International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 15 (2018) pp. 12396-12401


# Numerical Solutions of the Boundary-Layer Axisymmetric Flow over a Nonlinear Stretching Sheet by Spline Collocation Method

**Umesh Patel[1] and H.D.Doctor[2]**

[1]*Jagruti Vidyalaya Rumla, Navsari, Gujarat, India.*

[2]*Department of Mathematic Veer Narmad south Gujarat Uni. Surat-7, Gujarat, India.*

**Abstract**

We study numerically the boundary-layer flow of a viscous fluid over a nonlinear axisymmetric stretchingsheet. With the help of similarity transformation, the governing partial differential equations are reduced to an ordinary differential equation. The resulting ordinary differential equation is solved using Quartic spline collocation method. The numerical solutions are obtained in the form of velocity profiles and skin friction for various values of parameters.

**Keywords**: Boundary- layer axisymmetric flow, stretching sheet, spline collocation, Quartic spline.

## INTRODUCTION

The analysis of boundary-layer flow induced by a moving rigid surface due to Sakiadis [26] is well known. Suchboundary-layer behavior over a moving continuous surface is an important type of flow that occurs in severalengineering process. For example, materials manufactured by extrusion processes and heat-treated materialstraveling between a feed roll and a wind-up roll or on a conveyer belt possess the characteristics of a movingcontinuous surface. Crane [12] extended the analysis to flow induced by a stretching sheet. This problem haslater been extensively studied in various directions including Newtonian and non-Newtonian fluids, porousand nonporous space and hydrodynamic and magnetohydrodynamic (MHD) fluids. Some interesting recentinvestigations is mentioned in the references [5–9].

In all the aforementioned studies the flow is induced by a linear stretching sheet. In spite of the growingliterature on flow over stretching sheets and its obvious importance in the polymer and electrochemical industries,it is surprising to note that the corresponding analysis of nonlinear stretching sheet does not seem tohave received any adequate attention so far. Recently, Vajravelu [27] discussed the boundary-layer flow of aviscous fluid over a planar nonlinearly stretching sheet. He obtained the numerical solution of the problem.

Here we analyze axisymmetric flow over a nonlinear stretching sheet numerically. By using the similarity transformation [10], the partial differential equation governing the flow are transformed to ordinary differential equation, which is solved numerically by using Quartic spline collocation method. The solutions are obtained in the form of stream function$f$, velocity profile $f'$ and skin friction $f''(0)$, for different values of parameter involved.

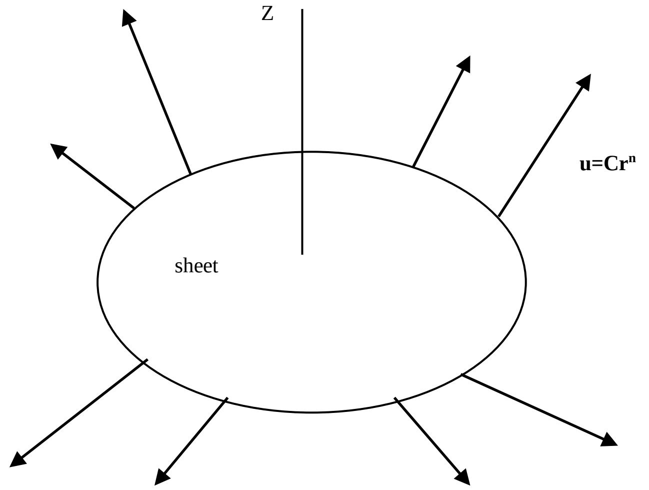


**Figure 1.** Geometry of the problem

## MATHEMATICAL FORMULATION:

Consider the steady, laminar flow of a viscous fluid over a nonlinearly stretching sheet. The sheet is in the plane$z = 0$ and the fluid occupies the half space $z > 0$. For the mathematical modeling we take the cylindrical polarcoordinate system *$(r, \theta, z)$* when flow occurs under the rotational symmetry. Thus all physical quantities areindependent of $\theta$ and the azimuthal component of velocity $v$ vanishes identically. The schematic diagram ofthe considered geometry is given in Fig. 1.

