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Weakly Integer Additive Set Indexers of Cartesian Product of Graphs

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Abstract

Let G(V, E) be a graph of order p and size q. An integer additive set-indexer(IASI) is defined as an injective function f: V(G) $\to 2^{Z^+}$ such that the induced function f^+: E(G) $\to 2^{Z^+}$ defined by f^+(uv) = f(u)+f(v) is also injective. If | f^+(uv)| =k for all uv∈ E(G) then f is said to k- uniform IASI . An IASI f is said to be weak IASI if | f^+(uv) |=max | |f(u)|, |f(v)| | for all uv∈ E(G).

In this paper, we study the admissibility of weak IASI for Cartesian Product of some graphs.

Keywords: Graph Labeling, Set Labeling, Set-Indexer, Cardinality of the Labeling Set, Integer Additive Set-Indexer, k-Uniform IASI, Weak(or Strong) IASI.

INTRODUCTION

Through out this paper by the word graph we mean a finite, undirected graph without loop and multiple edges having atleast one edge. The order and size of a graph used in this paper are p and q respectively. In our main research work we used a graph with labeling. So first we give some introduction of labeling and graph labeling and we also refer to the book by Harary and Berge respectively [1,2,3]. For notations and terminology we follow Bondy and Murthy[15,16].

Naming an object to identify it uniquely is a common human practice, continued from time immemorial. Labeling is a term used in technical sense for naming objects, drawn from any universe of discourse such as the set of numbers, algebraic groups and the power set of a non empty set. The objects requiring labeling could come from a variety of fields of human interest such as chemical elements, radio antennae, plants and animal species. The complexity involved in having a desired kind of labeling of a discrete structure generally arise due to the constraints inherent in the conditions required to the imposed as well as the nature of the universe of discourse from which the labels are drawn. Here graph labeling is an assignment of mathematical objects(mainly sets) to vertices and edges. Various type of graph labeling have been investigated during the last three decades. Some such labeling are magic labeling, antimagic labeling, cordial labeling, set magic labeling, harmonious labeling, prime labeling, strongly multiplicative labeling and orthogonal labeling etc[18]. There is an enormous amount of literature built up on several kinds of numerical labeling of graphs[4]. For terms that are not defined here are used in the sense of Harary[5,6].Acharya introduced a new type of graph labeling in which subsets of a set are assigned to graph elements, instead of numbers as in earlier literature on graph labeling and set-assignment on graphs mainly stem from behavior sciences. First evidence of using assignment of subsets of a given set to the arcs of a digraph appears in a seminal paper by Peay[11], also for a genres all reference of the set assignment notions, see [1,2,3,7]

Let G=(V, E) be a graph and X, Y, Z be any sets(may be finite or infinite). By a set assignment we may any of the functions f : V(G) $\to 2^X$, f : E(G) $\to 2^Y$ or f : V(G) U E(G) $\to 2^Z$ .A set assignment is called a set labeling if it is injective. The most survey of set assignment can be seen in [8,9,10].

Let G(V, E) be a graph having a non empty set X of cardinality n, a set-indexer of G is an injective set-valued function f : V(G) $\to 2^X$ such that the function $\hat{f}$: E(G) $\to 2^X$- {∅ } defined by $\hat{f}$(uv)= f(u) Δ f(v) is also injective for every edge uv ∈ E(G), where Δ denotes the symmetric difference of two sets. Every graph has a set-indexer, see[12]. In this paper, we are particularly interested in the Cartesian product of two graphs which admits weak IASI labeling. In such a situation, first we define Cartesian product of graphs. The purpose of this paper is to study the weak IASI labeling problem for the cartesian product of two graphs, which was defined by Whitehead and Russell in 1912. Given any two graphs G and H, the cartesian product of two graphs denoted by G×H, is defined by:

Let G and H be any two graphs then G×H, is the graph with vertex set V(G)×V(H), where the vertices (u, $\dot{u}$) and (v, $\dot{v}$) are adjacent iff u=v and $\dot{u}\dot{v}$ ∈E(H) or $\dot{u} = \dot{v}$ and uv ∈ E(G) (see Fig 1).

