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General Study on Polynomials Associated With Humbert Polynomials

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Abstract

In this paper, we address a polynomial associated with Humbert polynomials. The main objects of this paper is to provide a natural further step towards the unified presentation of a class of Humbert's polynomials which generalizes the well known class of Gegenbauer, Legendre, Pincherle, Horadam, Horadam-Pethe, Gould, Milovanovic Dordevic polynomials and many not so well known polynomials, we shall give some basic relations involving the generalized Humbert polynomials and then take up several generating functions, hypergeometric representations and finite series representations of some relatively more familiar polynomials Legendre, Gegenbauer, Hermite and Laguerre etc.

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INTRODUCTION

Gould [7] presented a systemetic study of an interesting generalization of Humbert, Gegenbauer and several other polynomial system defined by

$$\sum_{n=0}^{\infty} P_n(m, x, y, p, c)t^n = (c - mxt + yt^m)^p \qquad \ldots(1.1)$$

where m is a positive integer and other parameters are unrestricted in general. For the table of main special cases of (1.1) including Gegenbauer, Legendre, Pincherle and Humbert polynomials, see Gould [7], Pathan and Khan [8], Milovanovic and Dordevie [5] considered the polynomial $\{p_{n,m}^{\lambda}\}_{n=0}^{\infty}$ defined by the generating function

$$G_m^{\lambda}(x, t) = (1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} p(x)t^n \qquad \ldots(1.2)$$

where $m \in N$ and $\lambda > -1/2$. Note that,

for m = 1, $p_{n,1}^{\lambda}(x) = \dfrac{(\lambda)_n (2x-1)^n}{n!}$ (Horadam Polynomials [2]),

for m = 2, $p_{n,2}^{\lambda}(x) = C_n^{\lambda}(x)$ (Gegenbauer Polynomials),

for m =3, $p_{n,3}^{\lambda}(x) = p_{n+1}^{\lambda}(x)$ (Horadum-Pethe Polynomial [3]).

where

$$(\lambda)_0 = 1, (\lambda)_n = (\lambda+1)(\lambda+2)\ldots(\lambda+n-1); n = 1,2,3\ldots.$$

The explicit form of the polynomial $p_{n,m}^{\lambda}(x)$ is

$$p_{n,m}^{\lambda}(x) = \sum_{k=0}^{[n/m]} \frac{(-1)^k (\lambda)_{n-(m-1)k} (2x)^{n-mk}}{k!(n-mk)!} \qquad \ldots(1.3)$$

A generalization of various polynomials mentioned above is provided by the definition

$$\sum_{n=0}^{\infty} \Psi_n^{\nu}(x, y)t^n = (c - axt + yt^m)^{-\nu} \qquad \ldots(1.4)$$

where $m \in N$ and $\nu > -1/2$.

For c = y = 1, a = 2, $\nu = \lambda$ then (1.4) reduces to (1.2).

i.e. $\psi_n^{\nu}(x,1) = p_{n,m}^{\lambda}(x),$

for a = m, $\nu = -p$ then (1.4) reduces to (1.1)

i.e. $\psi_n^{\nu}(x, y) = p_n(m, x, y, p, c).$

Now with the help of [1],

$$(1-z)^{-a} = {}_1F_0[a; -; z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \qquad \ldots(1.5)$$

Since by [6; p.22(21)]

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k}; 0 \le k \le n \qquad \ldots(1.6)$$

And by [8; p.58 and p.55]

$$(n-mk)! = \frac{(-1)^{mk}\, n!}{(-n)_{mk}}; 0 \le mk \le n \qquad \ldots(1.7)$$

$$(-n)_{mk} = m^{mk} \prod_{s=0}^{m} \left(\frac{-n+s-1}{m}\right)_k \qquad \ldots(1.8)$$

$$(1-\nu-n)_{(m-1)k} = (m-1)^{(m-1)k} \prod_{p=1}^{m-1} \left(\frac{-\nu-n+p}{m-1}\right)_k ; \quad \ldots(1.9)$$

$$(t+\nu)^n = \sum_{k=0}^{n} \frac{n!\,t^k \nu^{n-k}}{k!(n-k)!}, k = 0,1,2,3,\ldots \quad \ldots(1.10)$$

In this paper we shall give some basic relations involving the generalized polynomial $\psi_n^\nu(x,y)$ and then take up several operational results, series representations, hypergeometric representations and generating functions of other polynomials, which are stated in terms of the generalized polynomials. This is interesting since as will be shown, the polynomials $\psi_n^\nu(x,y)$ contain [7], [10], [11] [4; p.164 (1)] as special cases.

