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Strongly Multiplicative Labeling in the Context of Various Graph Operations

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Abstract

A graph $G=\left(V(G),E(G)\right)$ with p vertices is said to be strongly multiplicative if the vertices of G can be labelled with p consecutive positive integers 1,2,…, p such that the label induced on the edges by the product of labels of end vertices are all distinct. In this paper we investigate strongly multiplicative labeling with respect to some standard graph operations. We prove that $B_{m,n}\odot K_1$ and shadow graph of $B_{m,n}$ are strongly multiplicative .We also prove that barycentric subdivision of $B_{m,n}$ and switching of a pendant vertex in $B_{m,n}$ are strongly multiplicative. Moreover the graph obtained by switching of a pendant vertex in path P_n, the graph obtain by switching of a vertex in cycle C_n, the graph obtained by switching of an apex vertex in helm H_n and the graph obtained by switching of a vertex having degree two in fan f_n are strongly multiplicative.

Keywords: Strongly Multiplicative Labeling; Bistar; Shadow Graph; Barycentric Subdivision; Switching of a Vertex

2010 Mathematics Subject Classification: 05C78.

INTRODUCTION

In this research article, by a graph we mean finite, connected, undirected, simple graph $G=\left(V(G),E(G)\right)$ of order $\left|V(G)\right|=p$ and size $\left|E(G)\right|=q$.

Definition 1.1. A *graph labeling* is an assignment of numbers to the vertices or edges or both subject to certain condition(s).

A latest survey of all the graph labeling techniques can be found in Gallian[2].

Definition 1.2. A graph $G=\left(V(G),E(G)\right)$ with p vertices is said to be *multiplicative* if the vertices of graph G can be labelled with p distinct positive integers such that the label induced on the edges by the product of end vertices are all distinct.

Multiplicative labeling was introduced by Beineke and Hegde[1] In the same paper they proved that every graph G admits multiplicative labeling and defined strongly multiplicative labeling as follows.

Definition 1.3. A graph $G=\left(V(G),E(G)\right)$ with p vertices is said to be *strongly multiplicative* if the vertices of graph G can be labelled with p consecutive positive integers $1,2,...,p$ such that the label induced on the edges by the product of labels of end vertices are all distinct.

In 2001 Beineke and Hegde [1] proved the following results:

- Every cycle C_n is strongly multiplicative.
- Every wheel W_n is strongly multiplicative.
- The complete graph K_n is strongly multiplicative $\Leftrightarrow n\le 5$.
- The complete bipartite graph $K_{n,n}$ is strongly multiplicative $\Leftrightarrow n\le 4$.
- Every spanning subgraph of a strongly multiplicative graph is strongly multiplicative.
- Every graph is an induced subgraph of a strongly multiplicative graph.

In 2015 Kanani and Chhaya [4] have discussed strongly multiplicative labeling of some path related graphs and proved the following results:

- The total graph $T\left(P_n\right)$ of the path P_n is strongly multiplicative.
- The splitting graph $S'\left(P_n\right)$ of the path P_n is strongly multiplicative.
- The shadow graph $D_2\left(P_n\right)$ of the path P_n is strongly multiplicative.
- The triangular snake TS_n is strongly multiplicative.

In this research paper we consider following standard definitions:

Definition 1.4. The *bistar* $B_{m,n}$ is the graph obtained by joining the apex vertices of two copies of star $K_{1,m}$ and $K_{1,n}$ by an edge.

Definition 1.5. Let *G* be a graph of order ***n***. The *corona product* of *G* with another graph *H*, $G \odot H$ is the graph obtained by taking one copy of *G* and n copies of *H* and joining the i[th] vertex of *G* with an edge to every vertex in i[th] copy of *H*.

Definition 1.6. The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of *G* namely G' and G'', join each vertex u' in G' to the neighbor of the corresponding vertex u'' in G''.

