International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 23 (2018) pp. 16418-16423


# On solutions of the II-D Navier-Stokes equations using Puknachev's subalgebra $< G_1, G_2 >$

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**Abstract**

The Lie symmetry analysis of the two-dimensional Navier-Stokes equations leads to a number of symmetries, from which follow an algebra and subalgebras, each associated with some solutions.

In this contribution, we determine solutions associated with the subalgebra $< G_1, G_2 >$, as outlined in the works of Pukhnachev, and do so through a new technique that we base on differentiable topological manifolds.

**Keywords:** Navier Stokes equations; Symmetry analysis; Differentiable manifolds.

## 1. INTRODUCTION

The solutions we determine are the ones that follow from one of the subalgebras Pukhnachev established for the two-dimensional Navier-Stokes equations

$$\rho\left( u_t + u\, u_x + v\, u_y\right) = \mu\nabla^2 u - p_x \qquad (1)$$

$$\rho\left( v_t + u\, v_x + v\, v_y\right) = \mu\nabla^2 v - p_y, \qquad (2)$$

$$u_x + v_y = 0. \qquad (3)$$

The unknowns $\mathbf{u} = (u,v)$ and $p$ are the spatial velocity vector and pressure. For a Newtonian fluid, the viscosity coefficient $\mu$ is a constant, while $\rho$, also a constant, is the mass density.

Pukhnachev's objective was to solve the equations through Lie symmetry group theoretical methods. Lie symmetry group theoretical analysis, or symmetry analysis, or simply symmetry, is a method introduced by Marius Sophus Lie (1842 – 1899), a Norwegian mathematician, through his now famous 1881 paper [1]. The theory has now been developed to new levels, as can be attested in [2], [3], [4], [5], [6] and [7].

It is a beautiful theory, but a closer look reveals that, in some cases, it does not really deliver in its pure form, and this is clearly evident in Pukhnachev's results. What he succeeded in doing was to transform the differential equations into integral equations. That is, he transformed a problem from one form into another. This is briefly discussed in Section 2.

Section 3 is on the theoretical basis of our differentiable topological manifolds approach. The solutions are identified as charts from one quotient space to another. Section 4 is on the solutions of the two-dimensional Navier-Stokes equations.

## 2. THE LIE SYMMETRY RESULTS

Sophus Lie's theory of symmetry groups applicable to differential equations, when applied to the equations (1), (2) and (3), involves an infinitesimal change of variables from

$(t,x,y,u,v,p)$ to $(\tilde{t},\tilde{x},\tilde{y},\tilde{u},\tilde{v},\tilde{p})$ so that

$$\tilde{t}^i = t^i + \epsilon\, \psi^i(t,x,y,u,v,p) + \mathbb{O}(\epsilon^2), \cdots i = 0,1,2,\cdots 5, \qquad (4)$$

where $t^0 = t,\ t^1 = x,\ t^2 = y,\ t^3 = u,\ t^4 = v,\ t^5 = p$, together with the symmetry generator $G$ given as

$$G = \phi^i(t,x,y,u,v,p)\,\frac{\partial}{\partial t^i}, \cdots i = 0,1,2,\cdots 5,$$

with

$$\phi^0 = \xi^0, \quad \phi^1 = \xi^1, \phi^2 = \xi^2,\ \phi^3 = \eta^1. \qquad (5)$$

That is,

$$G = \xi^0\frac{\partial}{\partial t} + \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v} + \eta^3\frac{\partial}{\partial p}. \qquad (6)$$

