International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 23 (2018) pp. 16708-16718
©Research India Publications. http://www.ripublication.com

Some Cyclic Codes of length $4p^n$ and their Minimum Distance Bounds

Jagbir Singh
Department of Mathematics, M.D. University, Rohtak 124001*, India.*

Sonika Ahlawat
Department of Mathematics, M.D. University, Rohtak 124001*, India.*

Abstract
The idempotents generating the minimal ideal in the semi-simple group algebra FC_{4p^n} of the cyclic group C_{4p^n} of order $4p^n$ over finite field F are obtained. Generating polynomials and minimum distance bounds for the corresponding cyclic codes of length $4p^n$ are also calculated.

Key Words: Group algebra, cyclotomic cosets, primitive idempotents, generating polynomials.

AMS Subject Classification: $11T71$, $11T55$, $22D20$.

1. Introduction

Let F be a Galois field of order q where q is some prime power of the form $4k+3$ and C_m be a cyclic group of order m such that $g.c.d.(q,m)=1$. Then the group algebra FC_m is semi-simple having finite cardinality of collection of primitive idempotents which equals the cardinality of collection of q-cyclotomic cosets modulo m. Let t be the multiplicative order of q modulo p^n, then $1 \le t \le \phi(p^n)$ [6]. Pruthi and Arora ([2, 8]) computed the primitive idempotents of minimal cyclic codes of length m in case, when $t=\phi(m)$ and $m=2,4,p^n,2p^n$. The primitive idempotents of length p^n with order of q modulo p^n is $\frac{\phi(p^n)}{2}$ were obtained by Arora et.al. [1] and minimal quadratic residue codes of length p^n by Batra and Arora [4]. Cyclic codes of length $2p^n$ over F, where order of q modulo $2p^n$ is $\frac{\phi(2p^n)}{2}$ were obtained by Batra and Arora [5]. Minimal cyclic codes of length p^nq, where p and q are distinct odd primes were obtained by Sahni and Sehgal [9]. The minimal cyclic codes of length p^nq were obtained by Bakshi and Raka [3]. Further, for $t=\phi(p^n)$ the minimal cyclic codes of length $8p^n$, were discussed by Singh and Arora [10]. F.Li et.al. obtained irreducible cyclic codes of length $4p^n$ and $8p^n$, where $q \equiv 3(mod\, 8)$ and $p/(q-1)$ [7].

In present paper, we obtained cyclic codes of length $4p^n$ over F where order of q modulo p^n is $\frac{\phi(p^n)}{2}$. The q-cyclotomic cosets modulo $4p^n$ are obtained in Section 2 and corresponding primitive idempotents in Section 3. In section 4, we discussed generating polynomials and dimensions for the corresponding cyclic codes of length $4p^n$. The minimum distance or the bounds for minimum distance of these codes are obtained in Section 5.

2. Cyclotomic Cosets

Lemma 2.1 *Suppose* $\frac{\phi(p^n)}{2}$ *be the order of* q *modulo* p^n. *Then the order of* q *modulo* p^{n-i} *is* $\frac{\phi(p^{n-i})}{2}$ *for all* $i,\ 0 \le i \le n-1$.

Proof. Proof is on similar lines as that of ([5], Theorem 2.5). □

Lemma 2.2 *If* $\frac{\phi(p^n)}{2}$ *is the order of* q *modulo* p^n *then for* $0 \le i \le n-1$,
(i) *order of* q *modulo* $2p^{n-i}$ *is* $\frac{\phi(p^{n-i})}{2}$.
(ii) *For* $p \equiv 1(mod\, 4)$, *order of* q *modulo* $4p^{n-i}$ *is* $\frac{\phi(p^{n-i})}{2}$.
(iii) *For* $p \equiv 3(mod\, 4)$, *order of* q *modulo* $4p^{n-i}$ *is* $\phi(p^{n-i})$. *Proof.* (i) *Since* $\frac{\phi(p^n)}{2}$ *is the order of* q *modulo* p^n *therefore by lemma* 2.1 *order of* q *modulo* p^{n-i} *is* $\frac{\phi(p^{n-i})}{2}, 1 \le i \le n-1$. *Hence*

$$q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\, p^{n-i}) \tag{1}$$

Since q *is of the form* $4k+1$ *therefore* $q \equiv 1(mod\ 2)$. *Hence,* $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 2)$. *As* $gcd(2,p^{n-i})=1$, *so*

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 23 (2018) pp. 16708-16718
©Research India Publications. http://www.ripublication.com

$q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 2p^{n-i})$, *since order of* q *modulo* p^{n-i} *is* $\frac{\phi(p^{n-i})}{2}$. *This implies that* $\frac{\phi(p^{n-i})}{2}$ *is the smallest integer for which* (2.1) *holds. Hence order of* q *modulo* $2p^{n-i}$ *is* $\frac{\phi(p^{n-i})}{2}$.
(ii) *Proof holds similar to that of* (i).
(iii) *If* $p \equiv 3(mod\ 4)$, *then* $q^{\frac{\phi(p^{n-i})}{2}} \equiv -1(mod\ 4)$ *so* $q^{\phi(p^{n-i})} \equiv 1(mod\ 4)$ *and* $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ p^{n-i})$ *so* $q^{\phi(p^{n-i})} \equiv 1(mod\, p^{n-i})$. *As,* $\gcd(4, p^{n-i}) = 1$ *therefore* $q^{\phi(p^{n-i})} \equiv 1(mod\, 4p^{n-i})$. *Further* $q^{\frac{\phi(p^{n-i})}{2}} \not\equiv 1(mod\, 4p^{n-i})$ *as* $q^{\frac{\phi(p^{n-i})}{2}} \equiv -1(mod\, 4)$. *Hence order of* q *modulo* $4p^{n-i}$ *is* $\phi(p^{n-i})$. □

Lemma 2.3 *For* $0 \le i \le n-1,$ *and* $0 \le k \le \frac{\phi(p^{n-i})}{2} - 1$, $1 + 2p^n \not\equiv q^k(mod 4p^{n-i})$.

Proof. Proof can be obtained by using lemma 2.1 and lemma 2.2. □

Lemma 2.4 *Let* p *be an odd prime. Then there exists an integer* g, $1 < g < 4p$ *such that* g *is primitive root modulo* p, *order of* g *modulo* 4 *is* 2. *Further, if* q *is any prime power and* $g.c.d.(q, p) = 1$, *then* $g \notin \{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$.

Proof. Consider the complete residue system, $S_p = \{0, 1, 2, ..., p-1\}$ modulo p, $S_2 = \{0, 1\}$ modulo 2, and $S_{2p} = \{0, 1, 2, ..., 2p-1\}$ modulo $2p$. Since $g.c.d.(2, p) = 1$. So there exist an integer $v \in S_p$ such that $2v - p = 1$. Let a be any primitive root mod p in S_p. For $p \equiv 1(mod\, 4)$, let $g \equiv 2av + 3p + 2ap(mod\, 4p)$. Then, $g \equiv a(mod\, p)$. Hence, g is primitive root modulo p. Now, $g \equiv 2av + 3p + 2ap(mod\, 4)$, so $g \equiv 3(mod\, 4)$, as p is of the form $4k+1$. Hence, order of g modulo 4 is 2. Further, for $p \equiv 3(mod\, 4)$, let $g \equiv 2av + p(mod\, 4p)$. Then, as for the case of $p \equiv 1(mod\, 4)$, g is primitive root modulo p and 4 both. Let $g \in \{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$. So $g = q^i$ for some $1 \le i \le \frac{\phi(p)}{2} - 1$ This implies $o(g) = o(q^i)$. Here, order of q modulo $4p$ is $\frac{\phi(p)}{2}$. So $o(q^i) \le \frac{\phi(p)}{2}$ modulo $4p$. This implies $o(g) \le \frac{\phi(p)}{2}$ modulo $4p$. But order of g mod $4p$ is $\phi(p)$. Hence $g \notin \{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$. □

Lemma 2.5 *If* $p \equiv 1(mod4)$, *there exist a fixed integer* g *satisfying* $gcd(g, 2pq) = 1, 1 < g < 4p, g \not\equiv q^k(modp)$ *where* $0 \le k \le \frac{\phi(p)}{2} - 1,$ *such that for* $0 \le j \le n-1,$ *the set* $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ *forms a reduced residue system modulo* p^{n-j} *and the set* $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}, \lambda, \lambda q, \lambda q^2,$ $..., \lambda q^{\frac{\phi(p^{n-j})}{2}-1}, \lambda g, \lambda gq, \lambda gq^2, ..., \lambda gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ *forms a reduced residue system modulo* $4p^{n-j}$, *where* $\lambda = 1 + 2p^n$.