Definition 1.7. Let $G=(V(G),E(G))$ be a graph. When every edge of *G* is subdivided then the resulting graph is called the *barycentric subdivision* of graph *G*.

Definition 1.8. The *switching of a vertex v* in a graph G means removing all the edges incident to *v* and adding edges joining *v* to every other vertex which are not adjacent to *v* in G. The graph obtained by switching of a vertex *v* in a graph G is denoted by G_v.

Definition 1.9. The *helm* $H_n (n \geq 3)$ is the graph obtained from wheel W_n by attaching a pendant edge at each rim vertex.

Definition 1.10. The *fan graph* f_n is obtained by taking $(n-3)$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called the *apex vertex*.

Main Theorems

Theorem 2.1 *$B_{m,n} \odot K_1$ is strongly multiplicative.*

Proof: Let $B_{m,n}$ be the bistar with vertex set $V(B_{m,n})=\{u_0,v_0,u_i,v_j : 1\leq i\leq m; 1\leq j\leq n\}$ where pendant vertices are $\{u_i,v_j\}$. Let $u_0',u_1',u_2',\ldots,u_m';v_0',v_1',v_2',\ldots,v_n'$ be the newly added vertices to obtain the graph $G=B_{m,n}\odot K_1$. The vertex set $V(G)=V(B_{m,n})\cup\{u_0',v_0',u_i',v_j' : 1\leq i\leq m; 1\leq j\leq n\}$ and the edge set $E(G)=E(B_{m,n})\cup\{u_0u_0',v_0v_0',u_iu_i',v_jv_j' : 1\leq i\leq m; 1\leq j\leq n\}$. It is noted that $|V(G)|=2m+2n+4$ and $|E(G)|=2m+2n+3$.

The vertex labeling $f:V(G)\rightarrow\{1,2,\ldots,2m+2n+4\}$ is defined as follows:

$f(u_0)=p_1$; where p_1 is the highest prime number such that, $5\leq p_1\leq 2m+2n+4$;

$f(v_0)=p_2$; where p_2 is the second highest prime number such that, $3\leq p_2<p_1$;

$f(u_0')=1$;

$f(v_0')=2$.

Now, label the remaining vertices starting from $u_1,u_2,\ldots,u_m;v_1,v_2,\ldots,v_n$ consecutively from the set $\{3,5,7,\ldots,2m+2n+3\}\setminus\{1,p_1,p_2\}$ and $u_1',u_2',\ldots,u_m';v_1',v_2',\ldots,v_n'$ consecutively from the set $\{4,6,8,\ldots,2m+2n+4\}\setminus\{2\}$.

The labeling pattern defined above covers all the possibilities and in each case the graph *G* under consideration admits strongly multiplicative labeling. That is, $B_{m,n}\odot K_1$ *is* strongly multiplicative.

Illustration 2.2. The $B_{3,4}\odot K_1$ and its strongly multiplicative labeling is shown in *Figure 1*.

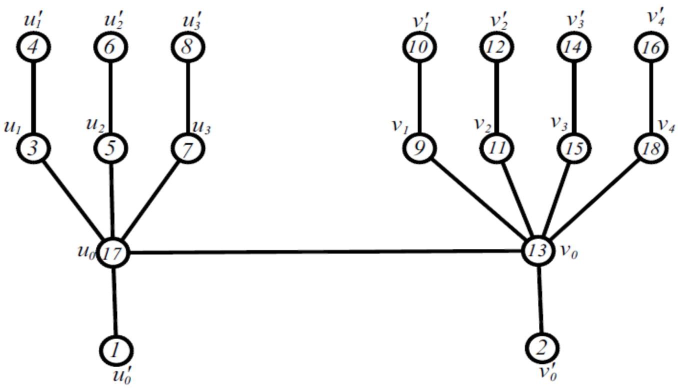

Figure-1: Strongly multipicative labeling of $B_{3,4}\odot K_1$.