The equations which govern the axisymmetric flow of a viscous fluid are the full Navier–Stokes equationsand are given by

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right) \tag{2}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{\partial^2 w}{\partial z^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{3}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, $\rho$ is the density, $p$ is the pressure, and $u$ and $w$ are the velocities inthe $r$ and $z$ directions, respectively. Since the flow is caused only due to the stretching of the sheet thereforethe pressure gradient can be neglected. By applying the usual boundary-layer approximations, the equationswhich govern the boundary-layer axisymmetric flow of a viscous fluid are Eq. 1 and

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = v\frac{\partial^2 u}{\partial z^2} \tag{4}$$

The boundary conditions are

$$u = ar^n,\ w=0 \text{ at } z = 0,$$
$$u \to 0 \text{ as } z \to \infty. \tag{5}$$

in which a >0 is the stretching constant and n is a positive integer. Introducing the following similaritytransformations

$$\eta = \sqrt{\frac{a(n+1)}{2v}}\, r^{\frac{n-1}{2}} z u = ar^n f'(\eta),$$

$$\omega = -ar^{\frac{n-1}{2}}\sqrt{\frac{2v}{a(n+1)}}\left[\frac{3n+1}{2}f(\eta) + \frac{n-1}{2}\eta f'(\eta)\right] \tag{6}$$

Eq. 1 is identically satisfied and Eq. 4 and conditions 5 gives the following equations.

$$f''' + \left(\frac{3n+1}{n+1}\right)ff'' - \left(\frac{2n}{n+1}\right)f'^2 = 0 \tag{7}$$

$$f = 0,\ f' = 1 \ \ at\ \ \eta = 0$$
$$f' \to 0 \quad \text{as } \eta \longrightarrow \infty \tag{8}$$

The physical quantity of interest is the skin friction coefficient $C_f$ which is defined as

$$C_f = \frac{\tau_w}{p(ar^n)^2} \tag{9}$$

where $\tau_w$is the shear stress at the wall and is given by

$$\tau_w = \mu \left.\frac{\partial u}{\partial z}\right|_{z=0} \tag{10}$$

Using Eq. 6, we get

$$Re_r^{\frac{1}{2}} C_f = f''(0), \tag{11}$$

where $Re_r^{\frac{1}{2}} = 2ar^{n+1}/(n+1)v$is the local Reynolds number. Note that for *n* = 1 one obtains the case ofthe linear stretching sheet

## QUARTIC SPLINE COLLOCATIO :

In this section, fourth-degree spline collocation is used to construct numerical solutions to boundaryvalue problems discussed in equation (7). A detailed description of spline functions generated by subdivision can be found in [17]

Consider equally spaced knots of a partition $\pi: a = \eta_0 < \eta_1 < \cdots < \eta_n = b$ on [a,b]. Let $S_4[\pi]$ be the space of continuously differentiable, piecewise, Quartic polynomials on $\pi$. That is, $S_4[\pi]$ is the space of Quartic polynomials on $\pi$. The Quartic splines is given by

$$S(\eta) = a_0 + b_0(\eta-\eta_0) + \frac{1}{2}c_0(\eta-\eta_0)^2 + \frac{1}{6}d_0(\eta-\eta_0)^3 + \frac{1}{24}\sum_{k=0}^{n-1} e_k (\eta-\eta_k)_+^{\,4} \tag{12}$$

Where the power function $(\eta-\eta_k)_+$ defined as

$$(\eta-\eta_k)_+ \begin{cases} \eta-\eta_k & \text{if } \eta > \eta_k \\ 0 & \text{if } \eta \le \eta_k \end{cases}$$

Consider a third order linear BVP of the form

$$u'''(\eta) + p(\eta)u''(\eta) + q(\eta)u'(\eta) + r(\eta)u(\eta) = t(\eta)\ ; a \le \eta \le b, \tag{13}$$

with boundary conditions $u(a) = k_1,\ u'(a) = k_2,\ u'(b) = k_3$

Where $u(\eta), p(\eta), q(\eta), r(\eta), s(\eta)$ are continuous functions defined in the interval $[a,b]$; $K_1, k_2$ *and* $k_3$ are finite real constants.

In this section, the spline solution of equations (13) is determined using a collocation method. Let (12) be an approximate solution of eq. (13), where $a_0, b_{0,} c_{0,} d_{0,} d_1 \dots d_{n-1}$ are unknown real coefficients to be determined.