### 2.1 The symmetries

In 1960, Puhnachev [8] investigated the equations and found eight symmetries:

$$G\_1 \& = \& \ 2\,t\,\partial/\partial t + x\ \partial/\partial x + \ y\ \partial/\partial y$$
$$- u\,\frac{\partial}{\partial u} - v\,\frac{\partial}{\partial v} - 2\,p\,\frac{\partial}{\partial p}, \tag{7}$$

$$G_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \tag{8}$$

$$G_3 = \frac{\partial}{\partial t}, \tag{9}$$

$$G_4 = \psi_1\frac{\partial}{\partial x} + \psi_1'\frac{\partial}{\partial u} - x\ \psi_1''\frac{\partial}{\partial p}, \tag{9}$$

$$G_5 = \frac{\partial}{\partial x}, \tag{10}$$

$$G_6 = \psi_2\ \frac{\partial}{\partial y} + \psi_2'\frac{\partial}{\partial v} - y\ \psi_2''\ \frac{\partial}{\partial p}, \tag{11}$$

$$G_7 = \frac{\partial}{\partial y}, \tag{12}$$

$$G_8 = \phi(t)\frac{\partial}{\partial p}. \tag{13}$$

### 2.2 The subalgebra $< \mathbf{G_1}, \mathbf{G_2} >$

For the subalgebra <G_1,G_2>, Pukhnachev established the solutions

$$u^r = \frac{C_1}{r}, \tag{14}$$

$$u^\varphi = -\frac{C_2}{r}\int^\xi e^{\left\{-\frac{z^2}{4}\right\}}z^{\{C_1+1\}}d_z + C_3, \tag{15}$$

$$p = \frac{1}{t}\int^\xi \frac{\{J_3^2(z)d\,z\}}{\{z^3\}}d\,z - \frac{C_1^2}{2\,r^2} + \frac{C_4}{t}, \tag{16}$$

$$\xi = \frac{r}{\sqrt{t}}, \tag{17}$$

where $C_1, C_2, C_3$ and $C_4$ are arbitrary constants, and

$$r = \sqrt{x^2+y^2},$$
$$\varphi = arctan\left(\frac{y}{x}\right),$$
$$r = u\cos\varphi + v\sin\varphi,$$
$$u^\varphi = v\cos\varphi - u\sin\varphi.$$

As evident from (16) and (17), there is still a need to resolve the integrals, so the process of solving the equations was not really completed. We start where Pukhnachev paused and finish, and do so through differentiable topological manifolds.

## 3. THE DIFFERENTIABLE TOPOLOGICAL MANIFOLDS BASIS

Our approach to the solution of the Navier-Stokes equations is borrowed from the method variation of parameters, a procedure popularly used to solve second-order non-homogeneous linear ordinary differential equations

$$a\ \frac{d^2y}{dx^2} + b\ \frac{dy}{dx} + c\,y = f(x), \tag{18}$$

with constant coefficients $a, b$ and $c$.

### 3.1 The variation of parameters method

The usual steps involved in solving (19) requires first setting

$$f(x) = 0, \tag{20}$$

so that

$$a\frac{d^2y}{d\,x^2} + b\ \frac{d\,y}{d\,x} + c\,y = 0, \tag{21}$$

the homogeneous case, from which the result

$$y_C = C_1y_1 + C_2y_2 \tag{19}$$

is obtained, known as the complementary solution. The constants $C_1$ and $C_2$, are the parameters that have to be varied. That is, at some stage we are going to have to set

$$v_i = C_i, \sim\sim i = 1,2. \tag{20}$$

Odd as it seems, but this is how the method of variation of parameters proceeds. These lead to the particular solution

$$y_P = v_1y_1 + v_2y_2, \tag{21}$$

so that

$$y = y_C + y_P, \tag{22}$$

the general solution.

We take these two assumptions to the next subsection and beyond. The assumption in (21) will be interpreted as points in quotient spaces, leading to (58). The second assumption, the one in (23), relates this space to the entire differentiable topological manifold. It leads to (31) and (32) which generate (59).

### 3.2 Differentiable Topological Manifolds

We start with a topological space $M = (X, J_X)$, a Hausdorff topology. That is, a set $X$ with a topology $J_X$. For it to be a differentiable topological manifold, or simply a differentiable manifold ve $DM$, we require an atlas A, so that $DM = (X, J_X, A)$.