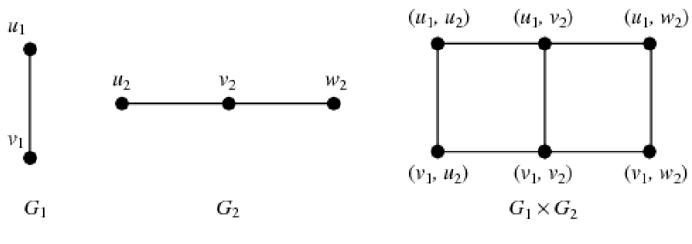

Figure 1: cartesian product of P_2 and P_3 i.e. $P_2 \times P_3$

An injective function f : V(G) $\to 2^{Z^+}$ such that the induced function f^+ : E(G) $\to 2^{Z^+}$ assigning to each edge the sum of the sets assigned to its end vertices is also injective is called an integer additive set- indexer (IASI). The sum of two sets A, B ⊆ Z^+ is denoted by A+B and is defined by A+B= { a+b; a∈A, b∈B }. The cardinality of the labeling set of an element (edge or vertex) of a graph is called the set-indexing number of that element. An IASI is said to be k-uniform if $|f^+(e)|$=k for all e∈E(G)(see Fig 2). Either we can say that a connected graph G have a k-uniform IASI if all of its edges have the same set-indexing number. Anandavally and Germina defined the concept of k-uniform IASI and studied the properties of 1-uniform IASI sum square graphs and presented the characterizations of 2-uniform IASI[13, 14].

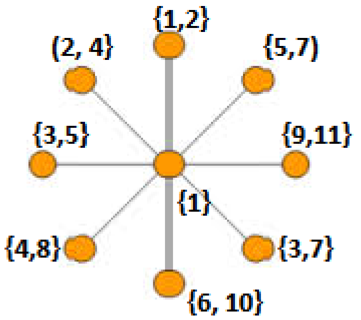

Figure 2: 2-uniform IASI graph

Germina and Anandavally[17] gave a necessary condition for a graph to be IASI. For competence, we include following lemma, theorems and definition independently :

Lemma 1.1: Let A, B⊆ Z^+, then max (|A|, |B|) ≤ |A+B| ≤ |A| |B|.

Thus, by above lemma we have for IASI f of a graph G max | |f(u)|, |f(v)| | ≤ | f^+(uv) |=|f(u)+f(v)| ≤ |f(u)| |f(v)|.

Definition 1.2: An IASI f of a graph G is said to be weak IASI if max | |f(u)|, |f(v)| |= | f^+(uv) |=|f(u)+f(v)| for all u, v in V(G). A graph having weak IASI is said to be a weak IASI graph and called weakly uniform IASI if | f^+(uv) |=k, where k is some positive integer. For example the star graph S_n is always weak IASI (see Fig 3).

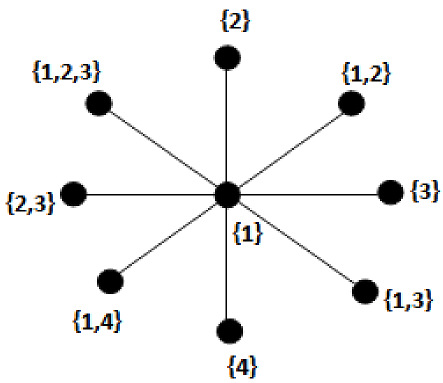

Figure 3: weak IASI labeling of S_8

Theorem 1.3: If G is weak IASI graph then every subgraph of G is also weak IASI.

Definition 1.4: Let f be an IASI on G , if for vertices u, v in G | f^+(uv) |=|f(u)+f(v)| = |f(u)| |f(v)|, then G is said to be strong IASI. A graph having strong IASI is said to be a strong IASI graph and called strong uniform IASI if | f^+(uv) |=k, where k is some positive integer. For example the star graph S_n is always weak IASI (see Fig 3).