Let $\eta_0, \eta_1 \dots, \eta_n$ be n+1 grid points in the interval [a,b], so that

$$\eta_i = a + ih,\ i = 0,1,\dots,n;\ \eta_0 = a,\ h = \frac{a-b}{h}. \tag{14}$$

It is required that approximate solution (12) satisfies the differential equation at the points $\eta = \eta_i$. Putting (12) in (13), we obtain the collocation equations as follows

$$a_0 r(\eta_i) + b_0[q(\eta_i) + r(\eta_i)\eta_i] + c_0\left[p(\eta_i) + q(\eta_i)\eta_i + \frac{1}{2}r(\eta_i){\eta_i}^2\right] + d_0\left[1 + p(\eta_i)\eta_i + \frac{1}{2}q(\eta_i){n_i}^2 + \frac{1}{6}r(\eta_i){\eta_i}^3\right] + \sum_{k=0}^{n-1} e_k\left[(\eta_i-\eta_k) + \frac{1}{2}p(\eta_i)(\eta_i-\eta_k)^2 + \frac{1}{6}q(\eta_i)(\eta_i-\eta_k)^3 + \frac{1}{24}r(\eta_i)(\eta_i-\eta_k)^4\right] = t(\eta_i) \quad \text{i= 0,1, 2, ….,n} \tag{15}$$

$$a_0 + b_0 a + c_0\left(\frac{1}{2}a^2\right) + d_0\left(\frac{1}{6}a^3\right) + \frac{1}{24}\sum_{k=0}^{n-1} e_k\,(\eta_i-\eta_k)_+^{\,4} = k_1 \tag{16}$$

$$b_0 + c_0 a + d_0\left(\frac{1}{2}a^2\right) + \frac{1}{6}\sum_{k=0}^{n-1} e_k\,(\eta_i-\eta_k)_+^{\,3} = k_2 \tag{17}$$

$$b_0 + c_0 b + d_0\left(\frac{1}{2}b^2\right) + \frac{1}{6}\sum_{k=0}^{n-1} e_k\,(\eta_i-\eta_k)_+^{\,3} = k_3 \tag{18}$$

Using the power function $(\eta-\eta_k)_+$ in the above equations a system of n+3 linear equations in n+3 unknowns $a_0, b_0, c_0, d_0, d_1, \dots, d_{n-1}$ is thus obtained. This system can be written in metrix-vector form as follows

$$AC = F \tag{19}$$

Where C = $[a_0, b_0, c_0, d_0, d_1, \dots, d_{n-1}]^T$

and $F = [k_{1,}, k_2\,, s(\eta_0),\ s(\eta_1), \dots, s(\eta_n)\,, k_3]^T$

where T denoting transpose.

The coefficient matrix A is an upper triangular Hessenberg matrix with a single lower subdiagonal, principal and upper diagonal having non-zero elements. Because of this nature of matrix A, the determination of the required quantities becomes simple and consume less time. The values of these constants ultimately yield the Quartic spline $S(\eta)$ in equation (12).

In case of nonlinear boundary value problem, the equations can be converted into linear form by any known method like quasilinearization [6 ] or Newton's linearization and hence this method can be used as iterative method. The procedure to obtain a spline approximation of $u_i$ ( i = 0, 1, 2, …, j; where j denotes the number of iteration) by an interative method starts with fitting a curve satisfying the end conditions and this curve is designated as $u_i$. We obtain the successive iterations $u_i$'s with the help of an algorithm described as above till the desired accuracy.

**QUARTIC SPLINE SOLUTION:**

Now the nonlinear boundary value problem, the equations (7), (8) can be converted into linear form by quasilinearization [6] method as follows

$f_{i+1}'''(\eta) + \left(\frac{3n+1}{n+1}\right) f_n(\eta) f_{i+1}''(\eta) - \left(\frac{4n}{n+1}\right) f_n'(\eta) f_{i+1}'(\eta) - \left(\frac{3n+1}{n+1}\right) f_n''(\eta) f_{i+1}(\eta) = -\left(\frac{2n}{n+1}\right)\left(f_n'(\eta)\right)^2 + \left(\frac{3n+1}{n+1}\right) f_n(\eta) f_n''(\eta)$; n = 0, 1, 2, … , (n-1) (20)

$$f_{i+1}(0) = 0,\ f_{i+1}'(0) = 1,\ f_{i+1}'(\infty) = 0 \quad (21)$$

For the numerical study the outer boundary is set at $\eta_\infty = 5$ and therefore the domain of the problem is restricted to [0, 5] and end condition $f'(\infty) = 0$ is considered as $f'(5) = 0$. The collocation equation corresponding to the equations (20), (21) are obtained as follows