Proof. By lemma 2.1, order of q modulo p is $\frac{\phi(p)}{2}$. Therefore the numbers $1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}$ are incongruent modulo p. But there are exactly $\phi(p)$ numbers in the reduced residue system modulo p. Therefore there exist a number g satisfying $gcd(g, 2pq) = 1, 1 < g < 4p, g \not\equiv q^k(modp)$ for $0 \le k \le \frac{\phi(p)}{2} - 1.$ Then the set $\{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p)}{2}-1}\}$ forms a reduced residue system modulo p. Since for $0 \le k \le \frac{\phi(p)}{2} - 1;$ $g \not\equiv q^k(modp)$. It follows that; $g \not\equiv q^k(modp^{n-j})$; for $0 \le k \le \frac{\phi(p^{n-j})}{2} - 1.$ Hence the set $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo $p^{n-j}, 0 \le j \le n-1.$
Similar result holds to show that the set $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}, \lambda, \lambda q, \lambda q^2, ..., \lambda q^{\frac{\phi(p^{n-j})}{2}-1}, \lambda g, \lambda gq, \lambda gq^2,$ $..., \lambda gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo $4p^{n-j}$. □

Lemma 2.6 *If* $p \equiv 3(mod4)$, *there exist a fixed integer* g *satisfying* $gcd(g, 2pq) = 1, 1 < g < 2p, g \not\equiv q^k(modp)$ *where* $0 \le k \le \frac{\phi(p)}{2} - 1,$ *such that for* $0 \le j \le n-1,$ *the set* $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\},$ *forms a reduced residue system modulo* p^{n-j} *and the set* $\{1, q, q^2, ..., q^{\phi(p^{n-j})-1}, g, gq, gq^2, ..., gq^{\phi(p^{n-j})-1}\},$ *forms a reduced residue system modulo* $4p^{n-j}$, *where* $\lambda = 1 + 2p^n$.

Proof. Proof is similar to that of lemma 2.5. □

Theorem 2.1 *If* $p \equiv 1(mod\ 4)$ *, then the* $(8n+3)$ q *-cyclotomic cosets modulo* $4p^n$ *are given by:*

$$\Omega_0 = \{0\},\ \Omega_{p^n} = \{p^n, p^n q\},\ \Omega_{2p^n} = \{2p^n\}$$

and for $0 \leq i \leq n-1$ *,*

$$\Omega_{sp^i} = \{sp^i, sp^i q, sp^i q^2, ..., sp^i q^{\frac{\phi(p^{n-i})}{2}-1}\} \text{ for } s = 1,2,4,\lambda,g,2g,4g,\lambda g.$$

Proof. $\Omega_0 = \{0\}$ is trivial.
Since q is of the form $4k+3$, so $p^n q^2 \equiv p^n (mod\ 4p^n)$ and hence $\Omega_{p^n} = \{p^n, p^n q\}$, $\Omega_{2p^n} = \{2p^n\}$.
By lemma 2.2 ; $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 4p^{n-i})$. Equivalently, $p^i q^{\frac{\phi(p^{n-i})}{2}} \equiv p^i(mod 4p^n)$. Therefore,
$\Omega_{p^i} = \{p^i, p^i q, p^i q^2, ..., p^i q^{\frac{\phi(p^{n-i})}{2}-1}\}$.
Similarly, $\Omega_{sp^i} = \{sp^i, sp^i q, sp^i q^2, ..., sp^i q^{\frac{\phi(p^{n-i})}{2}-1}\}$ for $s = 2,4,\lambda,g,2g,4g,\lambda g$.
Obviously, $|\Omega_0| = 1$. Also, $|\Omega_{p^n}| = 2$, $|\Omega_{2p^n}| = 1$, and
$|\Omega_{p^i}| = |\Omega_{2p^i}| = |\Omega_{4p^i}| = |\Omega_{\lambda p^i}| = |\Omega_{gp^i}| = |\Omega_{2gp^i}| = |\Omega_{4gp^i}| = |\Omega_{\lambda gp^i}| = \frac{\phi(p^{n-i})}{2}$.
Therefore, $\sum_{i=0}^{n-1} |\Omega_{p^i}| = \sum_{i=0}^{n-1} \frac{\phi(p^{n-i})}{2} = \frac{\phi(p^n)}{2} + \frac{\phi(p^{n-1})}{2} + \frac{\phi(p^{n-2})}{2} + ... + \frac{\phi(p)}{2} = \frac{p^n-1}{2}$.
Hence, $|\Omega_0| + |\Omega_{p^n}| + |\Omega_{2p^n}| + \sum_{i=0}^{n-1} \left\{ \sum_{t=1,2,4,\lambda,g,2g,4g,\lambda g} |\Omega_{tp^i}| \right\} = 4p^n.$ □

Theorem 2.2 *If* $p \equiv 3(mod\ 4)$ *, there are* $(6n+3)$ q *-cyclotomic cosets modulo* $4p^n$ *given by:*

$$\Omega_0 = \{0\},\ \Omega_{p^n} = \{p^n, p^n q\},\ \Omega_{2p^n} = \{2p^n\}$$

and for $0 \leq i \leq n-1$ *,*

$$\Omega_{p^i} = \{sp^i, sp^i q, sp^i q^2, ..., sp^i q^{\phi(p^{n-i})-1}\} \text{ for } s = 1,2,4,g,2g,4g.$$

3. Primitive Idempotents

Throughout this paper, we consider that α is $4p^n th$ root of unity in some extension field of F . Let M_s be the minimal ideal in $R_{4p^n} = \frac{F[x]}{<x^{4p^n}-1>} \equiv FC_{4p^n}$, generated by $\frac{(x^{4p^n}-1)}{m_s(x)}$, where $m_s(x)$ is the minimal polynomial for $\alpha^s,\ s \in \Omega_s$. We denote $\theta_s(x)$, the primitive idempotent in R_{4p^n} , corresponding to the minimal ideal M_s , given by $\theta_s(x) = \frac{1}{4p^n} \sum_{t=0}^{4p^n-1} \varepsilon_i^s x^t$ where $\varepsilon_i^t = \sum_{s \in \Omega_t} \alpha^{-is}$ and $\overline{C}_t = \sum_{s \in \Omega_t} x^s$.

Lemma 3.1 *For any odd prime* p *and a positive integer* k *, if* β *is primitive* $p^k th$ *root of unity in some extension field of* F *, then the following holds:*
(i) *If* q *is quadratic residue modulo* p^k *, then*

$$\sum_{t=0}^{\frac{\phi(p^k)}{2}-1} (\beta^{q^t} + \beta^{gq^t}) = \begin{cases} -1, & k = 1 \\ 0, & k \geq 2 \end{cases}.$$

(ii) *If* q *is primitive root modulo* p^k *, then*

$$\sum_{t=0}^{\phi(p^k)-1} \beta^{q^t} = \begin{cases} -1, & k = 1 \\ 0, & k \geq 2 \end{cases}$$

Proof. By lemma 2.5 , the set $\{1, q, q^2, ..., q^{\frac{\phi(p^k)}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^k)}{2}-1}\}$ is a reduced residue system $(mod p^k)$. Then;
$$\sum_{t=0}^{\frac{\phi(p^k)}{2}-1}(\beta^{q^t}+\beta^{gq^t}) = \sum_{t=0}^{p^k-1}\beta^t - \sum_{t=1,p/t}^{p^k}\beta^t = -\sum_{t=1}^{p^{k-1}}\beta^{pt}$$
If $k=1$, then $-\beta^p=-1$.
If $k\geq 2$, then $\beta^p\neq 1$, therefore
$$\sum_{t=1}^{p^{k-1}}\beta^{pt} = \beta^p(1+\beta^p+...+\beta^{pk-1}) = \beta^p\frac{(\beta^{pk}-1)}{\beta^p-1} = 0.$$
For the remaining part see [3, lemma 4]. □

Lemma 3.2 *For cyclotomic cosets* $\Omega_{p^i}, 0\leq i\leq n-1,\ \lambda^2\Omega_{p^i}=\Omega p^i=\lambda\Omega_{\lambda p^i}$.

