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New Subclass of Univalent Functions Defined By Using Generalized Al-Oboudi Differential Operator

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Abstract

The main object of this paper is investigating a new subclass of normalized analytic function in the open unit disk U which is defined by

Al-Oboudi Differential Operator. In which we obtain the co-efficient inequality and extreme points for this class.

Keywords: Univalent functions, differential operators, generalized Salagean differential operator, generalized Al-Oboudi differential operator.

INTRODUCTION

In the study of Geometric function theory, the univalent function is very attractive as we can see in recent years; many new articles are written in this area. The famous Biberbatch conjecture [1] in 1916 had given tremendous impact in the study of analytic univalent functions. This conjecture states that for every function $f \in S$, we have $|a_n| \le n$ for every n. Since f is normalized by $f(0)=0$ and $f'(0)=1$ then the second co-efficient $|a_2| \le 2$ if and only if f is the rotation of Koebe [2] function. He also conjectured that $|a_n| \le n$, (n = 2,3,.......) [8] which are generally valid.

Now operators of normalized analytic functions become very popular, namely for Differential and Integral. Many articles discuss on operators and new generalization of various authors. Perhaps Ruscheweyh [3] was leading the way in the differential operator who introduced on 1975. It followed by Salagean [4] in 1983 giving another version of differential and integral operator. Many properties have been discussed and studied many researchers for this two operators. In 2004, Al-Oboudi [5] generalized Salagean operator followed by S.Sumer Eker and S.Owa and S.Sumer and H.Ozelem ([6], [7] , [11]). In this study we use these operators ([9],[10]) to find another type of Differential Operators and obtain co-efficient inequalities and extreme points.

DEFINITIONS AND PRELIMINARIES

Definition 1.1

Let A denote the class of functions f normalized by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

Which are analytic in the open unit disc
$U = \{z \in C : |z| < 1\}$.

For $f \in A$, Al-Oboudi [4] introduces the following operator.

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D' f(z) = (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \ge 0 \tag{1.3}$$

$$D^n f(z) = D_\delta(D^{n-1} f(z)), \qquad n \in N = 1,2,3,.... \tag{1.4}$$

$$\therefore D^n f(z) = z + \sum_{j=2}^{\infty} \left[1+(j-1)\delta\right]^n a_j z^j, \quad n \in N_0 = N \cup \{0\}. \tag{1.5}$$

If $\delta = 1$, then we get Salagean [3] differential operator.

Definition: 1.2

Let $S(m,n,\alpha,\beta,\delta,b)$ denoted the sub class of A consisting of functions f which satisfy the inequality

$$\operatorname{Re}\left(1+\frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)\right) > \beta\left|\frac{D^m f(z)}{D^n f(z)}-1\right| + \alpha \tag{1.6}$$

For some $0 \le \alpha < 1, \beta \ge 0, b \in C - \{0\}, m \in N$, $n \in N \cup \{0\}, \delta\ (\delta \ge 0)$ and all z $\in$ U.

COEFFICIENT INEQUALITIES

Theorem 2.1

Let $f \in A$ satisfy

$$\phi(m,n,\alpha,\beta,\delta,b) \le 2(1-\alpha)b \qquad (2.1)$$

For some $0 \le \alpha < 1, \beta \ge 0, b \in C - \{0\}, m \in N$, $n \in N \cup \{0\}, \delta\ (\delta \ge 0)$ and all z $\in$ U .Then $f \in S(m,n,\alpha,\beta,\delta,b)$.

Where

$$\phi(m,n,\alpha,\beta,\delta,b) = \left(\begin{array}{l} \left|(1+(j-1)\delta)^m - (1+\alpha b)(1+(j-1)\delta)^n\right| \\ \quad + \left((1+(j-1)\delta)^m + ((2-\alpha)b-1)(1+(j-1)\delta)^n\right) \\ \quad + 2b\beta\left|(1+(j-1)\delta)^m - (1+(j-1)\delta)^n\right| \end{array} \right) \qquad (2.2)$$

Proof:

Suppose that $\phi(m,n,\alpha,\beta,\delta,b) \le 2(1-\alpha)b$ is true For some $0 \le \alpha < 1, \beta \ge 0, b \in C - \{0\}, m \in N$, $n \in N \cup \{0\}, \delta\ (\delta \ge 0)$ and all z $\in$ U, then It is suffices to prove that $\left|\frac{F(z)-1}{F(z)+1}\right| < 1$ .For $f \in A$, then define the function F(z) by

$$F(z) = \left(1 + \frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)} - 1\right)\right) - \beta\left|\frac{D^m f(z)}{D^n f(z)} - 1\right| - \alpha \qquad (2.3)$$