Definition 1.5: Caterpiller graph is a sequence of stars S_1 U S_2 U…..U S_r where each S_i is a star with central vertex c_i and n_i leaves for i=1, 2,….,r, and the leaves of S_i includes c_{i-1} and c_{i+1}, for i=2, 3, ….r-1. Or the caterpillar $Sn_1, n_2,\ldots,n_r$, having vertex set V($Sn_1, n_2,\ldots,n_r$)= { c_i :1≤ i ≤r} U $\bigcup_{i=2}^{r-1}\{x_i^j\ 2 \le j \le n_1 - 1\}$ U $\{x_i^j\ 2 \le j \le n_1 - 1\}$U $\{x_r^j\ 2 \le j \le n_r\}$ and the edge set E($Sn_1, n_2,\ldots,n_r$)= { $c_i\ c_{i+1}$:1≤ i ≤r-1} U $\bigcup_{i=2}^{r-1}\{\mathrm{ci}x_i^j\ 2 \le j \le n_1 - 1\}$ U $\{\mathrm{cr}x_i^j\ 2 \le j \le n_1 - 1\}$U $\{x_r^j\ 2 \le j \le n_r\}$.

The rest of the paper is organized as follows. In section 2 we investigate properties of weak IASI graphs.

WEAKLY INTEGTER ADDITIVE SET INDEXER GRAPHS

We first focus on Cartesian product of graphs, first considering the more general case with one arbitrary parameter and one fix. We know that for any two path P_n and P_n V($P_n \times P_n$) ={ v_{ij} : 1≤ i ≤ n and 1≤ j ≤ n } be the vertex set and E($P_n \times P_n$) ={ $v_{ij}\ v_{i+1\, j}$: 1≤ i ≤ n-1 and 1≤ j ≤ n }U{ $v_{ij}\ v_{i\, j+1}$: 1≤ i ≤ n and 1≤ j ≤ n-1 }be the edge set. Clearly | V($P_n \times P_n$) | = n^2 and |E($P_n \times P_n$) | =2(n^2–n).

Theorem.2.1: The Cartesian product $P_3 \times P_4$ is weak integer additive set indexer.

Proof. Define the required weak IASI in the following way f : V(G) →2^{Z^+}

$$f(a_{1,j}) = \begin{cases} j & if\ j = 1,3 \\ (1, j+1) & if\ j = 2,4 \end{cases}$$

$$f(a_{2,j}) = \begin{cases} (1, j+1) & if\ j = 1,3 \\ j & if\ j = 2,4 \end{cases}$$

$$f(a_{3,j}) = \begin{cases} (1{,}3 + (j+1)) & if\ j = 1,3 \\ (3 + j + 1) + 1 & if\ j = 2,4 \end{cases}$$

It can be easily seen that f is injective and the induced function f^+ : E(G) →2^{Z^+}defined by f^+(uv) = f(u)+f(v) is also injective. It also satisfies the condition of weak IASI which is | f^+(uv) | =max | |f(u)|, |f(v)| | for all uv∈ E(G). So $P_3 \times P_4$ is weak IASI graph.

It is not difficult to check that, in the above considered case, the function f consists of $\left\lceil \frac{n \times m}{2} \right\rceil$+3 consecutive integers (see Fig 4).

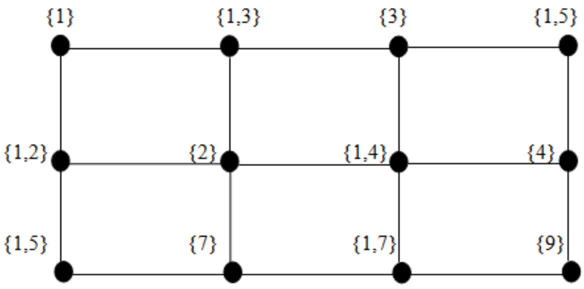

Figure 4: weak IASI labeling of $P_3 \times P_4$

Theorem 2.2: The Cartesian product $P_5 \times P_4$ admits a weak IASI labeling .