Proof. Since $\lambda^2\equiv 1(mod\ 4p^n)$, so the required result holds. □

Lemma 3.3 *(i) If* $p\equiv 1(mod\ 4)$ *, then* $\Omega_1=-\Omega_1$ *or* $\Omega_\lambda=-\Omega_1$ *according as* $\frac{\phi(p^n)}{4}$ *is odd or even.*
(ii) If $p\equiv 3(mod\ 4)$ *, then* $\Omega_g=-\Omega_1$.

Proof. Since p is an odd prime, so $\frac{\phi(p^n)}{2}$ is odd if and only if $p\equiv 3(mod\ 4)$.
(i) If $p\equiv 1(mod\ 4)$ and $q\equiv 3(mod\ 4)$
Clearly, $\lambda=1+2p^n\equiv -1(mod\ 4)$ and $q\equiv -1(mod\ 4)$
If $\frac{\phi(p^n)}{4}$ is odd, then $q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ 4)$. Also, $q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ p^n)$
Since, $(4,p^n)=1$. So, $q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ 4p^n)$. Hence, $\Omega_1=-\Omega_1$
If $\frac{\phi(p^n)}{4}$ is even, then $q^{\frac{\phi(p^n)}{4}}\equiv 1(mod\ 4)$ and $\lambda\equiv -1(mod\ 4)$. So, $\lambda q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ 4)$. Further, $q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ p^n)$. Thus, $\lambda q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ p^n)$.
Since, $(4,p^n)=1$. So, $\lambda q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ 4p^n)$. Hence, $\Omega_\lambda=-\Omega_1$
(ii) If $p\equiv 3(mod\ 4)$, then $g\equiv -1(mod\ 4)$ and for even k , $q^k\equiv 1(mod\ 4)$. So, $gq^k\equiv -1(mod\ 4)$.
Now, $q^{\frac{\phi(p^n)}{2}}\equiv 1(mod\ p^n)$. Here, $\frac{\phi(p^n)}{2}$ is odd. So, $q^k\not\equiv -1(mod\ p^n)$ for $0\leq k\leq\frac{\phi(p^n)}{2}-1$.
Also the set $\{1, q, q^2, ..., q^{\frac{\phi(p^n)}{2}-1}, g, gq, gq^2, ...,$
$gq^{\frac{\phi(p^n)}{2}-1}\}$, forms a reduced residue system modulo p^n , so $gq^k\equiv -1(mod\ p^n)$.
Since $(4,p^n)=1$, therefore
$gq^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ 4p^n)$. Hence $\Omega_g=-\Omega_1$ □

Notations: For $0\leq j\leq n-1$, define:
$A_j=p^j\sum_{s\varepsilon\Omega_{gp^j}}\alpha^s$, $B_j=p^j\sum_{s\varepsilon\Omega_{p^j}}\alpha^s$, $C_j=p^j\sum_{s\varepsilon\Omega_{2gp^j}}\alpha^s$, $D_j=p^j\sum_{s\varepsilon\Omega_{2p^j}}\alpha^s$, $E_j=p^j\sum_{s\varepsilon\Omega_{4gp^j}}\alpha^s$, $F_j=p^j\sum_{s\varepsilon\Omega_{4p^j}}\alpha^s$.
Here, $A_j^q=A_j$, so $A_j\in F$. Similarly, B_j , C_j , D_j , E_j and F_j all are in F .

Lemma 3.4 *For* $p\equiv 3(mod\ 4)$, A_j+B_j *=0 for all* j . *Proof.* *By definition,* $A_j+B_j = \sum_{t=0}^{\phi(p^n)-1}(\alpha^{gp^jq^t}+\alpha^{p^jq^t})$ = $\sum_{t=0}^{\phi(p^n)-1}(\beta^{gq^t}+\beta^{q^t})$*, where* $\beta=\alpha^{p^j}$.
Since, $p^jq^t\equiv p^jq^s(mod\ 4p^n)$ *if and only if* $q^t\equiv q^s(mod\ 4p^{n-j})$
if and only if $t\equiv s(mod\ \phi(p^{n-j}))$.
Thus, $\sum_{t=0}^{\phi(p^n)-1}(\beta^{gq^t}+\beta^{q^t})$ = $\frac{\phi(p^n)}{\phi(p^{n-j})}\sum_{t=0}^{\phi(p^{n-j})-1}(\beta^{gq^t}+\beta^{q^t})$ = $p^j\sum_{t=0}^{\phi(p^{n-j})-1}(\beta^{gq^t}+\beta^{q^t})$.
As the set $\{1, q, q^2, ..., q^{\phi(p^{n-j})-1}, g, gq, gq^2, ..., gq^{\phi(p^{n-j})-1}\}$ *forms a reduced residue system modulo* $4p^{n-j}$ *, therefore the above sum is:*

$$\sum_{t=0}^{\phi(p^n)-1}(\beta^{gq^t}+\beta^{q^t}) = p^j\Big[\sum_{t=1}^{4p^{n-j}}\beta^t-\sum_{t=1,p/t}^{4p^{n-j}}\beta^t-\sum_{t=1,2/t}^{4p^{n-j}}\beta^t+\sum_{t=1,2p/t}^{4p^{n-j}}\beta^t\Big].$$

Further, β *is* $4p^{n-j\ th}$ *root of unity, so* $\beta\neq 1, \beta^p\neq 1, \beta^2\neq 1, \beta^{2p}\neq 1$. *Thus,*

$$\sum_{t=1}^{4p^{n-j}}\beta^t=\sum_{t=1,p/t}^{4p^{n-j}}\beta^t=\sum_{t=1,2/t}^{4p^{n-j}}\beta^t=\sum_{t=1,2p/t}^{4p^{n-j}}\beta^t=0.$$

Hence; A_j+B_j *=0 for all* j. □

Lemma 3.5 (i) $C_j+D_j=\begin{cases} p^{n-1}, j=n-1\\ 0, otherwise.\end{cases}$

(ii) $E_j+F_j=\begin{cases} -p^{n-1}, j=n-1\\ 0, otherwise.\end{cases}$

Proof. Proof can be derived directly as of lemma 3.4 and using the facts that $\{1,q,q^2,...,q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2,...,gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ is a reduced residue system $modulo\ (2p^{n-j})$ and $modulo\ (p^{n-j})$ and using lemma 3.1 □

Lemma 3.6 *For* $0\leq i\leq n$*;* $0\leq j\leq n-1$

$$\sum_{s\epsilon\Omega_{p^j}}\alpha^{gp^is}=\sum_{s\epsilon\Omega_{\lambda p^j}}\alpha^{\lambda gp^is}=-\sum_{s\epsilon\Omega_{p^j}}\alpha^{\lambda gp^is}=\begin{cases} 0\ , & if\ i+j\geq n,\\ \frac{1}{p^j}A_{i+j}\ , & if\ i+j\leq n-1.\end{cases}$$

Proof. Here $\sum_{s\varepsilon\Omega_{\lambda p^j}}\alpha^{\lambda gp^is}=\sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1}\alpha^{(1+2p^n)^2gp^{i+j}q^t}=\sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1}\alpha^{gp^{i+j}q^t}=\sum_{s\varepsilon\Omega_{p^j}}\alpha^{gp^is}$

Let $\beta=\alpha^{p^{i+j}}$*. Then,* $\sum_{s\varepsilon\Omega_{p^j}}\alpha^{gp^is}=\sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1}\beta^{gq^t}$.

If $i+j\geq n$*, then* β *is* $4th$ *root of unity, then*

$$\sum_{s\varepsilon\Omega_{p^j}}\alpha^{gp^is}=\sum_{t=0}^{\phi(p^{n-j})-1}\beta^{gq^t}=0.$$

If $i+j\leq n-1$*, then* β *is* $4p^{n-i-j}th$ *root of unity, then*
$\beta^{gq^l}\equiv\beta^{gq^r}$ *if and only if* $gq^l\equiv gq^r(mod 4p^{n-i-j})$
if and only if $l\equiv r(mod\frac{\phi(p^{n-i-j})}{2})$.