We now consider two points $p \in U_p$ and $q \in U_q$, such that the sets $U_p$ and $U_q$ are elements of the same manifold. We can then build the sub-topologies $(U_p, J_X|\, U_p)$ and $(U_q, J_X|\, U_q)$. That is, the topology of $X$ restricted to $U_p$ and $U_q$. A mapping $\psi_p$, if it exists, then maps the space $(U_p, J_X|\, U_p)$ into the Euclidean space $(R^N, J_{R^N}$ |_( ψ_p (U_p))).

Similarly, $\psi_q$ maps $(U_q, J_X|\, U_q)$ into the Euclidean space $(R^N, J_{R^N}$ |_( ψ_q (U_q))) . If these mappings are homeomorphisms, then the set A, with

$$A = \{ (\{U_p\}, \{\psi_p\}), (U_q, \psi_q)\} \qquad (26)$$

is called an atlas, with $\psi_p, \psi_q$ called coordinates.

Our interest is in one of the charts mapping equivalence classes. That is,

$$A = \{ ([U_p], [\psi_p]), ( U_q, \psi_q)\}. \qquad (27)$$

Similarly, for manifolds in derivatives of $\psi$, we get the atlases

$$A^i = \{\left([U_p], \left[\psi_p^{\{(i)\}}\right]\right), \left(U_q, \psi_q^{\{(i)\}}\right)\}. \qquad (28)$$

#### 3.2.1 Transmission mappings

The mapping from $(R,\ J_R|\psi(U_p))$ to $\left(R, J_R\middle|\psi(Uq)\right)$, having stepped down from $R$^$N$ to $R$, is given by

$$\psi_p\left( \psi_q^{\{-1\}}( \psi_q([U_p]) ) \right), \qquad (29)$$

and it is called a transition mapping. Its inverse is

$$\psi_q\left( \psi_p^{\{-1\}}\left( \psi_p(U_q)\right) \right). \qquad (30)$$

We are interested in case where $[U_p]$ and $U_q$ overlap, so that there is a point $x$ in the neighbourhood of both $p$ and $q$

such that

$$[\psi[x]] = \psi(x). \qquad (31)$$

The transmission mappings in derivative spaces lead to

$$\frac{d^n[\psi[x]]}{d\,[x]^n} = \frac{d^n\psi(x)}{d\,x^n}, \qquad (32)$$

for $n = 1,2,3,\cdots$.

#### 3.2.2 Tangent Spaces

As indicated earlier, tangent spaces assist in establishing a function $f$, that allows for results to be projected onto the metric space.

A tangent space is a set

$$TP = \{ V\_\gamma, P \mid \gamma: \backslash R \to X \}, \qquad (33)$$

Such that

$$V_{\{\gamma,P\}f} = \left( f \circ \gamma^{\{-1\}}\right)[\gamma(\tau_0)], \qquad (34)$$

where $f \in C^\infty(X), V_{\{\gamma,P\}}: C^\infty(M) \to R,\ \gamma(\tau_0) = P$.

The tangent space $TP$ has the basis vectors $\{\partial Xi\}$. Any vector then can be represented in terms of it, so that

$$X = \xi\text{^}i\ \partial/\partial X\text{^}i\ \ |\_P. \qquad (35)$$

That is, $X \in TPX = TPM$.

#### 3.2.3 Cotangent Spaces

A tangent space is a vector space, and where there is one there should also be a co-vector space, hence the cotangent space. It is the set of all maps in the tangent space to R. That is,

$$\omega: T_pX \to R, \qquad (36)$$

with $\omega$ being an element of the cotangent space. The symbol $(df)_p$ represents a co-vector acting on mapping $f$ at $P$. A cotangent space, therefore, is

$$TP^* = \left\{ (d\,f)_p \middle| f \in C^\infty(X) \right\} \qquad (37)$$

and it is a vectors space, and is the dual of $TP$.

