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The Connected Total Monophonic Domination Number of a Graph

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Abstract

In this paper the concept of connected total monophonic domination number of a graph is introduced. A set of vertices M of a graph G is called a connected total monophonic dominating set if it is a total monophonic dominating set and its induced subgraph $\langle M\rangle$ is connected. The minimum cardinality of all connected total monophonic dominating sets of M is called the connected total monophonic domination number and is denoted by $\gamma_{c_{mt}}(G)$. It is shown that for every pair k, p of integers with $3 < k \leq p$, there exists a connected graph G of order p such that $\gamma_{c_{mt}}(G) = k$. Also, for any positive integers $2 < a < b < c$, there exists a connected graph G such that $m(G) = a, \gamma_m(G) = b$ and $\gamma_{cm_t}(G) = c$.

Keywords: Monophonic set, monophonic number, monophonic dominating set, monophonic domination number, total monophonic dominating set, total monophonic domination number, connected total monophonic dominating set, connected total monophonic domination number.

AMS Subject classification: 05C12, 05C05

INTRODUCTION

By a graph $G = (V, E)$ we consider a finite undirected graph without loops or multiple edges. The order and size of a graph are denoted by p and q respectively. For the basic graph theoretic notations and terminology we refer to Buckley and Harary [1]. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G . A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic.*

The *neighbourhood* of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is an *extreme vertex* if the subgraph induced by its neighbourhood is complete. A vertex v in a connected graph G is a *cut vertex* of G, if $G - v$ is disconnected. A vertex v in a connected graph G is said to be a *semi-extreme vertex* if $\Delta(< N(v) >) = |N(v)| - 1$. A graph G is said to be *semi-extreme graph* if every vertex of G is a semi-extreme vertex. An acyclic graph is called a *tree* [1].

A *chord* of a path $u_0, u_1, \dots, u_h$ is an edge $u_i u_j$, with $j \geq i + 2$. An $u - v$ path is called *monophonic path* if it is a chordless path. A *monophonic set* of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in M [ ]. Two vertices u and v are antipodal if $d(u, v) = diam\ G$ or $d(G)$.

For a subset D of vertices, we call D a dominating set if for each $x \in V(G) - D$, x is adjacent to at least one vertex of D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G [7]. A set of vertices M in G is called a monophonic dominating set if M is both a monophonic set and a dominating set. The minimum cardinality of a monophonic dominating set of G is its monophonic domination number and is denoted by $\gamma_m(G)$. A monophonic dominating set of size $\gamma_m(G)$ is said to be a γ_m set. A total monophonic domination set of a graph G is a monophonic domination set M such that the subgraph induced by M has no isolated vertices. The minimum cardinality among all the total monophonic domination set of G is called the total monophonic domination number and is denoted by $\gamma_{m_t}(G)$. [1]

The following theorems will be used in the sequel

Theorem 1.1 Each extreme vertex of a connected graph G belongs to every connected monophonic dominating set of G. [2]

Theorem 1.2 Every cut vertex of a connected graph G belongs to every connected monophonic dominating set of G. [2]

Theorem 1.3 For any non trivial tree T of order k, $\gamma_{mc}(T) = k$. [2]

Theorem 1.4 Each extreme vertex of a connected graph G belongs to every total monophonic dominating set of G. [1]

Corollary 1.1 For the complete graph K_p $(p \geq 2), \gamma_{mt}(G) = p$. [1]

Theorem 1.5 Let G be connected graph with cut – vertices and let M be the total monophonic domination set of G. If u is a cut-vertex of G then every component of $G - u$ contains an element of M. [1]

CONNECTED TOTAL MONOPHONIC DOMINATION NUMBER OF A GRAPH

Definition 2.1 A connected total monophonic domination set of a graph G is a total monophonic domination set M such that the subgraph induced by M is connected. The minimum cardinality among all the connected total monophonic domination set of G is called the connected total monophonic domination number and is denoted by $\gamma_{c_{mt}}(G)$. A connected total monophonic domination set of cardinality $\gamma_{c_{mt}}(G)$ is called $\gamma_{c_{mt}}$- set of G.