Therefore $\sum_{s\varepsilon\Omega_{p^j}}\alpha^{gp^is}=\sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1}\beta^{gq^t}=\frac{p^{i+j}}{p^j}\sum_{t=0}^{\frac{\phi(p^{n-i-j})}{2}-1}\beta^{gq^t}=\frac{1}{p^j}A_{i+j}.$ □

Proof of following lemmas can be obtained on similar lines as of lemma 3.6 and using lemma 3.2.

Lemma 3.7 *For* $0\leq i\leq n$*,* $0\leq j\leq n-1$

$$\sum_{s\epsilon\Omega_{p^j}}\alpha^{p^is}=\sum_{s\epsilon\Omega_{gp^j}}\alpha^{gp^is}=\sum_{s\epsilon\Omega_{\lambda gp^j}}\alpha^{\lambda gp^is}=\sum_{s\epsilon\Omega_{\lambda p^j}}\alpha^{\lambda p^is}=-\sum_{s\epsilon\Omega_{p^j}}\alpha^{\lambda p^is}$$
$$=-\sum_{s\epsilon\Omega_{gp^j}}\alpha^{\lambda gp^is}=\begin{cases} 0\ , & if\ i+j\geq n,\\ \frac{1}{p^j}B_{i+j}\ , & if\ i+j\leq n-1.\end{cases}$$

Lemma 3.8 *For* $0\leq i\leq n$*,* $0\leq j\leq n-1$
If $p\equiv 1(mod\ 4)$ *then,*

$$(i)\ \sum_{s\epsilon\Omega_{p^j}}\alpha^{2gp^is}=\sum_{s\epsilon\Omega_{2p^j}}\alpha^{\lambda gp^is}=\begin{cases} -\frac{\phi(p^{n-j})}{2}, if i+j\geq n\\ \frac{1}{p^j}C_{i+j}, if i+j\leq n-1.\end{cases}$$

$(ii)\ \sum_{s\epsilon\Omega_{p^j}} \alpha^{2p^is} = \sum_{s\epsilon\Omega_{gp^j}} \alpha^{2gp^is} = \sum_{s\epsilon\Omega_{2p^j}} \alpha^{\lambda p^is} = \sum_{s\epsilon\Omega_{2gp^j}} \alpha^{\lambda gp^is} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}D_{i+j}, if i+j \leq n-1. \end{cases}$

$(iii)\ \sum_{s\epsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{2p^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{\lambda gp^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}E_{i+j}, if i+j \leq n-1. \end{cases}$

$(iv)\ \sum_{s\epsilon\Omega_{p^j}} \alpha^{4p^is} = \sum_{s\epsilon\Omega_{2p^j}} \alpha^{4p^is} = \sum_{s\epsilon\Omega_{2gp^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{4p^is} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{\lambda p^is} = \sum_{s\epsilon\Omega_{gp^j}} \alpha^{4gp^is}$

$= \sum_{s\epsilon\Omega_{4gp^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{4gp^j}} \alpha^{\lambda gp^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}F_{i+j}, if i+j \leq n-1. \end{cases}$

Lemma 3.9 *For* $0 \leq i \leq n$, $0 \leq j \leq n-1$
If $p \equiv 3(mod\ 4)$, *then*

$(i).\ \sum_{s\epsilon\Omega_{p^j}} \alpha^{2gp^is} = \begin{cases} -\phi(p^{n-j}), if i+j \geq n \\ \frac{1}{p^j}C_{i+j}, if i+j \leq n-1. \end{cases}$

$(ii)\ \sum_{s\epsilon\Omega_{2p^j}} \alpha^{gp^is} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}C_{i+j}, if i+j \leq n-1. \end{cases}$

$(iii)\ \sum_{s\epsilon\Omega_{p^j}} \alpha^{2p^is} = \sum_{s\epsilon\Omega_{gp^j}} \alpha^{2gp^is} = \begin{cases} -\phi(p^{n-j}), if i+j \geq n \\ \frac{1}{p^j}D_{i+j}, if i+j \leq n-1. \end{cases}$

$(iv)\ \sum_{s\epsilon\Omega_{2p^j}} \alpha^{p^is} = \sum_{s\epsilon\Omega_{2gp^j}} \alpha^{\lambda gp^is} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}D_{i+j}, if i+j \leq n-1. \end{cases}$

$(v)\ \sum_{s\epsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{gp^j}} \alpha^{4p^is} = \begin{cases} \phi(p^{n-j}), if i+j \geq n \\ \frac{1}{p^j}E_{i+j}, if i+j \leq n-1. \end{cases}$

$(vi)\ \sum_{s\epsilon\Omega_{2p^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{\lambda gp^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}E_{i+j}, if i+j \leq n-1. \end{cases}$

$(vii)\ \sum_{s\epsilon\Omega_{p^j}} \alpha^{4p^is} = \sum_{s\epsilon\Omega_{gp^j}} \alpha^{4gp^is} = \begin{cases} \phi(p^{n-j}), if i+j \geq n \\ \frac{1}{p^j}F_{i+j}, if i+j \leq n-1. \end{cases}$

$(viii)\ \sum_{s\epsilon\Omega_{2p^j}} \alpha^{4p^is} = \sum_{s\epsilon\Omega_{2gp^j}} \alpha^{4gp^is} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{4p^is} = \sum_{s\epsilon\Omega_{4p^j}} \alpha^{\lambda p^is} = \sum_{s\epsilon\Omega_{4gp^j}} \alpha^{4gp^is}$

$= \sum_{s\epsilon\Omega_{4gp^j}} \alpha^{\lambda gp^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}F_{i+j}, if i+j \leq n-1. \end{cases}$

Theorem 3.1 *For* $p \equiv 1(mod\ 4)$, *the explicit expression for the* $(8n+3)$ *primitive idempotents in* R_{4p^n} *are given by*

$$\theta_0(x) = \tfrac{1}{4p^n}[\overline{C}_0 + \overline{C}_{p^n} + \overline{C}_{2p^n} + \sum_{i=0}^{n-1}\{\overline{C}_{p^i} + \overline{C}_{2p^i} + \overline{C}_{4p^i} + \overline{C}_{\lambda p^i} + \overline{C}_{gp^i} + \overline{C}_{2gp^i} + \overline{C}_{4gp^i} + \overline{C}_{\lambda gp^i}\}]$$

$$\theta_{p^n}(x) = \tfrac{1}{4p^n}[2\overline{C}_0 - 2\overline{C}_{2p^n} - \sum_{i=n-j}^{n-1}\{2\overline{C}_{2p^i} - 2\overline{C}_{4p^i} + 2\overline{C}_{2gp^i} - 2\overline{C}_{4gp^i}\}]$$

$$\theta_{2p^n}(x) = \tfrac{1}{4p^n}[\overline{C}_0 - \overline{C}_{p^n} + \overline{C}_{2p^n} - \sum_{i=0}^{n-1}\{\overline{C}_{p^i} - \overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{\lambda p^i} + \overline{C}_{gp^i} - \overline{C}_{2gp^i} - \overline{C}_{4gp^i} + \overline{C}_{\lambda gp^i}\}]$$

$$\theta_{2p^j}(x) = \tfrac{1}{4p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0 - \overline{C}_{p^n} + \overline{C}_{2p^n}\} - \tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{p^i} - \overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{\lambda p^i} + \overline{C}_{gp^i} - \overline{C}_{2gp^i} - \overline{C}_{4gp^i} + \overline{C}_{\lambda gp^i}\} +$$
$$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{D_{i+j}\overline{C}_{p^i} + F_{i+j}\overline{C}_{2p^i} + F_{i+j}\overline{C}_{4p^i} + D_{i+j}\overline{C}_{\lambda p^i} + C_{i+j}\overline{C}_{gp^i} + E_{i+j}\overline{C}_{2gp^i} + E_{i+j}\overline{C}_{4gp^i} + C_{i+j}\overline{C}_{\lambda gp^i}\}]$$