Finite Series Representation For ψ_n^ν (x,y)

Here we obtain the following two finite series representation for $\psi_n^\nu(x,y)$ viz.

$$\text{(i) } \psi_n^\nu(x,y) = \sum_{k=0}^{[n/m]} \frac{(-1)^k c^{-\nu-n+(m-1)k}}{k!(n-mk)!} (\nu)_{n+(1-m)k} (ax)^{n-mk} y^k \quad \ldots(2.1)$$

$$\text{(ii) } \psi_n^\nu(x,y) = \sum_{k=0}^{[n-(m-2)s/2]} \sum_{s=0}^{k} \frac{c^{-\nu-n+(m-1)s} (\nu)_k (-k)_s}{k!(n-2k-(m-2)s)!\,s!}$$

$$\times (2\nu+2k)_{n-2k-(m-2)s} \left(\frac{ax}{2}\right)^{n-ms} y^s \quad \ldots(2.2)$$

Proof of (2.1)

Let $$S = \sum_{n=0}^{\infty} \psi_n^\nu(x,y)\, t^n \quad \ldots(2.3)$$

Now using (1.4) and (1.5) in (2.3), we get

$$S = c^{-\nu} \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \left(\frac{axt - yt^m}{c}\right)^n$$

Again by (1.10) and by series rearrangement techniques, we get

$$S = c^{-\nu} \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(\nu)_{n-(m-1)k} (-1)^k (axt)^{n-mk} (yt^m)^k}{k!(n-mk)!\, c^{n-mk+k}} \quad \ldots(2.4)$$

Now equating (2.3) and (2.4) and comparing the coefficient of t^n on both side, we get the finite series representation (2.1) for $\psi_n^\nu(x,y)$.

Proof of (2.2)

From (1.4), we have

$$\sum_{n=0}^{\infty} \psi_n^{\nu}(x,y)\,t^n = c^{-\nu}\left[1-\frac{axt}{c}+\left(\frac{axt}{2c}\right)^2-\left(\frac{axt}{2c}\right)^2+\frac{yt^m}{c}\right]^{-\nu}$$

$$= c^{-\nu}\left[1-\frac{axt}{2c}\right]^{-2\nu}\left[1-\left\{\frac{\left(\frac{axt}{2c}\right)^2-\frac{yt^m}{c}}{\left(1-\frac{axt}{2c}\right)^2}\right\}\right]^{-\nu}$$

With the help of the equation (1.5), we get

$$= c^{-\nu}\sum_{k=0}^{\infty}\frac{(\nu)_k}{k!}\left[\left(\frac{axt}{2c}\right)^2-\frac{yt^m}{c}\right]^k\left[1-\frac{axt}{2c}\right]^{-2\nu-2k}$$

$$= c^{-\nu}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\nu)_k(2\nu+2k)_n}{k!\,n!}\left(\frac{axt}{2c}\right)^{n+2k}\left[1-\frac{\frac{yt^m}{c}}{(axt/2c)^2}\right]^k$$

$$= c^{-\nu}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{s=0}^{k}\frac{(\nu)_k(2\nu+2k)_n(-k)_s}{k!\,n!\,s!}\left(\frac{ax}{2c}\right)^{n+2k-2s}\left(\frac{y}{c}\right)^s t^{n+2k+(m-2)s}$$

Replacing n by n – 2k – (m – 2)s, we get

$$S = c^{-\nu}\sum_{n=0}^{\infty}\sum_{k=0}^{[(n-(m-2)s)/2]}\sum_{s=0}^{k}\frac{(\nu)_k(2\nu+2k)_{n-2k-(m-2)s}(-k)_s}{k!(n-2k-(m-2)s)!\,s!}\left(\frac{ax}{2c}\right)^{n-ms}\left(\frac{y}{c}\right)^s t^n$$

Comparing the coefficient of t^n on both side, we get the finite series representation (2.2) for $\psi_n^{\nu}(x,y)$.