Example 2.2 For the graph given in Fig 2.1, it is clear that $M_1 = \{v_1, v_8\}$ is the monophonic set of G so that $m(G) = 2$. It is verified that the set $M_2 = \{v_1, v_5, v_6, v_8\}$ is a minimum monophonic domination set of G so that $\gamma_m(G) = 4$. Also, it is clear that $M_3 = \{v_1, v_3, v_5, v_6, v_8\}$ is the minimum total monophonic domination set and $\gamma_{m_t}(G) = 5$ and $M_4 = \{v_1, v_3, v_5, v_6, v_7, v_8\}$ is the minimum connected total monophonic domination set and $\gamma_{cm_t}(G) = 6$. Thus the monophonic number, monophonic domination number, total monophonic domination number and connected total monophonic domination number are all different.

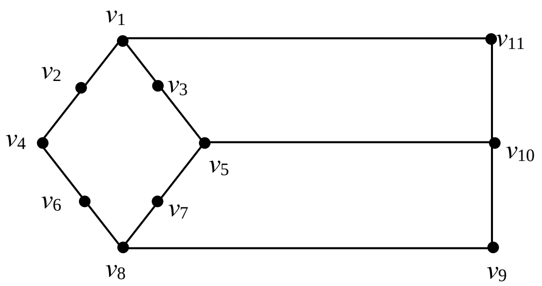

Fig: 2.1

Theorem 2.1 Each extreme vertex of a connected graph G belongs to every connected total monophonic dominating set of G.

Proof: Since each extreme vertex belongs to every total monophonic dominating set (Theorem 1.4), these extreme vertices also belong to every CTMD set.

Definition 2.2: A vertex v of a connected graph G is called a support vertex of G if it is adjacent to an end vertex of G.

Theorem 2.2: Each support vertex of a connected graph G belongs to every connected total monophonic domination set of G. If the set of all support vertices form a connected total monophonic domination set, then it is the unique minimum connected total monophonic domination set of G.

Proof: Since every connected total monophonic domination set is a total monophonic domination set, by Theorem 2.1, each extreme vertex belongs to every connected total monophonic domination set. Since a connected total monophonic domination set has no isolated vertices, it follows that each support vertex of G belongs to every connected total monophonic domination set.

Corollary 2.1: For the complete graph K_p $(p \geq 2), \gamma_{cm_t}(G) = p$.

Theorem 2.3: Let G be connected graph with cut – vertices and let M be the connected total monophonic domination set of G. If u is a cut-vertex of G then every component of $G- u$ contains an element of M.

Proof: Since every CTMD set is a TMD set, the result follows.

Theorem 2.4: For any non trivial tree T of order n , $\gamma_{cm_t}(G) = n$.

Proof: Since every vertex of T is either a cut- vertex or an end vertex, then the result follows from Theorem 2.3 and theorem 2.1.

Theorem 2.5: For any connected graph G of order $n \geq 2$, $2 \leq \gamma_{cm_t}(G) \leq n$.

Theorem 2.6 : For any connected graph G of order $n \geq 2$, $2 \leq \gamma_{mt}(G) \leq \gamma_{cm_t}(G) \leq n$.

Proof: Since any monophonic set contains at least two vertices, $2 \leq \gamma_{mt}(G)$. Again, every connected total monophonic domination set is an total monophonic domination set, $\gamma_{mt}(G) \leq \gamma_{cm_t}(G)$. Since the set of all vertices of G is always a connected total monophonic dominating set, $\gamma_{cm_t}(G) \leq n$.

Theorem 2.7: For any connected graph , $\gamma_{cm_t}(G) = 2$ if and only if $G = K_2$.

Proof: If $G = K_2$, then $\gamma_{cm_t}(G) = 2$. Conversely, let $\gamma_{cm_t}(G) = 2$. Let $M = \{u, v\}$ be a minimum connected total monophonic dominating set of G. Then uv is an edge. It is clear that a vertex different from u and v and cannot lie on a u-v monophonic and so $G = K_2$.

Theorem 2.9: If G is a connected graph of order $n \geq 2$ with every vertex of G is either a cut vertex or an extreme vertex, then , $\gamma_{cm_t}(G) = n$.

Proof: It follows from Theorems 2.1 and 2.4

Corollary 2.2 : Let G be a connected graph. If , $\gamma_{cm_t}(G) = 2$, then , $\gamma_{cm}(G) = 2$.