$$\theta_{4p^j}(x)=\tfrac{1}{4p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0+\overline{C}_{p^n}+\overline{C}_{2p^n}\}+\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{p^i}+\overline{C}_{2p^i}+\overline{C}_{4p^i}+\overline{C}_{\lambda p^i}+\overline{C}_{gp^i}+\overline{C}_{2gp^i}+\overline{C}_{4gp^i}+\overline{C}_{\lambda gp^i}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{F_{i+j}\overline{C}_{p^i}+F_{i+j}\overline{C}_{2p^i}+F_{i+j}\overline{C}_{4p^i}+F_{i+j}\overline{C}_{\lambda p^i}+E_{i+j}\overline{C}_{gp^i}+E_{i+j}\overline{C}_{2gp^i}+E_{i+j}\overline{C}_{4gp^i}+E_{i+j}\overline{C}_{\lambda gp^i}\}]$$

$$\theta_{2gp^j}(x)=\tfrac{1}{4p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0-\overline{C}_{p^n}+\overline{C}_{2p^n}\}-\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{p^i}-\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{\lambda p^i}+\overline{C}_{gp^i}-\overline{C}_{2gp^i}-\overline{C}_{4gp^i}+\overline{C}_{\lambda gp^i}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{C_{i+j}\overline{C}_{p^i}+E_{i+j}\overline{C}_{2p^i}+E_{i+j}\overline{C}_{4p^i}+C_{i+j}\overline{C}_{\lambda p^i}+D_{i+j}\overline{C}_{gp^i}+F_{i+j}\overline{C}_{2gp^i}+F_{i+j}\overline{C}_{4gp^i}+D_{i+j}\overline{C}_{\lambda gp^i}\}]$$

$$\theta_{4gp^j}(x)=\tfrac{1}{4p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0+\overline{C}_{p^n}+\overline{C}_{2p^n}\}+\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{p^i}+\overline{C}_{2p^i}+\overline{C}_{4p^i}+\overline{C}_{\lambda p^i}+\overline{C}_{gp^i}+\overline{C}_{2gp^i}+\overline{C}_{4gp^i}+\overline{C}_{\lambda gp^i}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{E_{i+j}\overline{C}_{p^i}+E_{i+j}\overline{C}_{2p^i}+E_{i+j}\overline{C}_{4p^i}+E_{i+j}\overline{C}_{\lambda p^i}+F_{i+j}\overline{C}_{gp^i}+F_{i+j}\overline{C}_{2gp^i}+F_{i+j}\overline{C}_{4gp^i}+F_{i+j}\overline{C}_{\lambda gp^i}\}]$$

for $-\Omega_{p^j}=\Omega_{p^j}$

$$\theta_{p^j}(x)=\tfrac{1}{4p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0-\overline{C}_{2p^n}\}-\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{2gp^i}-\overline{C}_{4gp^i}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{B_{i+j}\overline{C}_{p^i}+D_{i+j}\overline{C}_{2p^i}+F_{i+j}\overline{C}_{4p^i}-B_{i+j}\overline{C}_{\lambda p^i}+A_{i+j}\overline{C}_{gp^i}+C_{i+j}\overline{C}_{2gp^i}+E_{i+j}\overline{C}_{4gp^i}-A_{i+j}\overline{C}_{\lambda gp^i}\}]$$

$$\theta_{\lambda p^j}(x)=\tfrac{1}{4p^n}[-\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0-\overline{C}_{2p^n}\}+\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{2gp^i}-\overline{C}_{4gp^i}\}-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{B_{i+j}\overline{C}_{p^i}+D_{i+j}\overline{C}_{2p^i}+F_{i+j}\overline{C}_{4p^i}-B_{i+j}\overline{C}_{\lambda p^i}+A_{i+j}\overline{C}_{gp^i}+C_{i+j}\overline{C}_{2gp^i}+E_{i+j}\overline{C}_{4gp^i}-A_{i+j}\overline{C}_{\lambda gp^i}\}]$$

$$\theta_{gp^j}(x)=\tfrac{1}{4p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0-\overline{C}_{2p^n}\}-\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{2gp^i}-\overline{C}_{4gp^i}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{A_{i+j}\overline{C}_{p^i}+C_{i+j}\overline{C}_{2p^i}+E_{i+j}\overline{C}_{4p^i}-A_{i+j}\overline{C}_{\lambda p^i}+B_{i+j}\overline{C}_{gp^i}+D_{i+j}\overline{C}_{2gp^i}+F_{i+j}\overline{C}_{4gp^i}-B_{i+j}\overline{C}_{\lambda gp^i}\}]$$

$$\theta_{\lambda gp^j}(x)=\tfrac{1}{4p^n}[-\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0-\overline{C}_{2p^n}\}+\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{2gp^i}-\overline{C}_{4gp^i}\}-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{A_{i+j}\overline{C}_{p^i}+C_{i+j}\overline{C}_{2p^i}+E_{i+j}\overline{C}_{4p^i}-A_{i+j}\overline{C}_{\lambda p^i}+B_{i+j}\overline{C}_{gp^i}+D_{i+j}\overline{C}_{2gp^i}+F_{i+j}\overline{C}_{4gp^i}-B_{i+j}\overline{C}_{\lambda gp^i}\}]$$

and for $-\Omega_{p^j}=\Omega_{\lambda p^j}$

$$\theta_{p^j}(x)=\tfrac{1}{4p^n}[-\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0-\overline{C}_{2p^n}\}+\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{2gp^i}-\overline{C}_{4gp^i}\}-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{B_{i+j}\overline{C}_{p^i}+D_{i+j}\overline{C}_{2p^i}+F_{i+j}\overline{C}_{4p^i}-B_{i+j}\overline{C}_{\lambda p^i}+A_{i+j}\overline{C}_{gp^i}+C_{i+j}\overline{C}_{2gp^i}+E_{i+j}\overline{C}_{4gp^i}-A_{i+j}\overline{C}_{\lambda gp^i}\}]$$

$$\theta_{\lambda p^j}(x)=\tfrac{1}{4p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0-\overline{C}_{2p^n}\}-\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{2gp^i}-\overline{C}_{4gp^i}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{B_{i+j}\overline{C}_{p^i}+D_{i+j}\overline{C}_{2p^i}+F_{i+j}\overline{C}_{4p^i}-B_{i+j}\overline{C}_{\lambda p^i}+A_{i+j}\overline{C}_{gp^i}+C_{i+j}\overline{C}_{2gp^i}+E_{i+j}\overline{C}_{4gp^i}-A_{i+j}\overline{C}_{\lambda gp^i}\}]$$

$$\theta_{gp^j}(x)=\tfrac{1}{4p^n}[-\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0-\overline{C}_{2p^n}\}+\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{2gp^i}-\overline{C}_{4gp^i}\}-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{A_{i+j}\overline{C}_{p^i}+C_{i+j}\overline{C}_{2p^i}+E_{i+j}\overline{C}_{4p^i}-A_{i+j}\overline{C}_{\lambda p^i}+B_{i+j}\overline{C}_{gp^i}+D_{i+j}\overline{C}_{2gp^i}+F_{i+j}\overline{C}_{4gp^i}-B_{i+j}\overline{C}_{\lambda gp^i}\}]$$

$$\theta_{\lambda gp^j}(x)=\tfrac{1}{4p^n}[\tfrac{\phi(p^{n-j})}{2}\{\overline{C}_0-\overline{C}_{2p^n}\}-\tfrac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{2gp^i}-\overline{C}_{4gp^i}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{A_{i+j}\overline{C}_{p^i}+C_{i+j}\overline{C}_{2p^i}+E_{i+j}\overline{C}_{4p^i}-A_{i+j}\overline{C}_{\lambda p^i}+B_{i+j}\overline{C}_{gp^i}+D_{i+j}\overline{C}_{2gp^i}+F_{i+j}\overline{C}_{4gp^i}-B_{i+j}\overline{C}_{\lambda gp^i}\}]$$

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 23 (2018) pp. 16708-16718
©Research India Publications. http://www.ripublication.com

where A_{i+j} , B_{i+j} , C_{i+j} , D_{i+j} , E_{i+j} *and* F_{i+j} *are given by:*

$C_{n-1} = \frac{1}{2}p^{n-1}(\sqrt{p}+1)$, $\quad D_{n-1} = \frac{1}{2}p^{n-1}(1-\sqrt{p})$.

