q}$, T, p, ρ and $\vec{Q}$ in the form

$$\begin{aligned} \vec{q} = \vec{q}_b + \vec{q}', \rho = \rho_b + \rho', p = p_b + p', \\ T = T_b + T', \vec{Q} = \vec{Q}_b + \vec{Q}' \end{aligned} \tag{15}$$

where $\vec{q}'$, ρ', p', T', $\vec{Q}'$ represents the perturbed quantities which are assumed to be small. Substituting (15) in (2)-(6) and using the basic state equations we obtain the linearized equations for the infinitesimal perturbations as:

$$\nabla.\vec{q}' = 0, \tag{16}$$

$$\begin{aligned} \rho_0 \frac{\partial \vec{q}'}{\partial t} &= -\nabla p' + \mu \nabla^2 \vec{q}' - \mu' \nabla^4 \vec{q}' \\ &\quad - \rho' g_0 \,(1 + \varepsilon \cos(\omega t))\hat{k}, \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{\partial T'}{\partial t} - \frac{W' \,\Delta T \sqrt{Ri}}{d\,\sin(\sqrt{Ri})} \cos\left(\sqrt{Ri}(1 - \frac{z}{d})\right) \\ = -\nabla.\vec{Q}' + Q_1 T', \end{aligned} \tag{18}$$

$$\begin{aligned} \left(1 + \tau \frac{\partial}{\partial t}\right)\vec{Q}' = -\frac{\kappa\tau}{2} \frac{\Delta T \sqrt{Ri}}{d\,\sin(\sqrt{Ri})} \cos\left(\sqrt{Ri}(1 - \frac{z}{d})\right) \\ \times \left[\frac{\partial \vec{q}'}{\partial z} - \nabla W'\right] - \kappa \nabla T', \end{aligned} \tag{19}$$

$$\rho' = -\rho_0 \alpha T' \tag{20}$$

We eliminate p' from (17) by operating curl twice and eliminate $\vec{Q}'$ between (18) and (19). These equations are then non-dimensionalized using:

$$\begin{aligned} (x^*, y^*, z^*) = \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right);\ \vec{q}^* = \frac{\vec{q}'}{\left(\frac{\kappa}{d}\right)}; \\ t^* = \frac{t}{\left(\frac{d^2}{\kappa}\right)};\ T^* = \frac{T'}{\Delta T}; \omega^* = \frac{\omega}{\left(\frac{\kappa}{d^2}\right)}; \end{aligned} \tag{21}$$

to obtain, ignoring the asterisks:

$$\frac{1}{Pr}\frac{\partial}{\partial t}(\nabla^2 W) = \nabla^4 W - C\nabla^6 W + R\,(1 + \varepsilon f)\,\nabla_1^2 T, \tag{22}$$

$$\left.\begin{aligned} \frac{\sqrt{Ri}}{\sin(\sqrt{Ri})} \cos\left(\sqrt{Ri}(1 - z)\right)\left[1 + 2M\frac{\partial}{\partial t} - M\,\nabla^2\right]W \\ + \left[Ri\left(1 + 2M\frac{\partial}{\partial t}\right) - \left(1 + 2M\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t} + \nabla^2\right]T = 0. \end{aligned}\right\} \tag{23}$$

where $\nabla_1^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, $\nabla^2 = \nabla_1^2 + \dfrac{\partial^2}{\partial z^2}$,

$Pr = \dfrac{\mu}{\rho_0\,\kappa}$ (Prandtl number),

$R = \dfrac{\rho_0\,\alpha\,g\,d^3\,\Delta T}{\mu\,\kappa}$ (Rayleigh number),

$M = \dfrac{\tau\,\kappa}{2\,d^2}$ (Cattaneo number),

$C = \dfrac{\mu'}{\mu\,d^2}$ (Couple stress parameter).

(22) and (23) are solved for stress-free, isothermal, vanishing couple- stress boundary conditions ([14]) and hence we have

$$W = \frac{\partial^2 W}{\partial z^2} = T = 0 \quad \text{at } z = 0 \text{ and } z = 1.$$

Eliminating W from (22) and (23), we obtain

$$[R\,(1 + \varepsilon f)\,X_1 \nabla_1^2 + X_2\,X_3]T = 0 \tag{24}$$

with the boundary conditions

$$T = \frac{\partial^2 T}{\partial z^2} = \frac{\partial^4 T}{\partial z^4} = \frac{\partial^6 T}{\partial z^6} = 0 \quad \text{at } z = 0, 1. \tag{25}$$

where

$$Y = \frac{-4\,\pi^2}{Ri - 4\,\pi^2};$$

$$X_1 = Y\Big[1 + 2M\frac{\partial}{\partial t} - M\nabla^2\Big];$$

$$X_2 = \Big[\frac{1}{Pr}\frac{\partial}{\partial t}\nabla^2 - \nabla^4 + C\,\nabla^6\Big];$$

$$X_3 = \Big[Ri\Big(1 + 2M\frac{\partial}{\partial t}\Big) - \big(1 + 2M\frac{\partial}{\partial t}\big)\frac{\partial}{\partial t} + \nabla^2\Big].$$

