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Generalisation of Idempotent Fuzzy Matrices

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Abstract

In this paper, we introduce and study the concept of *k* – Idempotent fuzzy matrix as a generalization of Idempotent fuzzy matrix via permutations.

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INTRODUCTION

A fuzzy matrix $A = [a_{ij}]_{n\times n}$ is said to be idempotent if, and only if $A^2 = A$ .H. Y. Lee et. al. [1] has discussed the concept of idempotent fuzzy matrices. The notion, k – idempotent matrices introduced by Krishnamoorthy et. al.[3] as a generalization of idempotent matrices is associated and motivated by the parameter *k* – idempotent; we introduce and study a new characteristic k – idempotent fuzzy matrix in this paper. If a fuzzy matrix A is obtained by k – permuting the elements of A^2, then it is called *k* – idempotent. Here *k* is the fixed product of disjoint transposition in S_n – the symmetric group of order *n*. In this paper, some characterization of a k – idempotent fuzzy matrices are examined such as sum and product of two k-idempotent fuzzy matrices are k- idempotent. Furthermore, we show that some properties for k – idempotent fuzzy matrices which will be intended to provide further discussions. For a matrix $A = (a_{ij})_{n\times n}$, A^T, $adjA$ and $detA$ denotes the transpose, adjoin and determinant of the fuzzy matrix *A*. Let 'k' be a fixed product of disjoint transpositions in S_n, the set of all permutation on {1, 2,…, n}. Hence it is involuntary (that is k_2 = identity permutation). A square matrix is called a permutation matrix [4] if every row and every column contains exactly one '1' and all the other entries are '0'.In this paper, the index set $\{1,2,...n-1,n\}$ will be denoted by *N*. By the Prop. 2.4.5 in [4], $adjA = A^C$, where A^C is idempotent and $c \le n-1$. The operations +, . and – are defined as follows:

$$a+b = \max\{a,b\},\ a.b = \min\{a,b\}$$

$$\text{and } a-b = \begin{cases} a & if\ a > b \\ 0 & if\ a \le b \end{cases}$$

CHARACTERIZATIONS OF k – IDEMPOTENT FUZZY MATRICES

Definition 2.1

For a fixed product of disjoint transposition $k \in S_n$, a matrix $A = (a_{ij})_{n\times n}$ is said to be *k*-idempotent if $KA^2K = A$, where *K* is the associated permutation fuzzy matrix of '*k*'. The associated permutation fuzzy matrix K is a matrix with one on its southwest – northeast diagonal and zeros everywhere else.

That is, $K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Example 2.2

Let $A = \begin{pmatrix} 1 & 0.1 & 0.1 \\ 0.4 & 1 & 0.2 \\ 0.7 & 0.6 & 0.8 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

then $KA^2K = \begin{pmatrix} 1 & 0.1 & 0.1 \\ 0.4 & 1 & 0.2 \\ 0.7 & 0.6 & 0.8 \end{pmatrix} = A$

Hence *A* is a *k*- idempotent fuzzy matrix.

Remark 2.3

$KA^2K = A$ implies that $KAK = A^2$. From the definition, the following relations can also be obtained which would be useful in computational aspect.

$$K^2A = AK^2 = A;\ KA^3K = A^3 \text{ and } KA^2K = A^2$$

Theorem 2.4

A *k* – idempotent fuzzy matrix A is ciruculant [4] if and only if *AK* = *KA*.

Proof.

Assume that *AK* = *KA*

Pre multiplying by K, we have

$$KAK = A$$

But $A^2 = A$ {since A is k – idempotent

Hence A is k – idempotent.

The converse is also true by retracing the steps.

Next, we are examine some basic properties of idempotent fuzzy matrices. We know that all 1×1 fuzzy matrices are k – idempotent. Hence, in this paper, we deal only with square fuzzy matrix that dimension $n, n \geq 2$. Let F_I be the set of all idempotent fuzzy matrices.

Lemma 2.5

A fuzzy matrix is k - idempotent if and only if all its zero patterns [1] are idempotent.