$E_{n-1} = \frac{1}{2}p^{n-1}(\sqrt{p}-1)$, $\quad F_{n-1} = \frac{1}{2}p^{n-1}(-\sqrt{p}-1)$.

for $-\Omega_{p^j} = \Omega_{p^j}$.

$A_{n-1} = \sqrt{p^{2n-1}}$, $\quad B_{n-1} = 0$

for $-\Omega_{p^j} = \Omega_{\lambda p^j}$.

$A_{n-1} = [\sqrt{-3p^{2n-1}}]$, $\quad B_{n-1} = 0$

and for all $j \leq n-2$,

$A_j = B_j = C_j = D_j = E_j = F_j = 0$.

Proof. By definition,

$$\theta_s(x) = \frac{1}{4p^n}[\varepsilon_0^s\overline{C_0} + \varepsilon_{p^n}^s\overline{C_{p^n}} + \varepsilon_{2p^n}^s\overline{C_{2p^n}} + \sum_{i=0}^{n-1}\{\varepsilon_{p^i}^s\overline{C_{p^i}} + \varepsilon_{2p^i}^s\overline{C_{2p^i}} + \varepsilon_{4p^i}^s\overline{C_{4p^i}} + \varepsilon_{\lambda p^i}^s\overline{C_{\lambda p^i}} + \varepsilon_{gp^i}^s\overline{C_{gp^i}} + \varepsilon_{2gp^i}^s\overline{C_{2gp^i}} + \varepsilon_{4gp^i}^s\overline{C_{4gp^i}} + \varepsilon_{\lambda gp^i}^s\overline{C_{\lambda gp^i}}\}]$$

To evaluate $\theta_0(x)$, take $s=0$, then $\varepsilon_k^0 = \sum_{s\epsilon\Omega_0}\alpha^0 = 1$ for all $0 \leq k \leq 4p^n-1$. Therefore,

$$\theta_0(x) = \frac{1}{4p^n}[\overline{C}_0 + \overline{C}_{p^n} + \overline{C}_{2p^n} + \sum_{i=0}^{n-1}\{\overline{C}_{p^i} + \overline{C}_{2p^i} + \overline{C}_{4p^i} + \overline{C}_{\lambda p^i} + \overline{C}_{gp^i} + \overline{C}_{2gp^i} + \overline{C}_{4gp^i} + \overline{C}_{\lambda gp^i}\}]$$

For the evaluation of $\theta_{p^n}(x)$, take $s = p^n$. so we have to compute $\varepsilon_k^{p^n}$ for $k = 0, p^n, 2p^n, 3p^n, p^i, 2p^i, 4p^i,$ $\lambda p^i, gp^i, 2gp^i, 4gp^i, \lambda gp^i$.

Here, $\varepsilon_k^{p^n} = \sum_{s\epsilon\Omega_{p^n}}\alpha^{-ks} = \alpha^{-p^nk} = \alpha^{3p^nk}$. Therefore,

$\varepsilon_0^{p^n} = -\varepsilon_{2p^n}^{p^n} = -\varepsilon_{2p^i}^{p^n} = \varepsilon_{4p^i}^{p^n} = -\varepsilon_{2gp^i}^{p^n} = \varepsilon_{4gp^i}^{p^n} = 2$.

$\varepsilon_{p^n}^{p^n} = \varepsilon_{p^i}^{p^n} = -\varepsilon_{\lambda p^i}^{p^n} = \varepsilon_{gp^i}^{p^n} = \varepsilon_{\lambda gp^i}^{p^n} = 0.$

$$\theta_{p^n}(x) = \frac{1}{2p^n}[\overline{C}_0 - \overline{C}_{2p^n} - \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{2gp^i} - \overline{C}_{4gp^i}\}]$$

Similarly $\theta_{2p^n}(x)$ can be obtained.

Further to evaluate $\theta_{p^j}(x)$, take $s = p^j$ so we have to compute $\varepsilon_k^{p^j}$ for $k = 0, p^n, 2p^n, 3p^n, p^i, 2p^i, 4p^i, \lambda p^i,$ $gp^i, 2gp^i, 4gp^i, \lambda gp^i$.

$\varepsilon_k^{p^j} = \sum_{s\epsilon\Omega_{p^j}}\alpha^{-sk} = \sum_{s\epsilon\Omega_{p^j}}\alpha^{ks}$.

Therefore, using lemma $3.6-3.8$.

$\varepsilon_0^{p^j} = -\varepsilon_{2p^n}^{p^j} = \frac{\phi(p^{n-j})}{2}.$

$\varepsilon_{p^n}^{p^j} = 0.$

$$\sum_{s\epsilon\Omega_{gp^j}}\alpha^{p^is} = -\sum_{s\epsilon\Omega_{\lambda gp^j}}\alpha^{p^is} = \begin{cases} 0, & if\, i+j \geq n, \\ \frac{1}{p^j}A_{i+j}, & if\, i+j \leq n-1. \end{cases}$$

$$\sum_{s\epsilon\Omega_{p^j}}\alpha^{p^is} = -\sum_{s\epsilon\Omega_{\lambda p^j}}\alpha^{p^is} = \begin{cases} 0, & if\, i+j \geq n, \\ \frac{1}{p^j}B_{i+j}, & if\, i+j \leq n-1. \end{cases}$$

$$\sum_{s\epsilon\Omega_{2gp^j}}\alpha^{p^is} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}C_{i+j}, if i+j \leq n-1. \end{cases}$$

$$\sum_{s\epsilon\Omega_{2p^j}} \alpha^{p^is} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}D_{i+j}, if i+j \leq n-1. \end{cases}$$

$$\sum_{s\epsilon\Omega_{4gp^j}} \alpha^{p^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}E_{i+j}, if i+j \leq n-1. \end{cases}$$

$$\sum_{s\epsilon\Omega_{4p^j}} \alpha^{p^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}, if i+j \geq n \\ \frac{1}{p^j}F_{i+j}, if i+j \leq n-1. \end{cases}$$

So, $\theta_{p^j}(x) = \frac{1}{4p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{C}_0 - \overline{C}_{2p^n}\} - \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{2gp^i} - \overline{C}_{4gp^i}\} +$

$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{C_{i+j}\overline{C}_{p^i} + F_{i+j}\overline{C}_{2p^i} + H_{i+j}\overline{C}_{4p^i} - C_{i+j}\overline{C}_{\lambda p^i} + A_{i+j}\overline{C}_{gp^i} + E_{i+j}\overline{C}_{2gp^i} + G_{i+j}\overline{C}_{4gp^i} - A_{i+j}\overline{C}_{\lambda gp^i}\}]$

Similarly using lemma $3.6 - 3.8$, we can evaluate $\theta_{2p^j}(x)$, $\theta_{4p^j}(x)$, $\theta_{\lambda p^j}(x)$, $\theta_{gp^j}(x)$, $\theta_{2gp^j}(x)$, $\theta_{4gp^j}(x)$ and $\theta_{\lambda gp^j}(x)$. Further, the relations for A_j , B_j , C_j , D_j , E_j and F_j can be obtained by using the relation $\theta_{p^j}(\alpha^{p^j}) = 1$, $\theta_{p^j}(\alpha^{gp^j}) = 0$, $\theta_{2p^j}(\alpha^{2p^j}) = 1$ and $\theta_{4p^j}(\alpha^{4p^j}) = 1$ and lemma $3.4 - 3.9$.
Similarly we can find $\Omega_{p^j}(x)$ when $\Omega_{p^j} = -\Omega_{\lambda p^j}$. □

Expressions in the following theorem $3.11 - 3.12$ can be obtained on similar lines as in theorem 3.10 and using lemma $3.4 - 3.9$.