Proof. Consider the following injection

$$f : V(P_5 \times P_4) \rightarrow 2^{Z^+}$$

where

$$f(a_{1,j}) = \begin{cases} j & if\ j = 1,3 \\ (1, j+1) & if\ j = 2,4 \end{cases}$$

$$f(a_{2,j}) = \begin{cases} (1, j+1) & if\ j = 1,3 \\ j & if\ j = 2,4 \end{cases}$$

$$f(a_{3,j}) = \begin{cases} \left[\frac{3}{2}\right] 4 + j & if\ j = 1,3 \\ \left(1, \left(\left[\frac{3}{2}\right] 4 + j\right) + 1\right) & if\ j = 2,4 \end{cases}$$

$$f(a_{4,j}) = \begin{cases} \left(1, \left(\left[\frac{3}{2}\right] 4 + j\right) + 1\right) & if\ j = 1,3 \\ \left(\left[\frac{3}{2}\right] 4 + j\right) + 1 & if\ j = 2,4 \end{cases}$$

$$f(a_{5,j}) = \begin{cases} \left[\frac{5}{2}\right] 4 + j & if\ j = 1,3 \\ \left(1, \left(\left[\frac{5}{2}\right] 4 + j\right) + 1\right) & if\ j = 2,4 \end{cases}$$

We can see that f is a weak IASI and it uses $\left[\frac{5\times4}{2}\right]$+3 positive integers.

Next, we show that if P_n is a path of order n and P_4 is a path of order 4, then $P_n \times P_4$ is weak IASI.

Theorem 2.3: The Cartesian product $P_n \times P_4$ is weakly IASI for every $n \geq 3$.

Proof. Let $P_n \times P_4$ be the Cartesian product of two paths. We define the following vertex labeling

$f : V(P_n \times P_4) \rightarrow 2^{Z^+}$ as follows:

$$f(v_{i,j}) = \begin{cases} \left[\frac{i}{2}\right] 4 + j & 1 \leq i \leq n\ and\ i \not\equiv 0 (mod\ 2), j = 1,3 \\ \left(1, \left[\frac{i}{2}\right] 4 + j + 1\right) & 1 \leq i \leq n\ and\ i \not\equiv 0 (mod\ 2), j = 2,4 \end{cases}$$

$$f(v_{i,j}) =)= \begin{cases} \left(1, \left[\frac{i-1}{2}\right] 4 + j + 1\right) & 1 \leq i \leq n\ and\ i \equiv 0 (mod\ 2), j = 1,3 \\ \left[\frac{i-1}{2}\right] 4 + j & 1 \leq i \leq n\ and\ i \equiv 0 (mod\ 2), j = 2,4 \end{cases}$$

One can then verify that f is a weak IASI and in this case f requires atleast $\left[\frac{n\times4}{2}\right]$+3 positive integers.

In view of the above theorems, it is interesting to know more about weakly IASI for Cartesian product of graphs.

Theorem 2.4: The Cartesian product $P_n \times P_n$ is weakly IASI for every n.

Proof. Let G= $P_n \times P_n$ and V($P_n \times P_n$) ={ v_{ij} : $1\leq i \leq$ n and $1\leq j \leq$ n } be the vertex set with E($P_n \times P_n$) ={ $v_{ij}\, v_{i+1\, j}$: $1\leq i \leq$ n-1 and $1\leq j \leq$ n }U{ $v_{ij}\, v_{i\, j+1}$: $1\leq i \leq$ n and $1\leq j \leq$ n-1 } the edge set. Clearly | V($P_n \times P_n$) | = n^2 and |E($P_n \times P_n$) | =2(n^2–n).

Define the vertex labeling f : V(G) $\rightarrow 2^{Z^+}$ in the following way.

Case (i). If n is even and $1\leq i, j \leq$ n then we label the vertices in the following way

$$f(v_{i,j}) = \begin{cases} \left[\frac{i}{2}\right] n + j & if\ i\ and\ j\ are\ odd \\ \left(1, \left(\left[\frac{i}{2}\right] n + j\right) + 1\right) & if\ i\ is\ odd\ and\ j\ is\ even. \end{cases}$$

$$f(v_{i,j}) = \begin{cases} \left(1, \left(\left[\frac{i-1}{2}\right] n + j\right) + 1\right) & if\ i\ is\ even\ and\ j\ is\ odd \\ \left[\frac{i-1}{2}\right] n + j & if\ i\ and\ j\ are\ even\ . \end{cases}$$

Then f is a weakly IASI labeling of G. The case requires $\frac{n^2}{2}$+1 minimum positive integers.