Theorem 2.3. *The shadow graph $D_2(B_{m,n})$ of bistar $B_{m,n}$ is strongly multiplicative.*

Proof: Let G' and G'' be two copies of $B_{m,n}$. Let $V(G')=\{u_0', v_0', u_i', v_j' : 1\le i\le m; 1\le j\le n\}$ and $V(G'')=\{u_0'', v_0'', u_i'', v_j'' : 1\le i\le m; 1\le j\le n\}$. Let $G=D_2(B_{m,n})$. It is noted that $|V(G)|=2m+2n+4$ and $|E(G)|=4m+4n+4$.

The vertex labeling $f:V(G)\to\{1,2,...,2m+2n+4\}$ is defined as follows:

$f(u_0')=p_1$; where p_1 is the highest prime number such that, $7\le p_1\le 2m+2n+4$;

$f(u_0'')=p_2$; where p_2 is the second highest prime number such that, $5\le p_2<p_1$;

$f(v_0')=p_3$; where p_3 is the third highest prime number such that, $3\le p_3<p_2$;

$f(v_0'')=p_4$; where p_4 is the fourth highest prime number such that, $2\le p_4<p_3$.

Now, label the remaining vertices $u_1', u_2', ..., u_m'; u_1'', u_2'', ..., u_m''; v_1', v_2', ..., v_n'; v_1'', v_2'', ..., v_n''$ consecutively from the set $\{1,2,3,...,2m+2n+4\}\setminus\{p_1,p_2,p_3,p_4\}$.

The labeling pattern defined above covers all the possibilities and in each case the graph G under consideration admits strongly multiplicative labeling. That is, the shadow graph $D_2(B_{m,n})$ of bistar $B_{m,n}$ *is* strongly multiplicative.

Illustration 2.4. The shadow graph $D_2(B_{3,4})$ of bistar $B_{3,4}$ and its strongly multiplicative labeling is shown in *Figure 2*.

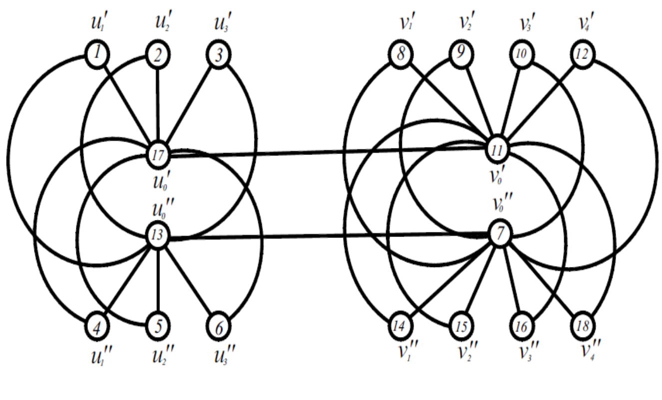

Figure-2: Strongly multipicative labeling of the graph $D_2(B_{3,4})$.

Theorem 2.5. *The barycentric subdivision* $S(B_{m,n})$ of bistar $B_{m,n}$ *is* strongly multiplicative.

Proof: Let $B_{m,n}$ be the bistar with vertex set $V(B_{m,n})=\{u_0, v_0, u_i, v_j : 1\le i\le m; 1\le j\le n\}$ where $\{u_i, v_j\}$ are pendant vertices. Let $e_0=u_0v_0$, $e_i=u_0u_i$ for $1\le i\le m$ and $e_i=v_0v_j$ for $1\le j\le n$. Let $w_0, w_1, w_2, ..., w_m; w_1', w_2', ..., w_n'$ be the newly added vertices to obtain the barycentric subdivision $G=S(B_{m,n})$. Where w_0 is added between u_0 and v_0, each w_i is added between u_0 and u_i, for $1\le i\le m$ and each w_j' is added between v_0 and v_j, for $1\le j\le n$. It is noted that $|V(G)|=2m+2n+3$ and $|E(G)|=2m+2n+2$.