HYPERGEOMETRIC REPRESENTATION FOR ψ_n^{ν} (x,y)

The finite series representation (2.1) for $\psi_n^{\nu}(x,y)$ is of particular interest to us in obtaining the following hypergeometric form for $\psi_n^{\nu}(x,y)$ viz.

$$\psi_n^{\nu}(x,y) = \frac{c^{-\nu-n}(\nu)_n(ax)^n}{n!}$$

$$\times\, {}_mF_{m-1}\left[\begin{matrix}\frac{-n}{m},\frac{-n+1}{m},\ldots,\frac{-n+m-1}{m};\\ \frac{-\nu-n+1}{m-1},\ldots,\frac{-\nu-n+m-1}{m-1};\end{matrix}\left(\frac{-m}{ax}\right)^m\frac{y}{c^{1-m}(m-1)^{m-1}}\right] \qquad \ldots(3.1)$$

Proof: With the help of equations (1.6) and (1.7), equation (2.1) reduces to

$$\psi_n^{\nu}(x, y = c^{-\nu-n} \sum_{k=0}^{[n/m]} \frac{(\nu)_n (-n)_{mk} (ax)^{n-mk} y^k}{(1-\nu-n)_{(m-1)k} (-1)^{mk}\, n!\, k!\, c^{(1-m)k}}$$

Again with the help of equations (1.8) and (1.9), we arrive at (3.1).

Generating Function For ψ_n^{ν} (x,y)

We now obtain the following additional generating function for $\psi_n^{\nu}(x, y)$ viz.

$$\sum_{n=0}^{\infty} \frac{(e)_n \psi_n^{\nu}(x, y) t^n}{(\nu)_n} = \sum_{n=0}^{\infty} \frac{(e)_n c^{\nu-n} (axt)^n}{n!}$$

$$\times\, {}_{m+1}F_m \left[\begin{matrix} \frac{e+n}{m}, \frac{e+n+1}{m}, \ldots, \frac{e+n+m-1}{m}; \nu+n; \\ \frac{-\nu+n}{m}, \frac{\nu+n+1}{m}, \ldots, \frac{\nu+n+m-1}{m}; \end{matrix} \frac{-yt^m}{c} \right] \quad \ldots(4.1)$$

Proof: From (2.1), we have

$$\sum_{n=0}^{\infty} \frac{(e)_n \psi_n^{\nu}(x, y) t^n}{(\nu)_n} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(e)_n c^{\nu-n} (\nu)_{n-(m-1)k} (-1)^k (ax)^{n-mk} y^k t^n}{(\nu)_n c^{(1-m)k} (n-mk)!\, k!}$$

Replacing n by n+mk, we get

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(e)_{n+mk} (-1)^k c^{-\nu-n-mk} (\nu)_{n+k} (ax)^n y^k t^{n+mk}}{(\nu)_{n+mk} c^{(1-m)k}\, n!\, k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(e)_n (e+n)_{mk} (-1)^k c^{-\nu-n-k} (\nu+n)_k (ax)^n y^k t^{n+mk}}{(\nu+n)_{mk}\, n!\, k!}$$

$$= \sum_{n=0}^{\infty} \frac{(e)_n c^{-\nu-n} (axt)^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k c^{-k} (e+n)_{mk} (\nu+n)_k y^k t^{mk}}{(\nu+n)_{mk}\, k!}$$

Now using (1.8), we arrive at

$$\sum_{n=0}^{\infty} \frac{(e)_n \psi_n^{\nu}(x, y) t^n}{(\nu)_n} = \sum_{n=0}^{\infty} \frac{(e)_n c^{-\nu-n} (axt)^n}{n!} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{m} \left(\frac{e+n+j-1}{m} \right)_k}{\prod_{p=1}^{m-1} \left(\frac{\nu+n+p-1)}{m} \right)_k} \left(\frac{-yt^m}{c} \right)^k$$

which is equivalent to (4.1).

Special Cases

I. For a = m; c = y = 1 then (2.1) and (2.2) reduces to

$$h_{n,m}^{\nu}(x)=\sum_{k=0}^{[n/m]}\frac{(-1)^k(\nu)_{n+(1-m)k}(mx)^{n-mk}}{k!(n-mk)!}, \quad \ldots(5.1)$$

$$h_{n,m}^{\nu}(x)=\sum_{k=0}^{[(n-(m-2)s)/2]}\sum_{s=0}^{k}\frac{(\nu)_k(-k)_s(2\nu+2k)_{n-2k-(m-2)s}}{k!(n-2k-(m-2)s)!\,s!}\left(\frac{mx}{2}\right)^{n-ms} \quad \ldots(5.2)$$

where $h_{n,m}^{\nu}(x)$ is Humbet polynomial [10].