Case (ii). If n is odd and $1\leq i, j \leq$ n then we label the vertices in the following way.

$$f(v_{i,j}) = \begin{cases} \left[\frac{i}{2}\right] n + j & if\ i\ and\ j\ are\ odd \\ \left(1, \left(\left[\frac{i}{2}\right] n + j\right) + 1\right) & if\ i\ is\ odd\ and\ j\ is\ even. \end{cases}$$

$$f(v_{i,j}) = \begin{cases} \left(1, \left(\left[\frac{i-1}{2}\right] n + j\right) + 1\right) & if\ i\ is\ even\ and\ j\ is\ odd \\ \left[\frac{i-1}{2}\right] n + j & if\ i\ and\ j\ are\ even\ . \end{cases}$$

Of course f is a weakly IASI labeling of G. The case requires $\frac{n^2}{2}$+$\frac{n+1}{2}$ minimum positive integers.

Remark 2.1: Theorem 2.4 is a generalization of theorem 2.1, 2.2 as well as theorem 2.3.

By theorem 2.4 cartesian product of two paths is weakly IASI. Thus there exists a injective vertex labeling such that f : $V(P_n \times P_n) \rightarrow 2^{Z^+}$ such that the induced edge labeling $f^+ : E(G) \rightarrow 2^{Z^+}$ is one to one.In particular, Cartesian product of two paths of same order n is weakly IASI. In fact, as we will see soon, the more general results on Cartesian product of two graphs obtained in this work.

In the above section weak IASI is defined for cartesian product of paths, which is formalised in the next section.

SECTION 3

Theorem 3.1: The graph $P_n \times C_n$ is weak IASI for every n≥3.

Proof. $P_2 \times C_2$ is not weak IASI, because in this case edge function $f^+ : E(G) \rightarrow 2^{Z^+}$ is not injective or we can say that in this assignment the edge function can never be injective.(see Fig 5).

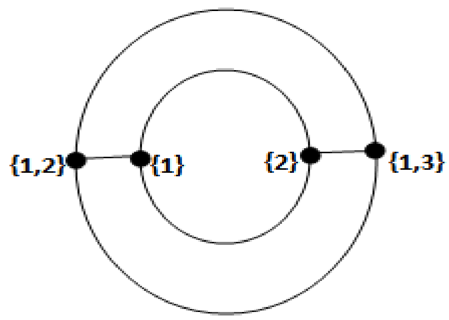

Figure 5: $P_2 \times C_2$, in this case $f^+ : E(G) \rightarrow 2^{Z^+}$ can never be injective

Now for n≥3, let G= $P_n \times C_n$ be a graph of order n^2 with size……..For graph G, let f : $V(G) \rightarrow 2^{Z^+}$ define by

$$f(u_i v_j) = \begin{cases} \left[\frac{i}{2}\right] n + j & \text{for } 1 \le i,j \le n \text{ and both odd} \\ \left(1, \left(\left[\frac{i}{2}\right] n + j\right) + 1\right) & \text{for } 1 \le i,j \le n \text{ and odd } i, \text{ even } j \end{cases}$$

$$f(u_i v_j) = \begin{cases} \left(1, \left(\left[\frac{i}{2}\right] n + j\right) + 1\right) & \text{for } 1 \le i,j \le n \text{ and even } i, \text{ odd } j \\ \left[\frac{i}{2}\right] n + j & \text{for } 1 \le i,j \le n \text{ and both even} \end{cases}$$

One can verify that f is weak IASI of graph G. It is not difficult to verify that the set of edge weights is injective(see Fig 6).