$\left(\frac{3n+1}{n+1}\right) f_n''(\eta) a_0 + \left[-\left(\frac{4n}{n+1}\right) f_n'(\eta) + \left(\frac{3n+1}{n+1}\right) f_n''(\eta)(\eta_n - \eta_0)\right] b_0 + \left[\left(\frac{3n+1}{n+1}\right) f_n(\eta) - \left(\frac{4n}{n+1}\right) f_n'(\eta)(\eta_n - \eta_0) + \frac{1}{2}\left(\frac{3n+1}{n+1}\right) f_n''(\eta)(\eta_n - \eta_0)^2\right] c_0 + \left[1 + \left(\frac{3n+1}{n+1}\right) f_n(\eta)(\eta_n - \eta_0) - \left(\frac{2n}{n+1}\right) f_n'(\eta)(\eta_n - \eta_0)^2 + \frac{1}{6}\left(\frac{3n+1}{n+1}\right) f_n''(\eta)(\eta_n - \eta_0)^3\right] d_0 + \sum_{k=0}^{n-1} e_k \left[(\eta_n - \eta_k) + \frac{1}{2}\left(\frac{3n+1}{n+1}\right) f_n(\eta)(\eta_n - \eta_k)^2 - \frac{1}{6}\left(\frac{4n}{n+1}\right) f_n'(\eta)(\eta_n - \eta_k)^3 + \frac{1}{24}\left(\frac{3n+1}{n+1}\right) f_n''(\eta)(\eta_n - \eta_k)^4\right] = -\left(\frac{2n}{n+1}\right)\left(f_n'(\eta)\right)^2 + \left(\frac{3n+1}{n+1}\right) f_n(\eta) f_n''(\eta)$
;n=0, 1, 2, …….,9 (22)

The conditions $f_{i+1}(0) = 0,\ f_{i+1}'(0) = 1, f_{i+1}'(5) = 0$ gives

$a_0 = 0$ , $b_0 = 1$ and

$$b_0 + 5c_0 + \frac{25}{2} d_0 + \frac{1}{6}\sum_{k=0}^{n-1} e_k\,(5 - \eta_k)_+{}^3 = 0 \qquad (23)$$

To obtain the complete solution of the problem. We solve equations (22), (23) for the unknown $a_0,\ b_0, c_0, d_0, e_0,\ e_1, \dots, e_9$ to obtain approximate solution for $f(\eta), f'(\eta)\ and\ f''(\eta)$ using equation (12). The results are obtained for different parameters involved in the problem and are presented in the table as well as in graph.

**Table: 1** Stream function f

| | **n =1** | **n =2** | **n =3** | **n =4** | **n =5** | **n =10** |
|---|---|---|---|---|---|---|
| $\boldsymbol{f(0.0)}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\boldsymbol{f(0.5)}$ | 0.3741596 | 0.3630976 | 0.3580974 | 0.3552388 | 0.3533867 | 0.3493248 |
| $\boldsymbol{f(1.0)}$ | 0.5679171 | 0.5394447 | 0.5269504 | 0.519913 | 0.5153949 | 0.5055987 |
| $\boldsymbol{f(1.5)}$ | 0.6636138 | 0.6220542 | 0.6042712 | 0.5943804 | 0.5880785 | 0.5745458 |
| $\boldsymbol{f(2.0)}$ | 0.7103019 | 0.6612198 | 0.6406458 | 0.6293181 | 0.6221444 | 0.6068561 |
| $\boldsymbol{f(2.5)}$ | 0.7335445 | 0.6809182 | 0.6592015 | 0.6473335 | 0.6398504 | 0.6239878 |
| $\boldsymbol{f(3.0)}$ | 0.7456315 | 0.6916784 | 0.669647 | 0.6576633 | 0.6501273 | 0.6342012 |
| $\boldsymbol{f(3.5)}$ | 0.7521318 | 0.6978053 | 0.6757462 | 0.6637736 | 0.6562527 | 0.6403773 |
| $\boldsymbol{f(4.0)}$ | 0.7554245 | 0.7009241 | 0.6788301 | 0.6668413 | 0.6593102 | 0.6434091 |
| $\boldsymbol{f(4.5)}$ | 0.7564549 | 0.7015859 | 0.6793114 | 0.6672112 | 0.6596038 | 0.6435229 |
| $\boldsymbol{f(5.0)}$ | 0.7555591 | 0.6999378 | 0.6772777 | 0.6649426 | 0.657177 | 0.6407321 |