By the above Lemma 2.4, we examine the properties of $(0,1)-$ fuzzy matrices and obtain a theorem and canonical form of the $(0,1)-$ fuzzy matrices. Thus we will be able to to charecterise the structure of the set of all idempotent fuzzy matrices, F_I.

Lemma 2.6

The set of all idempotent fuzzy matrices, F_I is closed under the following operations

(i) Permutation similarity

(ii) Transposition.

Remark 2.7

The product of the permutation matrix K must be identity.

i.e., $K^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} . \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$, the identity matrix.

In particular if $k(i) = i$, then the associated permutation matrix K reduces to identity matrix and k-idempotent fuzzy matrix reduces to idempotent fuzzy matrix.

Lemma 2.8

Let $A = (a_{ij})_{n\times n}$ be an k-idempotent fuzzy matrix. If $a_{ij} = 0$ for some i and j in N, then each product $a_{ik}a_{kj} = 0$ for all k in N.

Proof.

It is an immediate consequence of the fuzzy matrix product.

Remark 2.9

Let A be k – idempotent fuzzy matrix, then A^T is also k – idempotent.

Example 2.10

Let $A = \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.2 \\ 0.7 & 0.6 & 0.8 \end{pmatrix}$ be k-idempotent,

$$A^T = \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.2 \\ 0.7 & 0.6 & 0.8 \end{pmatrix}$$

and also $K\left(A^T\right)^2 K = A^T$.

Hence A^T is also k – idempotent.

Proposition 2.11

If the fuzzy matrix A is k – idempotent, then $adjA$ is also k – idempotent.

Proof.

We know that $AdjA = A^C$, where A^C is idempotent and $c \leq n-1.$ [4]

Since A is k-idempotent, $KA^2K = A$

Also, $K\left(A^C\right)^2 K = A^C$

Hence $A^C = adjA$ is k-idempotent.

Proposition 2.12

If A is k – idempotent, then the fuzzy matrix $AadjA$ is also k – idempotent.

Proof.

$$\begin{aligned} K(AadjA)^2 K &= K\left(A^2 (adjA)^2\right)K \\ &= KA^2K.K(adjA)^2 K \\ &= AadjA \end{aligned}$$

Hence *AadjA* is *k*-idempotent.

Lemma 2.13

Let A and B are two *k* – idempotent fuzzy matrices, then $\det(A)+\det(B)=\det(A+B)$ and $\det A.\det B=\det(AB)$.

Example 2.14

$$A=\begin{pmatrix}0.85 & 0.5\\ 0.5 & 0.85\end{pmatrix}; B=\begin{pmatrix}0.5 & 0.5\\ 0.5 & 0.5\end{pmatrix}$$

$$\det A=0.85 \ \& \ \det B=0.5$$

$$A+B=\begin{pmatrix}0.85 & 0.5\\ 0.5 & 0.85\end{pmatrix} \ \& \ AB=\begin{pmatrix}0.5 & 0.5\\ 0.5 & 0.5\end{pmatrix}$$

Now, $\det A+\det B=0.85+0.5$

$= 0.58$

$=\det(A+B)$

and $\det A.\det B=(0.85).(0.5)$

$= 0.5$

$=\det(AB)$

SOME OPERATIONS ON *k*-IDEMPOTENT FUZZY MATRICES

Proposition 3.1

Let *A* and *B* be two *k*-idempotent fuzzy matrices. Then *A+B* is *k* – idempotent fuzzy matrix.

Proposition 3.2

Let *A* and *B* be two *k*-idempotent fuzzy matrices. If *AB=BA,* then *AB* is also a *k*-idempotent fuzzy matrix

Proof.

$$K(AB)^2K=KA^2K.KB^2K$$
$$=A.B$$

Hence the fuzzy matrix *AB* is *k*-idempotent.

The following theorem gives the generalization of products of *k* – idempotent matrices.