SOLUTION

We assume the solution of (24) in the form

$$(T, R) = (T_0, R_0) + \varepsilon(T_1, R_1) + \varepsilon^2(T_2, R_2) + \cdots \tag{26}$$

where T_0 and R_0 are the eigenfunction and eigenvalue respectively for the unmodulated system and T_i and R_i , for $i \geq 1$, are the correction to T_0 and R_0 in the presence of modulation.

Substituting (26) in (24) and equating the corresponding terms, we obtain the following system of equations:

$$LT_0 = 0, \tag{27}$$

$$LT_1 = -(R_1 + R_0 f)X_1\nabla_1^2 T_0, \tag{28}$$

$$LT_2 = -(R_1 + R_0 f)X_1\nabla_1^2 T_1 - (R_2 + R_1 f)X_1\nabla_1^2 T_0, \tag{29}$$

where

$$L = R_0\,X_1\nabla_1^2 + X_2\,X_3. \tag{30}$$

The marginal stability solution for (27) is

$$T_0 = \sin(\pi z)\exp[i(lx + my)] \tag{31}$$

where l and m are the wave numbers in the x and y directions, respectively, such that $l^2 + m^2 = a^2$. Substituting (31) in (27), we obtain

$$R_0 = \frac{k_1^6\,\eta_1}{a^2\,Y\,\beta_1} - \frac{k_1^4\,\eta_1\,Ri}{a^2\,Y\,\beta_1}, \tag{32}$$

where $k_1^2 = \pi^2 + a^2$, $\eta_1 = 1 + C\,k_1^2$, $\beta_1 = (1 + Mk_1^2)$.

Then (28) for T_1 becomes

$$LT_1 = a^2(R_1 + R_0 f)Y\beta_1 T_0 \tag{33}$$

The solubility condition requires that the time independent part of the right-hand side of (33) should be orthogonal to T_0. Since f varies sinusoidally with time, the only time independent term in the right-hand side of (33) is $a^2R_1\beta_1\,T_0$. Hence $R_1 = 0$ and it follows that all the odd coefficients R_3, R_5 and so on in (26) are zero. Expanding the right-hand side of (33) using Fourier series and by inverting the operator L term by term we obtain T_1 as

$$T_1 = a^2R_0\beta_1 Y\;Re\Big\{\sum_{n=1}^{\infty}\frac{e^{-i\omega t}}{L_1(\omega, n)}\sin(n\pi z)\Big\}, \tag{34}$$

where

$$\begin{aligned} L_1(\omega, n) = \big[&-a^2R_0Y\beta_n - k_n^4\eta_n\gamma_n - k_n^4\eta_n Ri \\ &- \frac{\omega^2k_n^2}{Pr} + \frac{2M\;Ri\;\omega^2k_n^2}{Pr}\Big] \\ &+ i\omega\Big[2a^2M\;R_0Y + \frac{k_n^2\gamma_n}{Pr} \\ &+ \frac{k_n^2Ri}{Pr} - k_n^4\eta_n + 2M\;Ri\;k_n^4\eta_n\Big] \end{aligned} \tag{35}$$

$$k_n^2 = n^2\pi^2 + a^2;\ \eta_n = 1 + C\,k_n^2;$$
$$\beta_n = 1 + M\,k_n^2;\ \gamma_n = -k_n^2 + 2M\omega^2.$$

(29) for T_2 becomes

$$LT_2 = a^2R_0(\beta_n - i\,2M\omega)\,f\,T_1 + a^2R_2Y\beta_1T_0, \tag{36}$$

We use (36) to determine R_2, the first non-zero correction to R_0.

$$R_2 = -\frac{a^2\,R_0^2\,Y}{4}\star\sum_{n=1}^{\infty}\Big\{\frac{[N(\omega, n) + N^*(\omega, n)]}{|L_1(\omega, n)|^2}\Big\} \tag{37}$$

where $*$ denotes a complex conjugate and

$$N(\omega, n) = L_1^*(\omega, n)(\beta_n - i\,2M\omega)$$

RESULTS AND DISCUSSION

This paper presents an analytical study of the effects of gravity modulation and internal heat generation on the