The vertex labeling $f:V(G)\to\{1,2,...,2m+2n+3\}$ is defined as follows:

$f(u_0)=p_1$; where p_1 is the highest prime number such that, $3\le p_1\le 2m+2n+3$;

$f(v_0)=p_2$; where p_2 is the second highest prime number such that, $2\le p_2<p_1$;

$f(w_0)=1$.

Now, label the remaining vertices starting from $u_1, u_2, ..., u_m; v_1, v_2, ..., v_n$ consecutively from the set $\{3,5,7,...,2m+2n+3\}\setminus\{1,p_1,p_2\}$ and $w_1, w_2, ..., w_m; w_1', w_2', ..., w_n'$ consecutively from the set $\{4,6,8,...,2m+2n+4\}\setminus\{p_1,p_2\}$.

The labeling pattern defined above covers all the possibilities and in each case the graph G under consideration admits strongly multiplicative labeling. That is, the barycentric subdivision $S(B_{m,n})$ of bistar $B_{m,n}$ *is* strongly multiplicative.

Illustration 2.6. The barycentric subdivision $S(B_{4,6})$ of bistar $B_{4,6}$ and its strongly multiplicative labeling is shown in *Figure 3*.

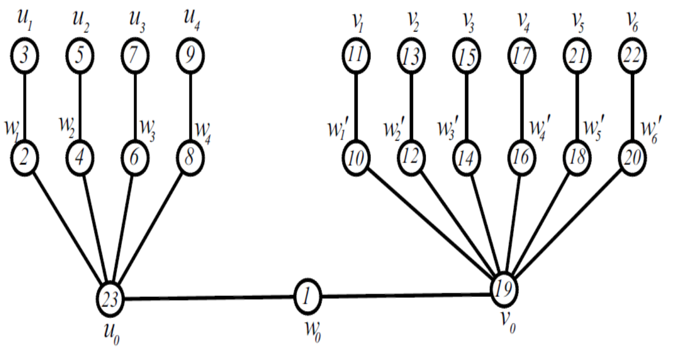

Figure 3: Strongly multipicative labeling of the graph $S(B_{4,6})$.

Theorem 2.7. *The graph obtained by switching of pendant vertex in bistar $B_{m,n}$ is strongly multiplicative.*

Proof: Let $B_{m,n}$ be the bistar with vertex set $V(B_{m,n}) = \{u_0, v_0, u_i, v_j : 1 \le i \le m; 1 \le j \le n\}$ where pendant vertices are $\{u_i, v_j\}$. Let G_u be the graph obtained by switching of pendant vertex *u*. Without loss of generality let the switched vertex be u_1. It is noted that $|V(G_{u_1})| = 2m + 2n + 3$ and $|E(G_{u_1})| = 2m + 2n + 2$.

The vertex labeling $f : V(G_{u_1}) \to \{1, 2, ..., 2m + 2n + 3\}$ is defined as follows:

$f(u_1) = p_1$; where p_1 is the highest prime number such that, $5 \le p_1 \le 2m + 2n + 3$;

$f(v_0) = p_2$; where p_2 is the second highest prime number such that, $3 \le p_2 < p_1$;

$f(u_0) = p_3$; where p_3 is the third highest prime number such that, $2 \le p_3 < p_2$.

Now, label the remaining vertices starting from $u_2, ..., u_m; v_1, v_2, ..., v_n$ consecutively from the set $\{1, 2, 3, ..., 2m + 2n + 3\} \setminus \{p_1, p_2, p_3\}$.

The labeling pattern defined above covers all the possibilities and in each case the graph under consideration admits strongly multiplicative labeling. That is, the graph obtained by switching of pendant vertex in bistar $B_{m,n}$ is strongly multiplicative.

Illustration2.8. The graph obtained by switching of pendant vertex in bistar $B_{5,3}$ and its strongly multiplicative labeling is shown in *Figure 4*.