Remark 2.1: For the complete graph $G = K_2$, $\gamma_{cm_t}(G) = 2$ and for the complete graph K_p , $\gamma_{cm_t}(G) = p$ so that the connected total monophonic domination number of a graph attains its least value 2 and largest value p.

Theorem 2.10: Let G be a connected graph with at least two vertices. Then $\gamma_{cm_t}(G) \leq 2\gamma_{cm}(G)$.

Proof: Let $R = \{r_1, r_2, \ldots, r_k\}$ be a minimum total monophonic domination set of G. Let $r_i \in N(y_i)$ for $i = 1,2, \ldots, k$ and $S = \{y_1, y_2, \ldots, y_k\}$. Then $R \cup S$ is a connected total monophonic domination set of G so that $\gamma_{cm_t}(G)) \leq |R \cup S| \leq 2\gamma_{cm}(G)$.

Theorem 2.11: For any connected graph G of order k , $2 \leq \gamma_m(G) \leq \gamma_{m_t}(G) \leq \gamma_{cm_t}(G) \leq k$.

Proof: Any monophonic domination set needs at least two vertices and so $\gamma_m(G) \leq 2$. Since every connected total monophonic domination set is a total monophonic domination set, it follows that $\gamma_{m_t}(G) \leq \gamma_{cm_t}(G)$. Since, $V(G)$ is connected total monophonic set of G , it is clear that $\gamma_{c_{mt}}(G) \leq k$. Hence, $\gamma_m(G) \leq \gamma_{m_t}(G) \leq \gamma_{cm_t}(G) \leq k$.

Result 2.2: For any connected graph G , girth of $G \leq$ connected total monophonic domination number.

CTMD number of Peterson Graph:

Consider the peterson graph G of form 2 given in Fig 2.2, Here $m(G) = 2$, $\gamma_m(G) = 3$, $\gamma_{m_t}(G) = \gamma_{cm_t}(G) = 4$.

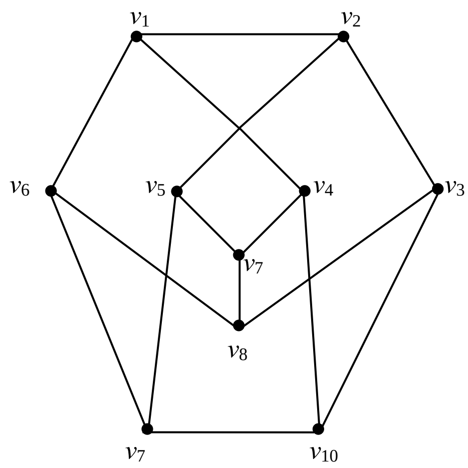

Fig. 2.2

Theorem 2.13: For complete bipartite graph $K_{r,s}$

$$\gamma_{c_{mt}}(G) = \begin{cases} 2 & if\ r = s = 1 \\ s & if\ s \geq 2, r = 1 \\ min\{r, s\} + 1 & if\ r, s \geq 2 \end{cases}$$

Proof: (i) Let $r = s = 1$, then $K_{r,s} = K_2$. Hence by corollary 2.1, $\gamma_{c_{mt}}(G) = 2$. (ii) Let $s \geq 2, r = 1$, then $K_{r,s}$ is a tree with s end vertices and these extreme vertices belongs to every TMD set and so every minimal TMD set is a CTMD set. (iii) Take $X = \{x_1, x_2, \dots, x_m\}$, $Y = \{y_1, y_2, \dots, y_n\}$ be a partition of G. Assume $m \leq n$. Consider $D = X$. Then D is a minimum monophonic set. Take $M = D \cup y_i$ for $1 \leq i \leq n$. Then C is a minimum CTMD set. Therefore $\gamma_{c_{mt}}(G) = |M| = m + 1 = \min\{m, n\} + 1$.

REALIZATION RESULTS

Theorem 3.1: For every pair k, p of integers with $3 \leq k \leq p$, there exists a connected graph of order p such that $\gamma_{cm_t}(G) = k$.

Proof: Take a copy of star $K_{1,a}$ with leaves $w, v_1, v_2, \dots, v_{k-4}$ and the support vertex y. Let $Q: u, v, w, x$ be a copy of C_4. Let G be the graph (Fig.3.1) obtained by adding new vertices $u_1, u_2, \dots, u_{p-k-1}$ and join each $u_i (1 \leq i \leq p - k - 1)$ with v and w.