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, Number 15 (2018) pp. 12396-12401


**Table 2:** Velocity profile $f'$

| | **n=1** | **n=2** | **n=3** | **n=4** | **n=5** | **n=10** |
|---|---|---|---|---|---|---|
| $\boldsymbol{f'(0.0)}$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\boldsymbol{f'(0.5)}$ | 0.534498 | 0.4999815 | 0.4847027 | 0.4760609 | 0.4704989 | 0.4584027 |
| $\boldsymbol{f'(1.0)}$ | 0.2679954 | 0.2361435 | 0.2228544 | 0.2155638 | 0.2109587 | 0.2011821 |
| $\boldsymbol{f'(1.5)}$ | 0.1306968 | 0.1104735 | 0.1026412 | 0.0985121 | 0.0959686 | 0.0907424 |
| $\boldsymbol{f'(2.0)}$ | 0.0641696 | 0.0538553 | 0.0502932 | 0.0485363 | 0.0475007 | 0.0454969 |
| $\boldsymbol{f'(2.5)}$ | 0.0327618 | 0.0286219 | 0.0275075 | 0.0270501 | 0.0268172 | 0.0264651 |
| $\boldsymbol{f'(3.0)}$ | 0.0177196 | 0.016703 | 0.0166798 | 0.01676 | 0.0168429 | 0.0170972 |
| $\boldsymbol{f'(3.5)}$ | 0.0099139 | 0.0100774 | 0.0103554 | 0.0105511 | 0.0106884 | 0.0110109 |
| $\boldsymbol{f'(4.0)}$ | 0.0051438 | 0.0054297 | 0.0056241 | 0.0057417 | 0.0058185 | 0.0059844 |
| $\boldsymbol{f'(4.5)}$ | 0.0015795 | 0.0014825 | 0.0014327 | 0.0013975 | 0.0013713 | 0.0013029 |
| $\boldsymbol{f'(5.0)}$ | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 3:** Skin friction $f''(0)$

| **n** | $\boldsymbol{f''(0)}$ |
|---|---|
| **1** | -1.16583349 |
| **2** | -1.30158215 |
| **3** | -1.36492236 |
| **4** | -1.40171394 |
| **5** | -1.4257801 |
| **10** | -1.47920937 |