The basis vectors of a cotangent space requires that

$$(d\,\omega^j)_p|\ \frac{\partial}{\partial x^i}\ |\_p = (\delta\text{^}j)\_i, \qquad (38)$$

so that

$$Basis(TP^*) = \{ \partial/\partial x\text{^}i \ \}\, |\_p. \qquad (39)$$

Therefore an element $\omega$ of $TP*$ can be written

$$\omega = \omega\_i\ (\ d\ x\text{^}i)\ |\_p. \qquad (40)$$

### 3.3 Quotient Spaces

Consider the general ordinary differential equation

$$f\left(x, \psi, \psi', \psi'', \psi^{\{(3)\}}, \cdots\right) \qquad (41)$$

with

$$\psi: X \to Y. \tag{42}$$

A set

$$S = \{ x_0, x_1, x_2, \cdots \} \subset X, \tag{43}$$

such that

$$x_i = P(x_j) = x_j + 2\pi k_s, \tag{44}$$

where $k_s$ is an integer, is called an equivalence class. This leads to an Quotient space $R/\sim$. It is the set of all equivalent classes in $R$, and is given by

$$R/\sim = \{ [x_0], [x_1], [x_2], \cdots \}. \tag{45}$$

It is a differentiable topological space. In our study, the image of this set, is also an equivalence class

$$\{ [\psi(x_0)], [\psi(x_1)], [\psi(x_2)], \cdots \}, \tag{46}$$

as such there is a homomorphism, and it extends to the derivative spaces

$$\{ [\psi^{\{(i)\}}(x_0)], [\psi^{\{(i)\}}(x_1)], [\psi^{\{(i)\}}(x_2)], \cdots \}, \tag{47}$$

for $i = 1,2,3,\cdots$.

## 4. AN ILLUSTRATIVE EXAMPLE: THE GAUSSIAN

The Gaussian is the integral

$$F(z) = \int_{\{-\infty\}}^{z} \frac{\{1\}}{\{\sqrt{\{2\pi\}}\}} e^{\left\{-\frac{\{s^2\}}{\{2\}}\right\}} d\, s, \tag{48}$$

where the function

$$F'(z) = \frac{\{1\}}{\{\sqrt{\{2\pi\}}\}} e^{\left\{-\frac{\{z^2\}}{\{2\}}\right\}}, \tag{49}$$

is called the anti-derivative. As a second-order:

$$F''(z) = -z\, F'(z) \tag{50}$$

The notion of anti-derivative derives from the fundamental theorem of calculus, credited to Isaac Newton (16421727) and Gottfried Leibniz (1646-1716). This was before Leonhard Euler (1707 – 1783) appeared on the scene, and introduced topology, though today topology is credited to the likes of Johann Benedict Listing (1808 - 1882), for coining the term *topology* and largely to Felix Hausdorff (1868 - 1942), of the Hausdorff topology.

The extension of topology to the study of differential equations, is an ongoing process, as evident from [9], [10], [11], [12] and [13].

Our contribution is to infinitely differentiable solutions, in particular those requiring the fundamental theorem, in this case, presented in the form

$$V_{\{f,z\}F(z)} = \frac{\{1\}}{\{2\pi\}} e^{\left\{-\frac{\{z^2\}}{\{2\}}\right\}}, \tag{51}$$

for some mapping $f$, with the velocity operator $Vf, z \in T\, P$, where

$$TP = \{ V\_\{f,z\} \mid f : R \to X \}, \tag{52}$$

a tangent vector space.

We begin with the trivial integration of (50). The first integration gives

$$\int F''(x) d\, x = -\int x\, F'\, d\, x - D_1. \tag{53}$$

That is,

$$F' = -\int x\, F'\, d\, x - D_1, \tag{54}$$

where $D_1$ is a constant of integration. The second integration gives

$$\int F' d\, x = -\int \left( \int x\, F'(x)\, d\, x \right) d\, x - D_1 x - D_2, \tag{55}$$

where $D_2$ is also a constant of integration. That is,

$$F = -\int \left( \int x\, F'(x)\, d\, x \right) d\, x - D_1 x - D_2, \tag{56}$$

or

$$\psi + \int ( \int x\, \psi'\ d\, x\ )\, d\, x + D\_1\, x + D\_2 = 0. \tag{57}$$

We turn to quotient spaces to resolve the remaining integral $\int \left( \int x\psi' dx \right) dx$, by generating equivalence classes, guided by the theory developed in the previous section.