Theorem3.3

If $A_1, A_2, ..., A_n$ be a *k*-idempotent fuzzy matrices belonging to a commuting family of matrices, then $\prod_{i=1}^{n} A_i$ is a *k* – idempotent fuzzy matrix.

Proof.

$$K\left(\prod_{i=1}^{n} A_i\right)^2 K = K\left(A_1A_2...A_n.A_1A_2...A_n\right)^2 K$$
$$=K\left(A_1^2A_2^2...A_n^2\right)K$$
$$=KA_1^2K.KA_2^2K....KA_n^2K$$
$$=A_1.A_2...A_n$$
$$=\prod_{i=1}^{n} A_i$$

Hence the fuzzy matrix $\prod_{i=1}^{n} A_i$ is a *k*-idempotent.

Definition 3.4

For a pair of *k* – idempotent fuzzy matrices *A* and *B*, the commutator of *A* and *B* is denoted by *[A,B]* and defined by $[A,B]=AB-BA$.

Remark 3.5

If *A* and *B* are two *k* – idempotent fuzzy matrices then *A+B* is *k* – idempotent if and only if $[A,B]=AB$.

If *A* and *B* are two *k* – idempotent fuzzy matrices then *AB* is *k* – idempotent if and only if $[A,B]=0$.

Example :3.6

Consider the idempotent fuzzy matrix $A=\begin{pmatrix}0.8 & 0 & 0.5\\ 0 & 1 & 0\\ 0.5 & 0 & 0.8\end{pmatrix}$

and it is also commutes with the associated permutation matrix $K=\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$, that is *AK = KA.*

We see that *A* is idempotent.

Lemma 2.4 fails if we relax the condition of commutability of matrix A and K.

Theorem 3.7

If A and B are two k – idempotent fuzzy matrices then $A(A+B)B$ commutes with the permutation matrix K.

Proof.

$$\begin{aligned} A(A+B)B &= A^2B+AB^2 \\ &= KA^2KB+AKB^2K \\ &= KAB^2K+KA^2BK \\ &= K(AB^2+A^2B)K \\ &= K(A^2B+AB^2) \\ &= KA(A+B)BK \end{aligned}$$

Hence $KA(A+B)B = A(A+B)BK$

Theorem 3.8

Let A and B are two commuting k – idempotent fuzzy matrices. The k – idempotency of $A(A+B)B$ necessarily implies that is a null matrix.

Proof.

For any two k – idempotent fuzzy matrices A and B, we have $A(A+B)B$ commutes with the permutation matrix K in proposition 3.2.

If $A(A+B)B$ is k – idempotent then by Lemma 3.6, it reduces to an idempotent matrix.

i.e., $[A(A+B)B]^2 = A(A+B)B$ -------- (3.1)

i.e., $[A^2B]+(AB^2)^2+A^2BAB^2+AB^2A^2B=A^2B+AB^2$

Since A and B are k – idempotent fuzzy matrices, we have $A^4=A$ and $B^4=B$.

Hence (3.1) becomes,

$$AB^2+ A^2B+A^3B^3+A^3B^3 = A^2B+AB^2$$

i.e., $2A^3B^3 = 0$

i.e., $A^3B^3 = 0$

i.e., $(AB)^3 = 0$ -------- (3.2)

Since A and B are commuting k – idempotent fuzzy matrices, AB is also k – idempotent by

Hence $(AB)^4=AB$

Pre – multiplying equation (3.2) by AB,

$$(AB)^4 = 0$$

i.e., $AB = 0$

It follows that $A(A+B)B=0$.

Remark 3.10

For example, the matrices A and B in Example 2.13 can be considered.

$$A(A+B)B = A^2B+AB^2$$

$$= \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

Clearly, the matrix $A(A+B)B$ commutes with the permutation matrix K. it can be easily verified that $A(A+B)B$ is not a k – idempotent.

CONCLUSION

Clearly, the study of this kind of canonical forms is important to develop the theory of a fuzzy matrix. The concept of k – idempotent fuzzy matrix is generalized to periodic matrices or n – potent fuzzy matrix (i.e., $KA^nK = A$).

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