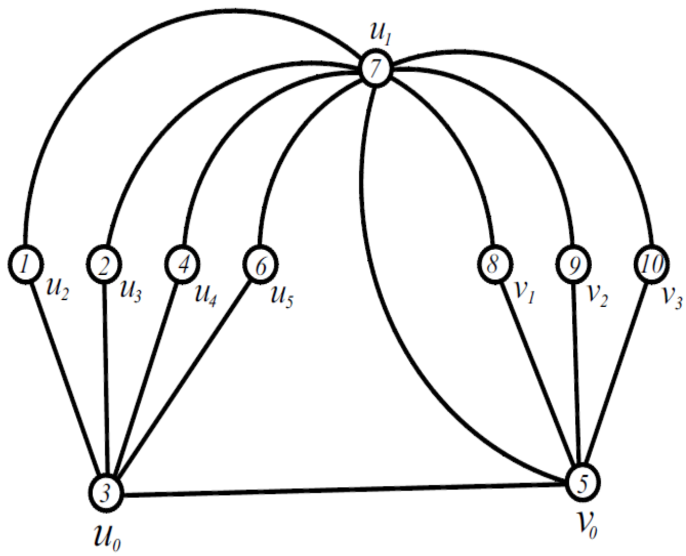

Figure 4: Strongly multipicative labeling of the graph obtained by switching of pendant vertex in bistar $B_{5,3}$.

Theorem2.9. *The graph obtained by switching of pendant vertex in path P_n is strongly multiplicative.*

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of path P_n and let G_v be the graph obtained by switching of a pendant vertex v of path P_n. Without loss of generality let the switched vertex be v_1. It is noted that $|V(G_{v_1})| = n$ and $|E(G_{v_1})| = 2n - 4$.

The vertex labeling $f : V(G_{v_1}) \to \{1, 2, ..., n\}$ is defined as follows:

$f(v_1) = p$; Where p is the highest prime number such that, $2 \le p \le n$.

Now, label the remaining vertices starting from $v_2, v_3 ..., v_n$ consecutively from the set $\{1, 2, 3, ..., n\} \setminus \{p\}$.

The labeling pattern defined above covers all the possibilities and in each case the graph under consideration admits strongly multiplicative labeling. That is, the graph obtained by switching of pendant vertex in path P_n is strongly multiplicative.

Illustration2.10. The graph G_{v_1} obtained by switching of pendant vertex v_1 in path P_7 and its strongly multiplicative labeling is shown in *Figure 5*.

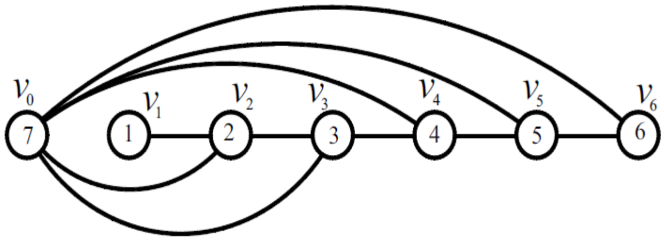

Figure 5: Strongly multipicative labeling of the graph obtained by switching of pendant vertex in path P_7.

Theorem 2.11. *The graph obtained by switching of a vertex in cycle* C_n *is strongly multiplicative.*

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of cycle C_n. Let G_v be the graph obtained by switching of a pendant vertex v of cycle C_n. Without loss of generality let the switched vertex be v_1. It is noted that $|V(G_{v_1})| = n$ and $|E(G_{v_1})| = 2n-5.$

The vertex labeling $f : V(G_{v_1}) \rightarrow \{1, 2, ..., n\}$ is defined as follows:

$f(v_1) = p;$ where p is the highest prime number such that, $2 \le p \le n.$

Now, label the remaining vertices starting from $v_2, v_3..., v_n$ consecutively in clockwise direction from the set $\{1, 2, 3, ..., n\} \setminus \{p\}$.