Equations (5.1) and (5.2) are known results [8; p.56 (2.6 and 2.7)].

II. For m = a = 3, ν = 1/2 in (5.1) and (5.2), we get

$$P_n(x)=\sum_{k=0}^{[n/3]}\frac{(-1)^k(1/2)_{n-2k}(3x)^{n-2k}}{k!(n-3k)!} \quad \ldots(5.3)$$

$$P_n(x)=\sum_{k=0}^{[(n-s)/2]}\sum_{s=0}^{k}\frac{(1/2)_k(-k)_s(1+2k)_{n+2k-s}}{(n-2k-s)!k!s!}\left(\frac{3x}{2}\right)^{n-3s} \quad \ldots(5.4)$$

where $P_n(x)$ is **Pincherle polynomial** [9].

Equations (5.3) and (5.4) are known results [8; p.56 (2.8 and (2.9)].

III. For a = m = 2 and ν = 1/2 then (5.1) give the finite series representation of Legendre polynomial [4; p.161(1)].

IV. For a = m = 2, then (5.1) give finite representation of Gegenbauer polynomial. [4; p.277(2)].

V. For a = m, y = c= 1 in (3.1), we get the following hypergeometric representation of Humbert polynomial

$$h_{n,m}^{\nu}(x)=\frac{(\nu)_n(mx)^n}{n!}\times{}_mF_{m-1}\left[\begin{matrix}\frac{-n}{m},\frac{-n+1}{m},\ldots,\frac{-n+m-1}{m};\\ \frac{-\nu-n+1}{m-1},\ldots,\frac{-\nu-n+m-1}{m-1};\end{matrix}\ \frac{(-1)^m}{(m-1)^{m-1}x^m}\right] \quad \ldots(5.5)$$

which is a known result [8; p.58 (3.6)].

VI. For m =2, (5.5) gives hypergeometric representation of Gegenbauer polynomial

$$C_n^{\nu}(x)=\frac{(\nu)_n(2x)^n}{n!}\,{}_2F_1\left[\begin{matrix}\frac{-n}{2},\frac{-n+1}{2};\\ 1-\nu-n\ \ ;\end{matrix}\ \frac{1}{x^2}\right] \quad \ldots(5.6)$$

which is a generalization of a known result [4; p.66(4)].

VII. For a = m, y = c = 1 in (4.1), we get the generating function for Humbert polynomial

$$\sum_{n=0}^{\infty}\frac{(e)_n h_{n,m}^{\nu}(x)\,t^n}{(\nu)_n}$$

$$=\sum_{n=0}^{\infty}\frac{(e)_n(mx)^n t^n}{(\nu)_n}\,{}_{m+1}F_m\left[\begin{matrix}\nu+n,\frac{e+n}{m},\ldots,\frac{e+n+m-1}{m};\\ \frac{\nu+n}{m},\frac{\nu+n+1}{m},\ldots,\frac{\nu+n+m-1}{m};\end{matrix}\ -t^m\right] \quad \ldots(5.7)$$

which is a known result [8; p.60(4.7)].

For e = ν, further reduces to a known result [6; p.86(26)].

VIII. For m = 3, ν =1/2 in (5.7), we get the generating function for Pincherle polynomial $P_n(x)$

$$\sum_{n=0}^{\infty}\frac{(e)_n\,p_n(x)t^n}{(1/2)_n}$$

$$=\sum_{n=0}^{\infty}\frac{(e)_n(3xt)^n}{n!}\,{}_4F_3\left[\begin{matrix}\frac{1}{2}+n,\frac{e+n}{3},\frac{e+n+1}{3},\frac{e+n+2}{3};\\ \frac{\frac{1}{2}+n}{3},\frac{\frac{3}{2}+n}{3},\frac{\frac{5}{2}+n}{3};\end{matrix}\ -t^3\right]$$

which is a known result [8; p.60(4.8)].

For e = 1/2 which further reduces to a known result [7; p.697].

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