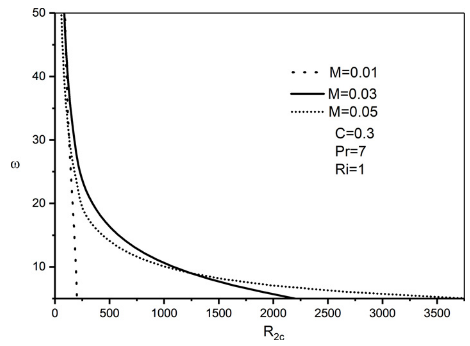

FIGURE 2. Variation of R_{2c} with ω for various values of Cattaneo number

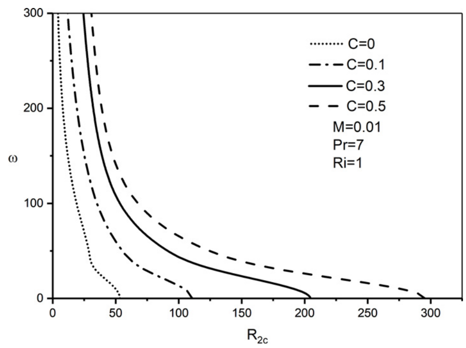

FIGURE 3. Variation of R_{2c} with ω for various values of couple stress parameter

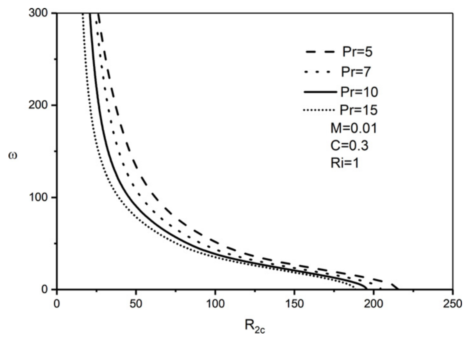

FIGURE 4. Variation of R_{2c} with ω for various values of Prandtl number

onset of convection in a couple stress fluid layer employing Maxwell-Cattaneo law. The analysis presented is based on the assumptions that the amplitude of the gravity modulation is very small compared to the mean gravity and that the convective currents are weak so that nonlinear effects may be neglected. The violation of these assumptions would alter the results significantly when the modulating frequency, ω, is low. This is because the perturbation method imposes the condition that the amplitude of εT_1 should not exceed that of T_0 which in turn gives the condition $\omega > \varepsilon$. Thus the validity of the results depends on the value of the modulating frequency, ω. When $\omega \ll 1$, the period of modulation is large and it affects the entire volume of the fluid. For large frequencies the effect of modulation disappears because the buoyancy force takes a mean value leading to the equilibrium state of the unmodulated case. Thus the effect of modulation is significant only for the small and moderate values of ω. ([14])

Figures 2, 3, 4 and 5 show the variation of R_{2c} with respect to ω for various governing parameters. We see from these figures that R_{2c} is positive over the entire range of values of ω indicating that the effects of gravity modulation and internal heat generation is to stabilize the system i.e., convection occurs at a later point compared to the unmodulated system.

Figure 2 shows the variation of R_{2c} with the modulating frequency ω for different values of Cattaneo number, M. Cattaneo number has stabilizing effects for small values of the frequency. Cattaneo number is proportional to the relaxation time and the speed of the temperature propagation will decrease for increasing relaxation time and hence delay the onset of convection. However this effect reverses and becomes destabilizing for moderate values because of the decrease in the size of the convection cells. The value of ω at which stabilizing effect reverses is dependent on the values of the other governing parameters. The effect of modulation vanishes for large frequencies.

The effect of couple stress parameter on the onset of convection is shown in Figure 3. It can be seen that the effect of increasing the couple stress parameter is to make the system more stable. C is indicative of the concentration of the suspended particles and the effect of suspended particles is to increase the viscosity of the fluid and thereby having an stabilizing influence on the system.

Figure 4 depicts the variation of R_{2c} with ω for different values of Prandtl number, Pr. We find that the magnitude of the correction Rayleigh number decreases with increase in Prandtl number. This indicates that Pr reduces the stabilizing effect. This effect is more pronounced for moderate frequencies.

Figure 5 shows the variation of R_{2c} with ω for different values of internal Rayleigh number, Ri. We observe that R_{2c} decreases with increase in Ri; thus destbilizing the system. Increase in Ri amounts to increase in energy supply to the system. This results in deviations from the basic state temperature distribution and thus in the enhancement of the thermal disturbances.