We know that,

$$F(z) - 1 = \left(\frac{D^m f(z) - (1+\alpha b)D^n f(z) - b\beta\left|D^m f(z) - D^n f(z)\right|}{bD^n f(z)}\right) \qquad (2.4)$$

and

$$F(z) + 1 = \frac{D^m f(z) - [1+(\alpha-2)b]D^n f(z) - b\beta\left|D^m f(z) - D^n f(z)\right|}{bD^n f(z)} \qquad (2.5)$$

$$\therefore \left|\frac{F(z)-1}{F(z)+1}\right| = \left|\frac{D^m f(z) - (1+\alpha b)D^n f(z) - b\beta\left|D^m f(z) - D^n f(z)\right|}{D^m f(z) - [1+(\alpha-2)b]D^n f(z) - b\beta\left|D^m f(z) - D^n f(z)\right|}\right| < 1$$

$$\sum_{j=p+1}^{\infty} \left(\begin{array}{l} \left|(1+(j-1)\delta)^m - (1+\alpha b)(1+(j-1)\delta)^n\right| \\ \quad + \left((1+(j-1)\delta)^m + ((2-\alpha)b-1)(1+(j-1)\delta)^n\right) \\ \quad + 2b\beta\left|(1+(j-1)\delta)^m - (1+(j-1)\delta)^n\right| \end{array} \right) \left|a_j\right| \le 2(1-\alpha)b$$

$$\therefore \quad \phi(m,n,\alpha,\beta,\delta,b) \le 2(1-\alpha)b$$

This completes the theorem.

Remarks:

On specializing the parameters we obtain well known classes of analytic function,

1. If we let b=1 and $\beta = 0$ then the class $S(m,n,\alpha,\beta,\delta,b)$ reduces to the form $\mathrm{Re}\left(\frac{D^m f(z)}{D^n f(z)}\right) > \alpha$ which is analogs to the class $S_{m,n,\delta}(\alpha)$ introduced by Sevtap Sumer Eker and H. Ozlem Guney [11] .
2. If we let $\beta = 0$ then the class $S(m,n,\alpha,\beta,\delta,b)$ reduces to the form $\mathrm{Re}\left(1 + \frac{1}{b}\left(\frac{D^m f(z)}{D^n f(z)} - 1\right)\right) > \alpha$ which is analogs to the class $S_{m,n,b,\delta}(\alpha)$ introduced by M.Thirucheran and T.Stalin, [12] .

EXTREME POINTS

If we define the new subclass $\tilde{S}(m,n,\alpha,\beta,\delta,b) \subset S(m,n,\alpha,\beta,\delta,b)$, which consists of the function

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, (a_j \ge 0) \qquad (3.1)$$

Whose Taylor-Maclaurin coefficient satisfy the inequality (2.1)

Now we determine the extreme points of the subclass $\tilde{S}(m,n,\alpha,\beta,\delta,b)$

Theorem 3.1

Let $f_1(z) = z$ and

$$f_j(z) = z + \sum_{j=2}^{\infty} \eta_j \frac{2(1-\alpha)b}{\phi(m,n,\alpha,\beta,\delta,b)} z^j, (j = 2,3,4,...) \qquad (3.2)$$

Then $f \in S(m,n,\alpha,\beta,\delta,b)$

if it can be expressed in the form

$$f(z) = \sum_{j=1}^{\infty} \eta_j f_j(z) \qquad (3.3)$$

Where $\eta_j > 0$ and $\sum_{j=1}^{\infty} \eta_j = 1$.

Proof:

Suppose that

$$f(z) = \sum_{j=1}^{\infty} \eta_j f_j(z) \qquad (3.4)$$

$$= z + \sum_{j=2}^{\infty} \eta_j \frac{2(1-\alpha)b}{\phi(m,n,\alpha,\beta,\delta,b)} z^j$$

$$=\sum_{j=2}^{\infty}\eta_j \frac{2(1-\alpha)b}{\phi(m,n,\alpha,\beta,\delta,b)}\phi(m,n,\alpha,\beta,\delta,b)$$

$$=2(1-\alpha)b\sum_{j=2}^{\infty}\eta_j$$

$$=2(1-\alpha)b(1-\eta_1)$$

$$\leq 2(1-\alpha)b \qquad (3.5)$$

Which is shows that f satisfies the condition (2.1)

$$f\in\widetilde{S}(m,n,\alpha,\beta,\delta,b).$$

Conversely ,

Suppose that $f\in\widetilde{S}(m,n,\alpha,\beta,\delta,b)$.