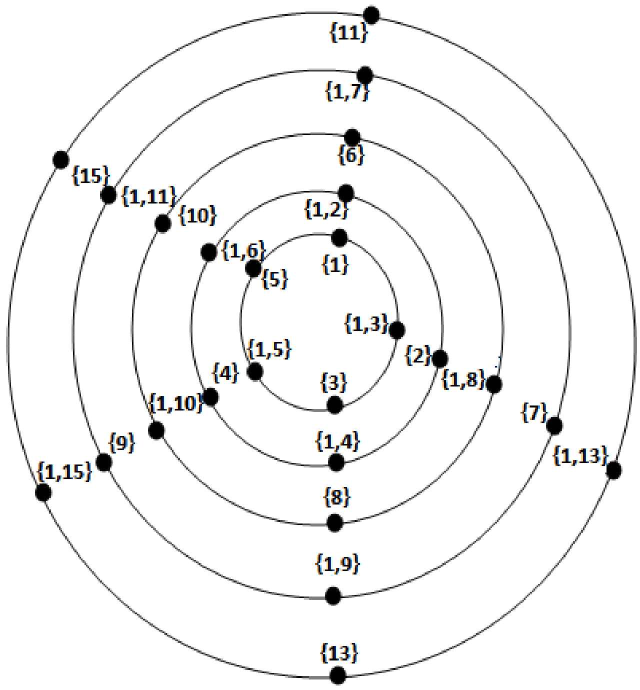

Figure 6: weak IASI labeling of $P_5 \times C_5$

Theorem 3.2: There is a weak IASI labeling for every caterpiller $Sn_1, n_2, \ldots, n_r$.

Proof. Let $Sn_1, n_2, \ldots, n_r$ be the caterpillar with the order of vertices $c_1, x_1^1, x_1^2, x_1^3, \ldots, x_1^{n_1-1}, c_2, x_2^2, x_2^3, \ldots, x_2^{n_2-1}, c_3, x_3^2, x_3^3, \ldots, x_3^{n_3-1}, \ldots\ldots\ldots c_r, x_r^2, x_r^3, \ldots, x_r^{n_r}$. Now, we describe a vertex labeling $V(Sn_1, n_2, \ldots, n_r) \rightarrow 2^{Z^+}$ as follows :

Case (i). Choose the vertices c_i and x_i in sequence, if the chosen vertex is c_i then we label it by the singleton set {i} and we label consecutive sets {1,2}, {1, 3}……{1, n_1} to the leaves of tree S_1(i.e. to $x_1^1, x_1^2, x_1^3, \ldots, x_1^{n_1-1}$), then we label the leaves of S_2 $x_2^2, x_2^3, \ldots, x_2^{n_2-1}$ by the consecutive singleton sets {r+1}, {r+2}…….so on. After this leaves of S_3 with consecutive sets {1, n_1+1}, {1, n_1+2}……so on, until the central vertex c_r or all the leaves of S_r has been labeled (see Fig 7).

Note that x_i^j (1≤ i ≤r) are labeled with sets of order two for odd i and with singletons for even i.

Case (ii). If the centeral vertices c_i' s are labeled with singleton in case (i), we start by labeling the c_i by {1,2}, {1,3}……{1,r+1} in order and we label consecutive sets {1},{2}…….{n_1-1} to leaves of tree S_1, then then we label the leaves of S_2 $x_2^2, x_2^3, \ldots, x_2^{n_2-1}$ by the consecutive singleton sets {1,r+2},{1,r+3}…..so on, until the central vertex c_r or all the leaves of S_r has been labeled (see Fig 8).

Note that in this case x_i^j (1≤ i ≤r) are labeled with singleton sets for odd i and with sets of order 2 for even i.

Remark 3.1: Note that all the vertices of caterpiller Sn_1, n_2,……,n_r can be labeled with singleton sets or sets of order two, but in that case graph uniform IASI graph.

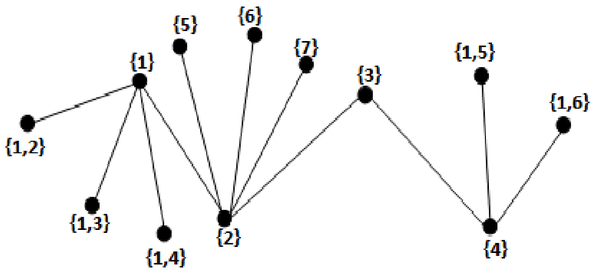

Figure 7: $S_{4,5,2,3}$ with weak IASI labeling

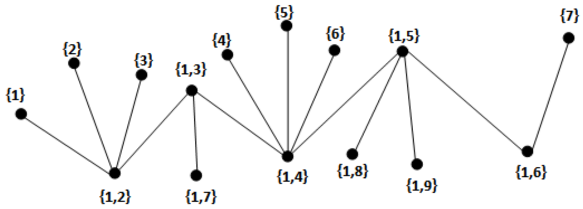

Figure 8: $S_{4,3,5,4,2}$ with weak IASI labeling

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