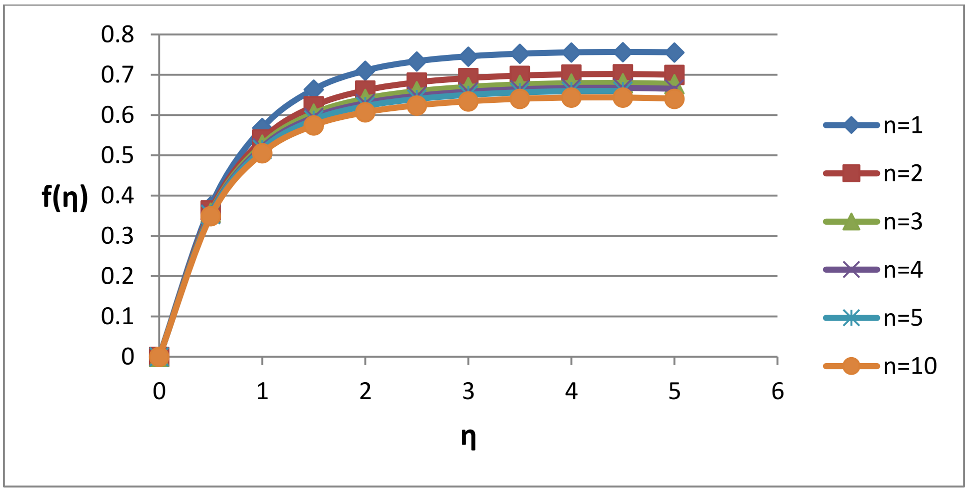


**Figure 2.** Variation of f with increasing of the parameter n

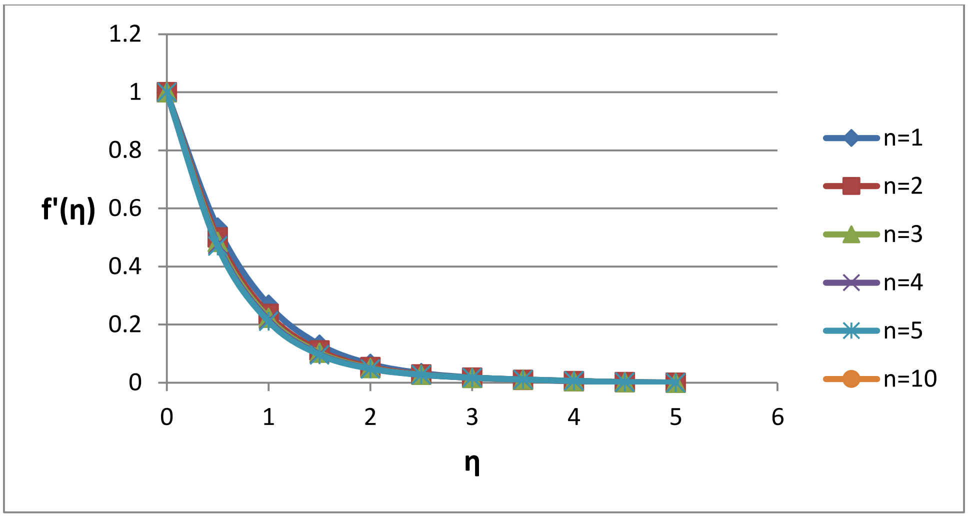


**Figure 3.** Variation of $f'$ for increasing value of the parameter n

**DISCUSSION OF RESULTS AND CONCLUSIONS:**

The graphs of the function $f(\eta)$ and $f'(\eta)$ are drawn against $\eta$ for different values of the parameter n in fig. 2 and 3. Here the parameter n is a measure of nonlinearity of the stretching sheet.

It is shown in Fig. 3 that the velocity $f'$ and boundary-layer thickness decreases by increasing the parameter n. Fig. 2 shows that the $f$ decreases as $\eta$ increases whereas the boundary-layer thickness increases. It is also observed that the flows does not noticeably depend on n for n≥ 15. The values of the skin friction and fractional darg coefficient are given in Table 3. It is clear form table 3 that the magnitude of skin friction coefficient increases with an increase in n. Also the first value in table 3 for the linear stretching case (n=1) is exactly the same when compared with the numerical and perturbation solutions presented in [3]. This shows that the Quartic spline collocation results agree well with the numerical and analytical ones.

In this paper, analysis of the axisymmetric flow of a viscous fluid over a nonlinearly stretching sheet is carried out. The numerical solutions are presented using the Quartic spline. The effect of the parameter n on the velocity are presented graphically and discussed. It is found that the effect of the parameter n is small for n≥ 15. Moreover, the results of the linear stretching sheet can be derived form the presented solution for n=1 and are quite comparable with the existing results as presented in [3]

**REFERENCES**

[1] Abbas, Z., Sajid, M., Hayat, T.: MHD boundary layer flow of an upper-convected Maxwell fluid in a porous channel. Theor. Comput. Fluid Dyn. **20**, 229–238 (2006)

[2] Abbasbandy, S.: The application of homotopy analysis method to nonlinear equations arising in heat transfer. Phys. Lett. A **360**, 109–113 (2006)

[3] Ariel, P.D.: Axisymmetric flow of a second grade fluid past a stretching sheet. Int.J. Eng. Sci. Vol 39, pp. 529-553(2001).

[4] Ariel, P.D.: Generalized three dimensional flow due to a stretching sheet. Z. Angew. Math. Mech. **83**, 844–852 (2003)

[5] Ariel, P.D., Hayat, T., Asghar, S.: The flow of an elastico-viscous fluid past a stretching sheet with slip. Acta Mech. **187**,

[6] Bellam, R. E. and Kalaba, R. E., Quasilinezaration and nonlinear boundary value problem, American Elswevir Publishing company, INC, New York. 1965.