Let $M = \{v_1, v_2, \dots, v_{k-4}\}$ be the set of all extreme vertices of G. By theorem 1.4, every total monophonic dominating set contains M. Clearly, M is not a total monophonic dominating set. It is clear that, $M_1 = M \cup \{u, v, y\}$ is the unique minimum total monophonic dominating set of G. Also, $M_2 = M_1 \cup \{w\}$ is a connected total monophonic dominating set of G, so that $\gamma_{cm_t}(G) = k - 1 + 1 = k$.

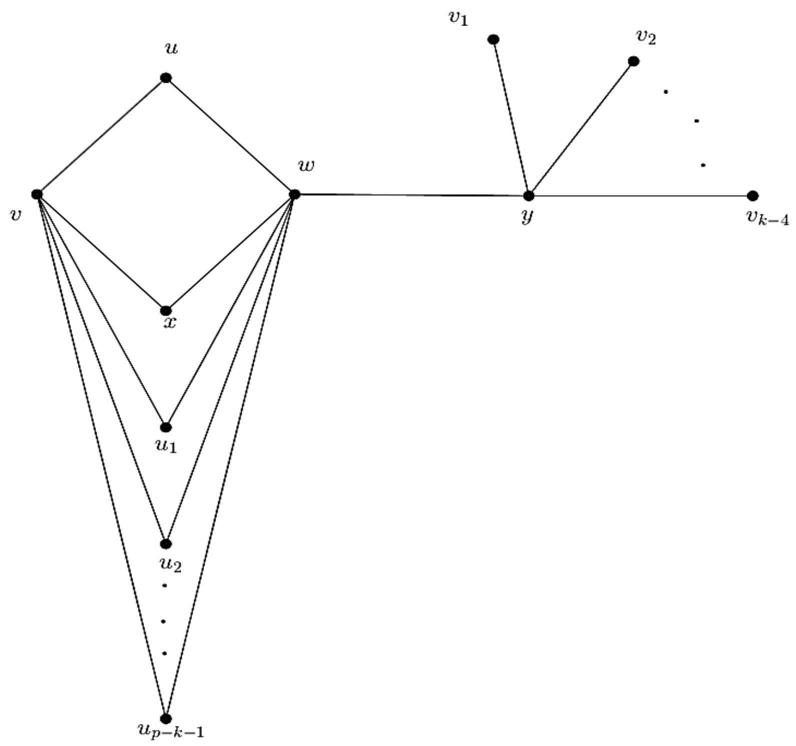

Fig. 3.1

Theorem 3.2: For any positive integers $2 < a < b < c$, there exists a connected graph G, such that $m(G) = a$, $\gamma_m(G) = b$ and $\gamma_{cm_t}(G) = c$.

Proof: For each integer i, with $1 \leq i \leq b-a-1$, let $F_i: u_i, v_i, w_i$ be a path of order 3. Let $C: z_1, t, s, r, w, z, z_1$ be a cycle of order 6. Let H be a graph obtained from F_i and C_6 by joining the vertex w of C_6 to the vertices $w_i (1 \leq i \leq b-a-1)$. Take a copy of star $K_{1,a}$ with leaves $u, z, x_1, x_2, \dots, x_{a-2}, w_{b-a-1}$ and the support vertex w. Subdivide the edges wx_i where $1 \leq i \leq a-2$, calling the new vertices $y_1, y_2, \dots, y_{c-b-3}$ where x_i is adjacent to y_i for all $i \in \{1,2,\dots,c-b-3\}$. Join wu with v is adjacent to u and w. Join each u_i to the vertex u. Let G be the graph obtained by adding two vertices z_2 and z_3 to both z and t. The graph G is shown in Figure 3.2.