Theorem 3.2 *For $p \equiv 3(mod\, 4)$ the explicit expression for the $(6n+3)$ primitive idempotents in R_{4p^n} are given by*

$\theta_0(x) = \frac{1}{4p^n}[\overline{C}_0 + \overline{C}_{p^n} + \overline{C}_{2p^n} + \sum_{i=0}^{n-1}\{\overline{C}_{p^i} + \overline{C}_{2p^i} + \overline{C}_{4p^i} + \overline{C}_{gp^i} + \overline{C}_{2gp^i} + \overline{C}_{4gp^i}\}]$

$\theta_{p^n}(x) = \frac{1}{4p^n}[2\overline{C}_0 - 2\overline{C}_{2p^n} - \sum_{i=0}^{n-1}\{2\overline{C}_{2p^i} - 2\overline{C}_{4p^i} + 2\overline{C}_{2gp^i} - 2\overline{C}_{4gp^i}\}]$

$\theta_{2p^n}(x) = \frac{1}{4p^n}[\overline{C}_0 - \overline{C}_{p^n} + \overline{C}_{2p^n} - \sum_{i=0}^{n-1}\{\overline{C}_{p^i} - \overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{gp^i} - \overline{C}_{2gp^i} - \overline{C}_{4gp^i}\}]$

$\theta_{p^j}(x) = \frac{1}{4p^n}[\phi(p^{n-j})\{\overline{C}_0 - \overline{C}_{2p^n}\} - \phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{2gp^i} - \overline{C}_{4gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{A_{i+j}\overline{C}_{p^i} + 2C_{i+j}\overline{C}_{2p^i} + 2E_{i+j}\overline{C}_{4p^i} + B_{i+j}\overline{C}_{gp^i} + 2D_{i+j}\overline{C}_{2gp^i} + 2F_{i+j}\overline{C}_{4gp^i}\}]$

$\theta_{2p^j}(x) = \frac{1}{4p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{C}_0 - \overline{C}_{p^n} + \overline{C}_{2p^n}\} - \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{p^i} - \overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{gp^i} - \overline{C}_{2gp^i} - \overline{C}_{4gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{C_{i+j}\overline{C}_{p^i} + E_{i+j}\overline{C}_{2p^i} + E_{i+j}\overline{C}_{4p^i} + D_{i+j}\overline{C}_{gp^i} + F_{i+j}\overline{C}_{2gp^i} + F_{i+j}\overline{C}_{4gp^i}\}]$

$\theta_{4p^j}(x) = \frac{1}{4p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{C}_0 + \overline{C}_{p^n} + \overline{C}_{2p^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{p^i} + \overline{C}_{2p^i} + \overline{C}_{4p^i} + \overline{C}_{gp^i} + \overline{C}_{2gp^i} + \overline{C}_{4gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{E_{i+j}\overline{C}_{p^i} + E_{i+j}\overline{C}_{2p^i} + E_{i+j}\overline{C}_{4p^i} + F_{i+j}\overline{C}_{gp^i} + F_{i+j}\overline{C}_{2gp^i} + F_{i+j}\overline{C}_{4gp^i}\}]$

$\theta_{gp^j}(x) = \frac{1}{4p^n}[\phi(p^{n-j})\{\overline{C}_0 - \overline{C}_{2p^n}\} - \phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{2gp^i} - \overline{C}_{4gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{B_{i+j}\overline{C}_{p^i} + 2D_{i+j}\overline{C}_{2p^i} + 2F_{i+j}\overline{C}_{4p^i} + A_{i+j}\overline{C}_{gp^i} + 2C_{i+j}\overline{C}_{2gp^i} + 2E_{i+j}\overline{C}_{4gp^i}\}]$

$\theta_{2gp^j}(x) = \frac{1}{4p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{C}_0 - \overline{C}_{p^n} + \overline{C}_{2p^n}\} - \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{p^i} - \overline{C}_{2p^i} - \overline{C}_{4p^i} + \overline{C}_{gp^i} - \overline{C}_{2gp^i} - \overline{C}_{4gp^i}\} +$

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 23 (2018) pp. 16708-16718
©Research India Publications. http://www.ripublication.com

$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{D_{i+j}\overline{C}_{p^i}+F_{i+j}\overline{C}_{2p^i}+F_{i+j}\overline{C}_{4p^i}+C_{i+j}\overline{C}_{gp^i}+E_{i+j}\overline{C}_{2gp^i}+E_{i+j}\overline{C}_{4gp^i}\}]$

$\theta_{4gp^j}(x) = \frac{1}{4p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{C}_0+\overline{C}_{p^n}+\overline{C}_{2p^n}\}+\frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{C}_{p^i}+\overline{C}_{2p^i}+\overline{C}_{4p^i}+\overline{C}_{gp^i}+\overline{C}_{2gp^i}+\overline{C}_{4gp^i}\}+$
$\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{F_{i+j}\overline{C}_{p^i}+F_{i+j}\overline{C}_{2p^i}+F_{i+j}\overline{C}_{4p^i}+E_{i+j}\overline{C}_{gp^i}+E_{i+j}\overline{C}_{2gp^i}+E_{i+j}\overline{C}_{4gp^i}\}]$

$A_{n-1}=0,\ B_{n-1}=0,\ C_{n-1}=\frac{p^{n-1}}{2}(1+\sqrt{-p}),\ D_{n-1}=\frac{p^{n-1}}{2}(1-\sqrt{-p}),$
$E_{n-1}=\frac{p^{n-1}}{2}(\sqrt{-p}-1),\ F_{n-1}=-\frac{p^{n-1}}{2}(\sqrt{-p}+1).$
and for all $j\leq n-2,\ A_j=B_j=C_j=D_j=E_j=F_j=0.$

4. Dimension and Generating Polynomials

If α is primitive $4p^n th$ root of unity in some extension field of F, then $m_s(x)=\prod_{s\epsilon\Omega_s}(x-\alpha^s)$ denote the minimal polynomial for α^s.
If $m_s(x)$ denote the minimal polynomial for $\alpha^s, s\epsilon\Omega_s$, then the generating polynomial for cyclic code M_s of length $4p^n$ corresponding to the cyclotomic coset Ω_s is $\frac{x^{4p^n}-1}{m_s(x)}$ and the dimension of minimal cyclic code M_s is equal to the cardinality of the class Ω_s [11].
Thus the dimensions of the codes $M_0, M_{p^n}, M_{2p^n}, M_{p^i}, M_{2p^i}, M_{4p^i}, M_{\lambda p^i}$, $M_{gp^i}, M_{2gp^i}, M_{4gp^i}$ and $M_{\lambda gp^i}$ are $1,2,1,\frac{\phi(p^n)}{2},\frac{\phi(p^n)}{2},\frac{\phi(p^n)}{2},\frac{\phi(p^n)}{2}$, $\frac{\phi(p^n)}{2},\frac{\phi(p^n)}{2},\frac{\phi(p^n)}{2}$ and $\frac{\phi(p^n)}{2}$ respectively.

Theorem 4.1 (i) *The generating polynomial for the codes* M_0, M_{p^n} *and* M_{2p^n} *are* $(1+x+x^2+...+x^{4p^n-1})$, $(x^2-1)(1+x^4+...+x^{4(p^n-1)})$ *and* $(x^2+1)(x-1)(1+x^4+...+x^{4(p^n-1)})$ *respectively.*
(ii) *The generating polynomial for* $M_{2p^i}\oplus M_{2gp^i}$ *and* $M_{4p^i}\oplus M_{4gp^i}$ *are* $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(1+x^{4p^{n-i}}+...x^{4p^{n-i(p^i-1)}})$ *and* $(x^{p^{n-i-1}}-1)(x^{p^{n-i}}+1)(x^{2p^{n-i}}+1)(1+x^{4p^{n-i}}+...x^{4p^{n-i(p^i-1)}})$ *respectively.*
(iii) *For* $p\equiv 1(mod\, 4)$, *the generating polynomial for* $M_{p^i}\oplus M_{\lambda p^i}\oplus M_{gp^i}\oplus M_{\lambda gp^i}$ *is* $(x^{2p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{4p^{n-i}}+...x^{4p^{n-i(p^i-1)}})$.
(iv) *For* $p\equiv 3(mod\, 4)$, *the generating polynomial for* $M_{p^i}\oplus M_{gp^i}$ *is* $(x^{2p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{4p^{n-i}}+...x^{4p^{n-i(p^i-1)}})$.