The labeling pattern defined above covers all the possibilities and in each case the graph under consideration admits strongly multiplicative labeling. That is, the graph obtained by switching of a vertex in cycle C_n is strongly multiplicative.

Illustration2.12. The graph G_{v_1} obtained by switching of a vertex v_1 in cycle C_7 and its strongly multiplicative labeling is shown in *Figure 6*.

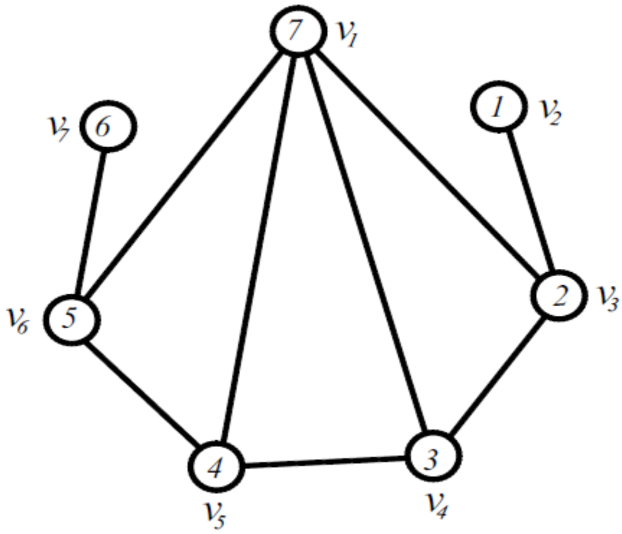

Figure 6: Strongly multipicative labeling of the graph obtained by switching of a vertex in cycle C_7.

Theorem2.13. *The graph obtained by switching of an apex vertex in helm* H_n *is strongly multiplicative.*

Proof: Let H_n be a helm with v_0 as the apex vertex and $v_1, v_2, ..., v_n$ be the vertices of cycle C_n and $u_1, u_2, ..., u_n$ be the pendant vertices. Let G_v be the graph obtained by switching of an apex vertex v_0 of helm H_n. Without loss of generality let the switched vertex be v_0. It is noted that $|V(G_{v_0})| = 2n+1$ and $|E(G_{v_0})| = 3n.$

The vertex labeling $f : V(G_{v_1}) \rightarrow \{1, 2, ..., n\}$ is defined as follows:

$f(v_1) = p;$ where p is the highest prime number such that, $2 \le p \le 2n+1.$

Now, label the remaining vertices starting from $v_1, v_2, v_3..., v_n$ consecutively in clockwise direction from the set $\{1, 3, 5, ..., 2n+1\} \setminus \{p\}$ and $u_1, u_2, ..., u_n$ consecutively in clockwise direction from the set $\{2, 4, 6, ..., 2n\}$.

The labeling pattern defined above covers all the possibilities and in each case the graph under consideration admits strongly multiplicative labeling. That is, the graph obtained by switching of an apex vertex in helm H_n is strongly multiplicative.

Illustration 2.14. The graph G_{v_0} obtained by switching of apex vertex v_0 in helm H_5 and its strongly multiplicative labeling is shown in *Figure 7*.