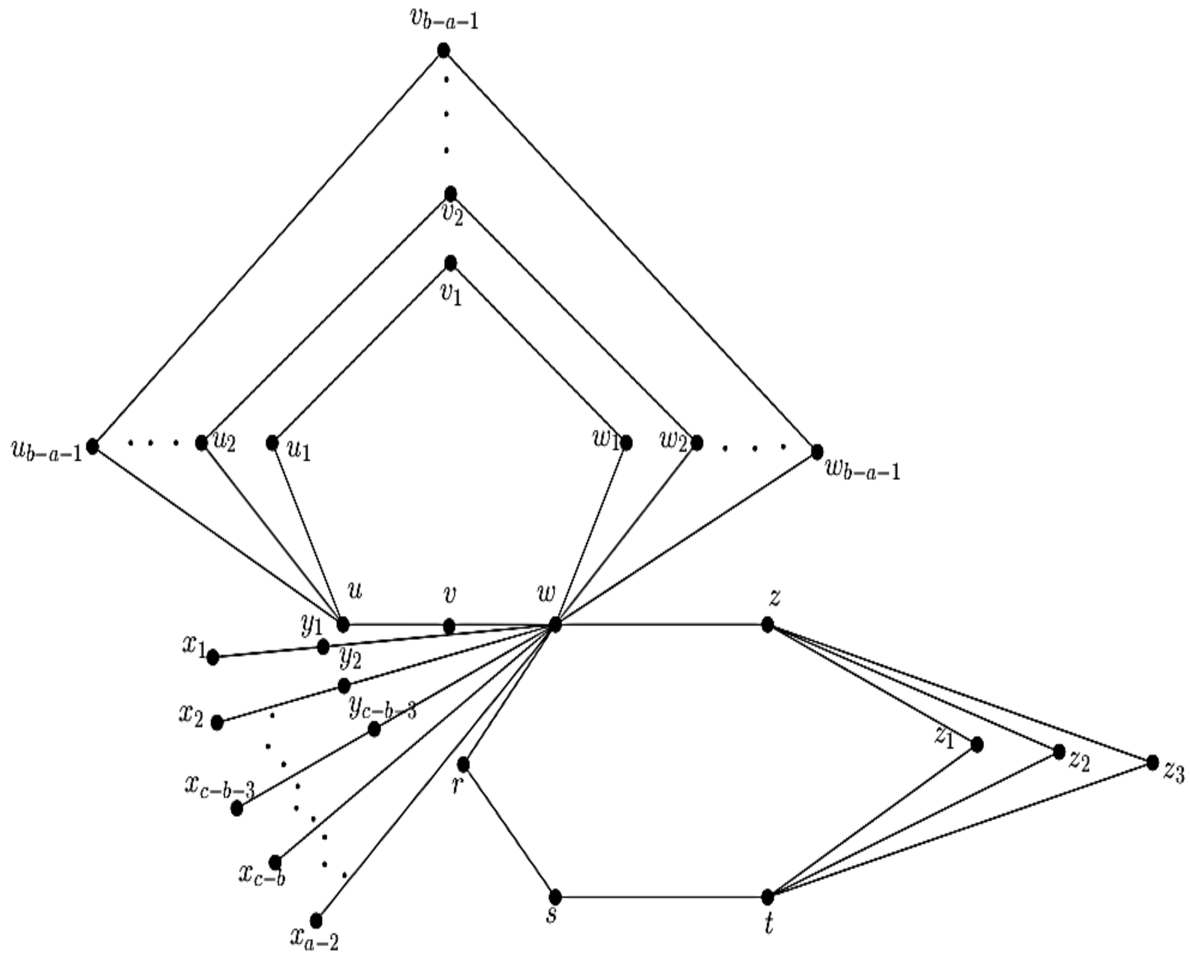

Fig. 3.2

The set of extreme vertices $M_1 = \{x_1, x, \dots, x_{a-2}\}$ is contained in a minimum monophonic set of G. It is clear that $u, t \notin M_1$, $M_2 = M_1 \cup \{u, t\}$ is a monophonic set of G. Therefore, $m(G) = a - 2 + 2 = a$. Let $M_3 = M_2 \cup \{w, w_1, w_2, \dots, w_{b-a-1}\}$ is the minimum monophonic domination set of G, so that $\gamma_m(G) = a + 1 + b - a - 1 = b$. Let $M_4 = M_3 \cup \{y_1, y_2, \dots, y_{c-b-3}, v, s\}$ is the minimum total monophonic domination set of G. Let $M_5 = M_4 \cup \{r\}$, note that , every connected total monophonic domination set of G contains M_1. It is clear that M_5 is a connected total monophonic domination set of G, so that $\gamma_{cm_t}(G) = c$.

CONCLUSION

We can extend the concept of connected total monophonic domination number to find the connected total edge monophonic domination number, upper connected total monophonic domination number of composition of graphs and so on.

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