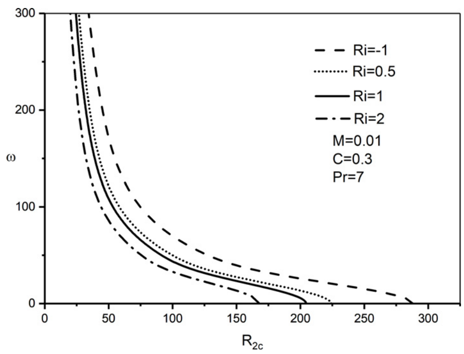

FIGURE 5. Variation of R_{2c} with ω for various values of internal Rayleigh number Ri

CONCLUSION

The effect of gravity modulation and internal heat generation on the onset of convection in a couple stress fluid with Maxwell-Cattaneo law is studied using a linear stability analysis and the following conclusions are drawn:

- Cattaneo number is stabilizing for small frequencies and destabilizing for moderate frequencies.
- Couple stress parameter stabilizes the system.
- Prandtl number enhances the destabilizing effect of modulation.
- Internal Rayleigh number destabilizes the system.

Low frequency gravity modulation and internal heat generation have a significant influence on stability of the system. The study is significant in areas of low temperature fluids, granular flows and liquid metals. The problem gives insight into external means of controlling the onset convection.

ACKNOWLEDGEMENT

The authors sincerely thank the management of CHRIST (Deemed to be University) for their incessant support and Dr. S Pranesh for his encouragement and insightful comments.

REFERENCES

[1] Bissell, J. J., 2015, "On oscillatory convection with the Cattaneo-Christov hyperbolic heat-flow model," Proc. R. Soc. A Math. Phys. Eng. Sci., 471(2175), pp. 20140845.

[2] Bissell, J. J., 2016, "Thermal convection in a magnetized conducting fluid with the Cattaneo–Christov heat-flow model," Proc. R. Soc. A., 472(2195), pp. 20160649.

[3] Stranges, D. E., Khayat, R. E., and Debruyn, J., 2016, "Finite thermal convection of non-Fourier fluids," Int J Therm Sci., 104, pp. 437–447.

[4] Pranesh, S., and Kiran, R. V., 2015, 2015, "Rayleigh-Bénard Chandrasekhar convection in an electrically conducting fluid using Maxwell-Cattaneo law with temperature modulation of the boundaries," Int J Eng Research & Tech, 4(10), pp. 174–184.

[5] Shivakumara, I. S., Ravisha, M., Ng CO, and Varun, V. L., 2015, "A thermal non-equilibrium model with Cattaneo effect for convection in a Brinkman porous layer," Int J Non Linear Mech, 71, pp. 39–47.

[6] Stokes, V. K., 1966, "Couple stresses in fluids," Phys Fluids, 9(9), pp. 1709–1715.

[7] Pranesh, S., and Sangeetha George, 2014, "Effect of imposed time periodic boundary temperature on the onset of Rayleigh-Bènard convection in a dielectric couple stress fluid," Int. J. Appl. Math. Comput., 5(4), pp. 1–13.

[8] Bhadauria, B. S., Siddheshwar, P. G., Singh, A. K., and Vinod K. Gupta, 2016, "A local nonlinear stability analysis of modulated double diffusive stationary convection in a couple stress liquid," J. Appl. Fluid Mech., 9(3), pp. 1255–1264.

[9] Shankar, B. M., Jai Kumar, and Shivakumara, I. S., 2016, "Stability of natural convection in a vertical dielectric couple stress fluid layer in the presence of a horizontal ac electric field," Appl Math Modelling, 40(9), pp. 5462–5481.

[10] Bhadauria, B. S., and Palle Kiran, 2015, "Chaotic and oscillatory magneto-convection in a binary viscoelastic fluid under G-jitter Int J Heat Mass Transfer," 84, pp. 610–624.

[11] Sameena Tarannum,and Pranesh, S., 2016, "Effect of gravity modulation on the onset of Rayleigh-Bènard Convection in a weak electrically conducting couple stress fluid with saturated porous layer," Int J Eng Res, 5, pp. 914–928.

[12] Vasudha Yekasi, Pranesh, S., and Shahnaz Bathul, 2016, "Effects of gravity modulation and internal heat generation on the onset of Rayleigh-Bènard convection in a micropolar fluid," J Advances Math, 12(6), pp. 6270–6285.

[13] Siddheshwar, P. G., and Pranesh, S., 2004, "An Analytical Study of Linear and Non-linear Convection in Boussinesq–Stokes Suspensions," Int J Non-Linear Mech, 39(1), pp. 165–172.

[14] Malashetty, M. S., and Swamy, M. S., 2011, "Effect of gravity modulation on the onset of thermal convection in rotating fluid and porous layer," Phys. Fluids, 23, pp. 064108.