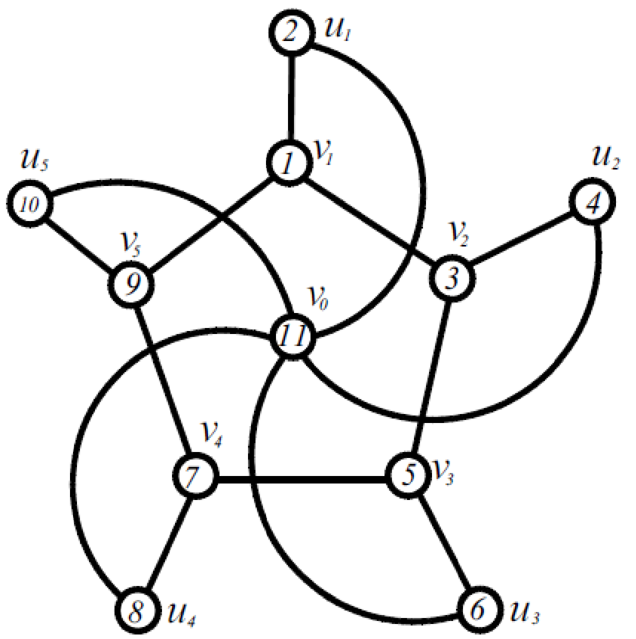

Figure 7: Strongly multipicative labeling of the graph obtained by switching of an apex vertex in helm H_5.

Theorem 2.15. *The graph obtained by switching of a vertex having degree 2 in fan* f_n *is strongly multiplicative.*

Proof: Let f_n be the fan with v_0 as the apex vertex and $v_1, v_2, ..., v_n$ be the vertices of fan f_n. Let G_{v_1} be the graph obtained by switching of a vertex v_1 having degree 2 of $G = f_n$. It is noted that $|V(G_{v_1})| = n+1$ and $|E(G_{v_1})| = 3n-5$.

The vertex labeling $f : V(G_{v_1}) \rightarrow \{1, 2, ..., n+1\}$ is defined as follows:

$f(v_0) = p_1$; where p_1 is the highest prime number such that, $2 \le p_1 \le n+1$;

$f(v_1) = p_2$; where p_2 is the second highest prime number such that, $p_2 < p_1 \le n+1$.

Now, label the remaining vertices starting from $v_2, v_3 ..., v_n$ consecutively from the set $\{1, 2, 3, ..., n+1\} \setminus \{p_1, p_2\}$.

The labeling pattern defined above covers all the possibilities and in each case the graph under consideration admits strongly multiplicative labeling. That is, the graph obtained by switching of a vertex having degree 2 in fan f_n is strongly multiplicative.

Illustration 2.16. The graph G_{v_1} obtained by switching of a vertex v_1 having degree 2 in fan f_7 and its strongly multiplicative labeling is shown in *Figure 8*.

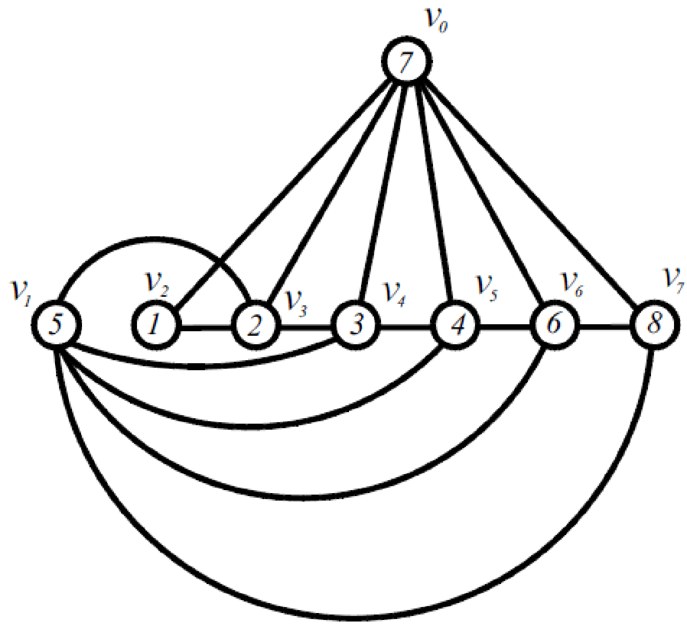

Figure 8: Strongly multipicative labeling of the graph obtained by switching of vertex having degree 2 in fan f_7.

CONCLUDING REMARKS:

Here, we have derived eight new results related to strongly multiplicative labeling. To derive similar results for other graph families is an open problem.

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