### *4.1 The equivalence classes*

Note again that $\psi$ and $\psi^{00}$ share the same infinite zeroes, and that this set of zeroes constitutes an equivalence class, in both $x$ and $\psi$, and its derivatives. Hence,

$$\frac{\{[\psi]^{\{(3)\}}\}}{\{[\psi]'\}} = \frac{\{[\psi]^{\{(4)\}}\}}{\{[\psi]''\}}, \tag{58}$$

which has the solution

$$[\psi]' = \frac{\{[a]\backslash sin\,(\,i\,[\omega]([x] + [\phi]))\}}{\{i\,[\omega]\}}, \tag{59}$$

where $a$ and $\omega$ are constants within the quotient space, but not necessarily outside of it. To determine $a$ and $\omega$ through (57) we note that

$$[\psi]'' = \psi'' \;, \tag{60}$$

or

$$[\psi]' = \psi' + b \;\;, \tag{61}$$

where the $b$ is also constant within the quotient space.

These lead to

$$\psi = -4\, e^{\{-2\,\pi\,x^2\}\sqrt{\{\pi\}x}} \frac{\{sec\,[\sqrt{\{\pi\}x}\sqrt{\{\,(4\,\pi\,x^2 - 2\,)\}}]\}}{\{-2\,e^{\{-\pi\,x^2\}} + 4\,e^{\{-\pi\,x^2\}\pi}x^2\}} + \frac{\{2\,e^{\{-2\,\pi\,x^2\}\sqrt{\{\pi\}x}}\}}{\{-2\,e^{\{-\pi\,x^2\}} + 4\,e^{\{-\pi\,x^2\}\pi}x^2\}} \tag{62}$$

,

and is plotted in Figure 2, and compares favourably with the numerically obtained result in Figure 1.