Since ,

$$a_j \leq \frac{2(1-\alpha)b}{\phi(m,n,\alpha,\beta,\delta,b)},(j=2,3,...) \qquad (3.6)$$

Let

$$\eta_j \leq \frac{\phi(m,n,\alpha,\beta,\delta,b)}{2(1-\alpha)b}a_j \text{ and}$$

$$\eta_1 = 1-\sum_{j=2}^{\infty}\eta_j$$

Then we obtain

$$f(z)=\sum_{j=1}^{\infty}\eta_j f_j(z)$$

This is completes the proof of theorem 3.1

INTEGRAL MEANS OF INEQUALITIES

Definition 4.1

Let two functions f and g are analytic in U, *we* say that $f(z)$ is subordinate to $g(z)$ if there exist a function $w(z)$ analytic in U satisfying $w(0)=0$ and $|w(z)|<1$ such that $f(z)=g(w(z))$

It is denoted by $f(z)\prec g(z)$.

Lemma 4.2

Let $g(z)$ is univalent in U. Then $f(z)\prec g(z)$ if and only if $f(0)=g(0)$ and $f(U)\subset g(U)$.(Duran[7]).

Theorem 4.3

If f and g are analytic in U with $f(z)\prec g(z)$, then $\int_0^{2\pi}|f(z)|^{\mu}d\theta \leq \int_0^{2\pi}|g(z)|^{\mu}d\theta$ for $\mu>0$ and $z=re^{i\theta},0<r<1$.(Littlewood [8])

Theorem 4.4

Let $f\in\widetilde{S}(m,n,\alpha,\beta,\delta,b)$ and suppose that $f(z)$ is defined by

$$f(z)=z+\frac{2(1-\alpha)b\,\varepsilon_j}{\phi(m,n,\alpha,\beta,\delta,b)}z^j,(j=2,3,........,|\varepsilon_j|=1$$

. If there exist an analytic function $w(z)$ given by $\{w(z)\}^{j-1}=\frac{\phi(m,n,\alpha,\beta,\delta,b)}{2(1-\alpha)b\varepsilon_j}\sum_{j=2}^{\infty}a_jz^{j-1}$, then for $z=re^{i\theta},0<r<1$,

$$\int_0^{2\pi}|f(re^{i\theta})|^{\mu}d\theta \leq \int_0^{2\pi}|g(re^{i\theta})|^{\mu}d\theta,\mu>0 .$$

Proof

We must show that

$$\int_0^{2\pi}\left|1+\sum_{j=2}^{\infty}a_jz^{j-1}\right|^{\mu}d\theta \leq \int_0^{2\pi}\left|1+\frac{2(1-\alpha)b\,\varepsilon_j}{\phi(m,n,\alpha,\beta,\delta,b)}z^{j-1}\right|^{\mu}d\theta$$

By the help of Littlewood subordination theorem, its suffices to shoe that

$$1+\sum_{j=2}^{\infty}a_jz^{j-1}\prec 1+\frac{2(1-\alpha)b\,\varepsilon_j}{\phi(m,n,\alpha,\beta,\delta,b)}z^{j-1}$$

Let $1+\sum_{j=2}^{\infty}a_jz^{j-1}=1+\frac{2(1-\alpha)b\,\varepsilon_j}{\phi(m,n,\alpha,\beta,\delta,b)}(w(z))^{j-1}$

Therefore

$$(w(z))^{j-1}=\frac{\phi(m,n,\alpha,\beta,\delta,b)}{2(1-\alpha)b\varepsilon_j}\sum_{j=2}^{\infty}a_jz^{j-1}$$

Hence $w(0)=0$.

Furthermore, using (2.1)

$$|(w(z))|^{j-1}=\left|\frac{\phi(m,n,\alpha,\beta,\delta,b)}{2(1-\alpha)b\varepsilon_j}\sum_{j=2}^{\infty}a_jz\right|^{j-1}$$

$$\leq\frac{\phi(m,n,\alpha,\beta,\delta,b)}{2(1-\alpha)b}\sum_{j=2}^{\infty}|a_j||z|^{j-1}$$

$\leq |z| < 1.$

Hence theorem completed.

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