Proof. (i) The minimal polynomial for α^0, α^{p^n} and α^{2p^n} are $(x-1)$, $(x^2-\mu^2)$, $(x+1)$ and $(x+\mu)$ respectively. The corresponding generating polynomials are $(1+x+x^2+...+x^{4p^n-1})$, $(x^2-1)(1+x^4+...+x^{4(p^n-1)})$ and $(x^2+1)(x-1)(1+x^4+...+x^{4(p^n-1)})$.
(ii) The product of minimal polynomial satisfied by α^{2p^i} and α^{2gp^i} is $\frac{x^{p^{n-i}}+1}{x^{p^{n-i-1}}+1}$. Therefore, the generating polynomial for $M_{2p^i}\oplus M_{2gp^i}$ is $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(1+x^{4p^{n-i}}+...x^{4p^{n-i(p^i-1)}})$. The product of minimal polynomial satisfied by α^{4p^i} and α^{4gp^i} is $\frac{x^{p^{n-i}}-1}{x^{p^{n-i-1}}-1}$. Therefore, the generating polynomial for $M_{4p^i}\oplus M_{4gp^i}$ is $(x^{p^{n-i-1}}-1)(x^{p^{n-i}}+1)(x^{2p^{n-i}}+1)(1+x^{4p^{n-i}}+...x^{4p^{n-i(p^i-1)}})$. Also the product of minimal polynomial satisfied by α^{p^i}, α^{gp^i}, $\alpha^{\lambda p^i}$ and $\alpha^{\lambda gp^i}$ is $\frac{x^{2p^{n-i}}+1}{x^{2p^{n-i-1}}+1}$. Therefore, the generating polynomial for $M_{p^i}\oplus M_{\lambda p^i}\oplus M_{gp^i}\oplus M_{\lambda gp^i}$ is $(x^{2p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{4p^{n-i}}+...x^{4p^{n-i(p^i-1)}})$. □

5. Minimum Distance Bounds

Lemma 5.1 *If* l *is a cyclic code of length* m *generated by* $g(x)$ *and its minimum distance is* d, *then the code* $\bar{l}$ *of length* mk *generated by* $g(x)(1+x^m+x^{2m}+...+x^{(k-1)m})$ *is a repetition code of* l *repeated* k *times and its minimum distance is* dk. [2]

Here, we find the minimum distance of the minimal cyclic code M_s of length $4p^n$, generated by the primitive idempotent θ_s.

Theorem 5.1 *The minimum distance of the codes* M_0, M_{p^n} *and* M_{2p^n} *are* $4p^n, 2p^n, 4p^n$ *respectively. For* $0 \le i \le n-1$*, the minimum distance of the cyclic codes* $M_{2p^i}, M_{2gp^i}, M_{4p^i}$ *and* M_{4gp^i} *are greater than equal to* $8p^i$ *and for the codes* $M_{p^i}, M_{gp^i}, M_{\lambda p^i}$ *and* $M_{\lambda gp^i}$ *are greater than equal to* $4p^i$.

Proof. Since generating polynomial for the code M_0 is $(1 + x + x^2 + ... + x^{4p^n-1})$, which is itself a polynomial of length $4p^n$, hence its minimum distance is $4p^n$.
The minimum distance of the cyclic code M_{p^n} with generating polynomial $(x^2 - 1)(1 + x^4 + ... + x^{4(p^n-1)})$ is $2p^n$ as it is repetition code of length 4 with generating polynomial $(x^2 - 1)$, whose minimum distance is 2 repeated p^n times.
The minimum distance of the cyclic code M_{2p^n} with generating polynomial $(x^3 - x^2 + x - 1)(1 + x^4 + ... + x^{4(p^n-1)})$ is $4p^n$ as it is repetition of the cyclic code of length 4 with generating polynomial $(x^3 - x^2 + x - 1)$ whose minimum distance is 4, repeated p^n times.
Since the product of generating polynomial for the cyclic codes M_{2p^i} and M_{2gp^i} is $(x^{p^{n-i-1}} + 1)$, then the minimum distance of this code say C is 2. Now consider the cyclic code C_1 of length $2p^{n-i}$ generated by the polynomial $(x^{p^{n-i-1}} + 1)(x^{p^{n-i}} - 1)$,and then minimum distance of this code is 4, as it is 2 time repetition of the code C. Further, the minimum distance of the code C_2 of length $4p^{n-i}$ generated by the polynomial $(x^{p^{n-i-1}} + 1)(x^{p^{n-i}} - 1)(x^{2p^{n-i}} + 1)$ is 8, as it is again 2 time repetition of the code C_1. Since the cyclic code of length $4p^n$ generated by the polynomial $(x^{p^{n-i-1}} + 1)(x^{p^{n-i}} - 1)(x^{2p^{n-i}} + 1)(1 + x^{4p^{n-i}} + ...x^{4p^{n-i}(p^i-1)})$ is a repetition code of the code C_2, repeated p^i times. Hence its minimum distance is $8p^i$, by lemma 5.1. The codes corresponding to Ω_{2p^i} and Ω_{2gp^i} are the sub codes of above code so, by [3, 5.4] their minimum distance are greater than or equal to $8p^i$.
The product of generating polynomial for the cyclic codes M_{4p^i} and $4gp^i$ is $(x^{p^{n-i-1}} - 1)(x^{p^{n-i}} + 1)(x^{2p^{n-i}} + 1)(1 + x^{4p^{n-i}} + ...x^{4p^{n-i}(p^i-1)})$. Hence, similarly as above minimum distance of cyclic codes M_{4p^i} and $4gp^i$ are greater than or equal to $8p^i$.
Now the product of generating polynomial for the cyclic codes $M_{p^i}, M_{gp^i}, M_{\lambda p^i}$ and $M_{\lambda gp^i}$ is $(x^{2p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{4p^{n-i}}+...x^{4p^{n-i}(p^i-1)})$, therefore, if we take a code C of length $4p^{n-i}$ generated by the polynomial $(x^{2p^{n-i-1}}+1)(x^{2p^{n-i}}-1)$, then the minimum distance of this code C_1 of length $4p^n$ generated by the polynomial $(x^{2p^{n-i-1}} + 1)(x^{2p^{n-i}} - 1)(1 + x^{4p^{n-i}} + ...x^{4p^{n-i}(p^i-1)})$ is a repetition code of the code C, repeated p^i times. Hence its minimum distance is $4p^i$. The codes corresponding to Ω_{p^i}, Ω_{gp^i}, $\Omega_{\lambda p^i}$ and $\Omega_{\lambda gp^i}$ are the sub codes of above codes so, their minimum distances are greater than or equal to $4p^i$. □

References

[1] Arora S.K., Batra S., Cohen S.D., Pruthi M., *The Primitive Idempotents of a Cyclic Group Algebra*, Southest Asian Bulletin of Mathematics, **26** (2002), $549 - 557$.

[2] Arora S.K., Pruthi M., *Minimal cyclic codes of length* $2p^n$, Finite Field Appl., **5**, (1999), $177 - 187$.

[3] Bakshi G.K. ,Raka M., *Minimal Cyclic Codes of Length* $p^n q$, Finite Fields Appl.,**9**, (2003), 432-448.

[4] Batra S. ,Arora S.K., *Minimal Quadratic Cyclic Codes of Length* p^n *(* p *odd prime)*, Korean J. Comput. & Appli. Math., **8** (2001), 531-547.

[5] Batra S. , Arora S.K., *Some Cyclic Codes of Length* $2p^n$, Des. Codes Cryptogr.,**61**, (2011), 41-69.

[6] Burton D.M., *Elementary Number Theory*, Wm. C. Brown Publisher, Dubuque(Reprinted, New Delhi: University Book Stall, 1990.

[7] Li F., Yue Q. ,Li C., *Irreducible Cyclic codes of length* $4p^n$ *and* $8p^n$, Finite Fields Appli., **34**, (2015), 208-234.

[8] Pruthi M. ,Arora S.K., *Minimal codes of prime power length*, Finite Fields Appli., **3**, (1997), 99-113.

[9] Sahni A. , Sehgal P.T., *Minimal Cyclic Codes of Length* $p^n q$, Finite Fields Appli., **18**, (2012), 1017-1036.

[10] Singh J. , Arora S.K., *Minimal Cyclic Codes of Length* $8p^n$ *over* $GF(q)$, J.Appl. Math.Comput., **48**, (2015), 55-69.

[11] Pless V., *Introduction of the theory of Error Correcting Codes*,Wiley, New York 1981.