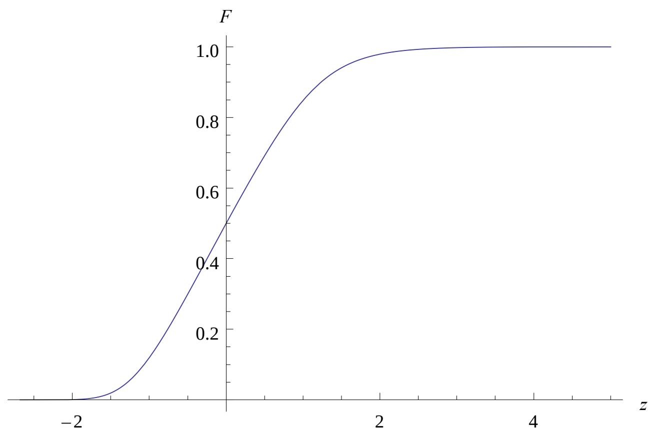


**Fig. 1.** A plot of the Gaussian (48) obtained through Numerical techniques.

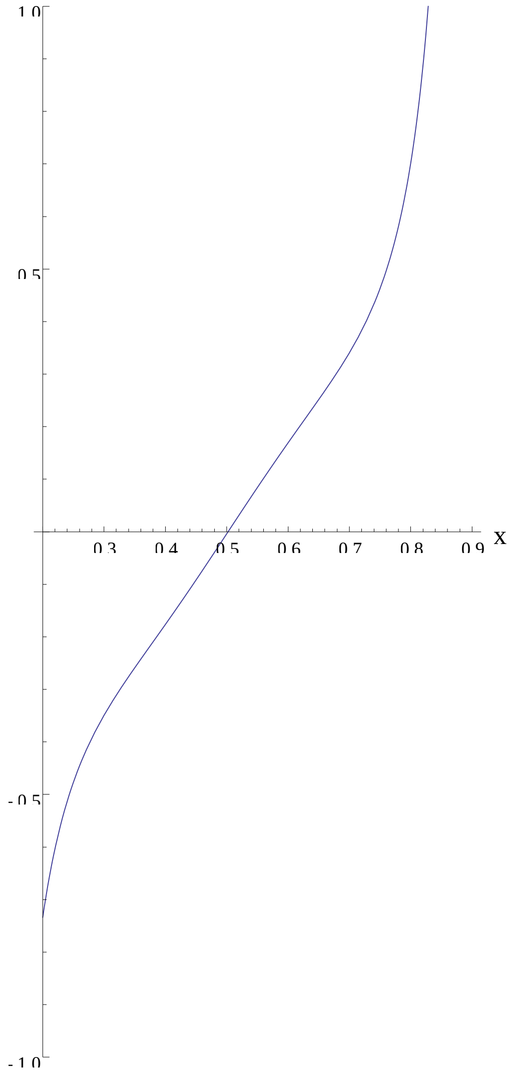


**Fig. 2.** A plot of the solution in (62).

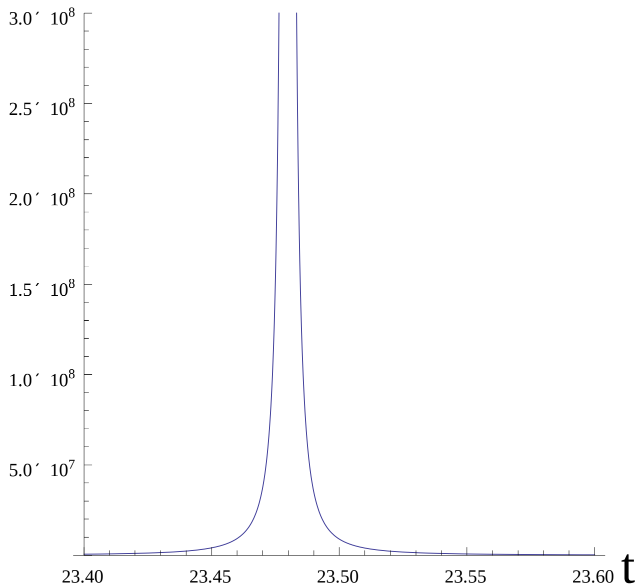


**Fig. 3.** A plot of velocity $u = u^{\varphi}$ in (65) against time t, with $\tan\varphi = 0$.

## 5. SOLVING THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS

Consider (16) in the form

$$u^{\varphi} = -\frac{\{C_2\}}{\{r\}} \int^{\xi} e^{\left\{-\frac{\alpha z^2}{4}\right\}} z^{\{C_1+1\}} d\,z + C\_3. \quad (63)$$

Differentiating with respect to $\alpha$ to the order $(C_1+1)/2$ gives

$$u^{\varphi} = -\frac{C_2}{r} \frac{\partial^{\left\{\frac{C_1+1}{2}\right\}}}{\partial\,\alpha^{\left\{\frac{(C_1+1)}{2}\right\}}} \int^{\xi} e^{\left\{-\frac{\alpha z^2}{4}\right\}} d\,z + C\_3. \quad (64)$$

Letting $s^2 = \frac{\alpha z 2}{2}$ converts this expression to

$$u^{\varphi} = -\frac{C_2}{r} \frac{\partial^{\left\{\frac{C_1+1}{2}\right\}}}{\partial\,\alpha^{\left\{\frac{C_1+1}{2}\right\}}} \sqrt{\left\{\frac{\{2\}}{\{\alpha\}}\right\}} \int e^{\left\{-\frac{\alpha s^2}{s}\right\}} d\,s + C\_3, \quad (65)$$

which is not different from (48), implying it is solved by (62), evaluated at $\alpha = 1/2$.

## 6. CONCLUSION

In this contribution we set out to determine the solution of the two-dimensional Navier-Stokes equations using a modified form of the method of variation of parameters. The method relies heavily on Hausdorff topologies, in particular differentiable manifolds arising from them. It was first tested successfully on the Gaussian, an integral well-known for its un-integrability. The resulting solution of the integral was found to be at the core of the Navier-Stokes equations, which enabled the solution to the latter to result easily through fractional derivatives, plotted in Figures 3.

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