[7] Cortell, R.: A note on magnetohydrodynamic flow of a power-law fluid over a stretching sheet. Appl. Math. Comp. **168**, 557–566 (2005a)

[8] Cortell, R.: Effects of viscous dissipation and work done by deformation on the MHD flow and heat transfer of a viscoelastic fluid over a stretching sheet. Phys. Lett. A **357**, 298–305 (2006)

[9] Cortell, R.: Flow and heat transfer of an electrically conducting fluid of second grande over a stretching sheet subject to suction and to a transverse magnetic field. Int. J. Heat Mass transf. 49, 1851-1856(2006)

[10] Cortell, R.: Flow and heat transfer of a fluid through a porous medium over a stretching surface with internal heat generation/ absorption and suction/blowing. Fluid Dyn. Res. **37**, 231–245 (2005)

[11] Cortell, R.: Viscous flow and heat transfer over a nonlinearly stretching sheet. Appl. Math. Comp. Vol. 184, Issue 2, 15, pp. 864-873 (2007)

[12] Crane, L.J.: Flow past a stretching plate. Z. Angew. Math. Phys. (ZAMP) **21**, 645–647 (1970)

[13] Hayat, T., Abbas, Z., Sajid, M.: Series solution for the upper-convected Maxwell fluid over a porous stretching plate. Phys. Lett. A **358**, 396–403 (2006)

[14] Hayat, T., Khan, M., Asghar, S.: Homotopy analysis of MHD flows of an Oldroyd 8-constant fluid. ActaMech. **168**, 213–232 (2004)

[15] Hayat, T., Khan, M., Ayub, M.: Couette and Poiseuille flows of an Oldroyd 6-constant fluid with magnetic field. J. Math. Anal. Appl. **298**, 225–244 (2004)

[16] Hayat, T., Khan, M., Ayub, M.: On the explicit analytic solutions of an Oldroyd 6-constant fluid. Int. J. Eng. Sci. **42**, 123–135 (2004)

[17] Joshi D. C. Numerical study of a viscoelastic heat transfer flow over a semi-infinite conducting plate with a pressure gradient through collocation using Quintic B-spline, International Journal of Applied Mathematics and Mechanics, Vol. 7, No. 20, pp. 89- 102 (2011)

[18] Liao, S.J.: A new branch of solutions of boundary-layer flows over an impermeable stretched plate. Int. J. Heat Mass Transf. **48**, 2529–2539 (2005)

[19] Liao, S.J.: A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate. J. Fluid Mech. **385**, 101–128 (1999)

[20] Liao, S.J.: An analytic solution of unsteady boundary-layer flows caused by an impulsively stretching plate. Commun. Nonlin.Sci. Numer. Simm. **11**, 326–339 (2006)

[21] Liao, S.J.: Beyond perturbation: introduction to homotopy analysismethod. Chapman&Hall/CRC Press, Boca Raton (2003)

[22] Liao, S.J., Pop, I.: Explicit analytic solution for similarity boundary layer equations.

[23] Int. J. Heat Mass Transf. **46**(10), 1855–1860 (2004)

[24] Liao, S.J.: On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet. J. Fluid Mech. **488**, 189–212 (2003)

[25] Liao, S.J.: On the homotopy analysis method for nonlinear problems. Appl. Math. Comput. **147**, 499–513 (2004)

[26] Sajid, M., Hayat, T., Asghar, S.: On the analytic solution of the steady flow of a fourth grade fluid. Phys. Lett. A **355**, 18–24 (2006)

[27] Sakiadis, B.C.: Boundary layer behavior on continuous solid surfaces. AICHE J. **7**, 26–28 (1961)

[28] Vajravelu, K.: Viscous flow over a nonlinearly stretching sheet. Appl. Math. Comput. **124**, 281–288 (2001)

[29] Wu, W., Liao, S.J.: Solving solitary waves with discontinuity by means of the homotopy analysis method. Chaos Solitons Fractals **26**, 177–185 (2005)

[30] Wu, Y.Y.,Wang, C., Liao, S.J.: Solving the one loop solution of the Vakhnenko equation by means of the homotopy analysis method. Chaos Solitons Fractals **23**, 1733–1740 (2005)

[31] Xu,H., Liao, S.J.: Series solutions of unsteadymagnetohydrodynamic flows of non-Newtonian fluids caused by an impulsively stretching plate. J. Non-Newtonian Fluid Mech. **129**, 46–55 (2005)

[32] Yang, C., Liao, S.J.: On the explicit purely analytic solution of Von Karman swirling viscous flow. Comm. Nonlin. Sci. Numer. Simul. **11**, 83–93 (2006)

[33] Zhu, S.P.: A closed-form analytical solution for the valuation of convertable bonds with constant dividend yield. Anziam J. **47**, 477–494 (2006)

[34] Zhu, S.P.: An exact and explicit solution for the valuation of American put options. Quan. Financ. **6**, 229–242 (2006)