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A natural topology on the tensor power of real groups

Sung Myung
Department of Mathematics Education, Inha University,
100 Inharo, Nam-gu, Incheon, 22212 Korea.

Abstract: The tensor products and tensor powers of abelian groups play significant roles whenever biadditivity or multiadditivity is involved in mathematical models. The tensor products and powers are very well understood and researched algebraically, but rarely regarded as a topological spaces even when the abelian groups have natural structures of topological abelian groups. In the present article, we give a foundation to define a natural topology on the tensor product of a topological abelian group and study some of the topological properties when the underlying abelian group is related with the group of real or complex numbers. The most interesting case is the tensor powers of abelian groups which are the multiplicative group of units of topological fields because the Milnor's K-groups show up as their quotients. In these cases, we compare the topologies of the tensor powers and the Milnor's K-groups whose topologies are given in our earlier work.

Keywords: tensor product, topology, K-theory

INTRODUCTION

For two abelian groups $(A,+)$ and $(B,+)$, the tensor product $A\otimes B$ is defined to be the factor group F/J where F is the free abelian group generated by the set $\{(a,b) \mid a\in A, b\in B\}$ of pairs of elements whose first coordinates are in A and second coordinate coordinates are in B and J is the subgroup of F generated by the set of elements of the following forms (c.f., [1] or [8]):
$(a_1+a_2,b)-(a_1,b)-(a_2,b)$
$(a,b_1+b_2)-(a,b_1)-(a,b_2)$
for $a_1,a_2,a\in A$ and $b,b_1,b_2\in B$.

A tensor $a\otimes b$ denotes the element of $A\otimes B$ which is the coset of J containing the element (a,b). It can be easily deduced that $(na)\otimes b = a\times(nb) = n(a\otimes b)$ for every $a\in A$, $b\in B$ and $n\in\mathbb{Z}$ where $\mathbb{Z}$ is the ring of rational integers.

An abelian group $(A,+)$ is called a topological abelian group if the underlying set A is equipped with a topology such that its binary operation $A\times A\to A$ ($(a,b)\mapsto a+b$ and the inverse $A\to A$ ($a\mapsto -a$) are both continuous (c.f., Chapter 10 of [1]). Some of the most natural and easiest examples are the additive group $(\mathbb{R},+)$ of real numbers and the multiplicative group $(\mathbb{R}^{\times},\cdot)$ of nonzero real numbers.

Tensor product of abelian groups play such a tremendous role whenever there are bi-additive or multi-additive mathematical gadgets and the additive group of real numbers and the multiplicative group of real numbers are most basic algebraic structures for the purpose of engineering or scientific applications. But, one seldom bother to define a topology on the tensor product of these algebraic structures (over $\mathbb{Z}$). The one of difficulties is that a general element of $A\otimes B$ is not of the single form $a\otimes b$ but is rather a $\mathbb{Z}$-linear combination $\sum_{i=1}^{k} n_i(a_i\otimes b_i)$ where $a_i\in A$, $b_i\in B$ and $n_i\in\mathbb{Z}$. In any cases, we will have to require at least that $\mathbb{R}\times\mathbb{R}\to\mathbb{R}\otimes\mathbb{R}$ where (a,b) is mapped to $a\otimes b$ is continuous for the topology to be natural.

In case of $\mathbb{R}\otimes\mathbb{R}$, one plausible way to formulate a topology on the tensor product seems to be by considering it as an infinite dimensional $\mathbb{R}$-vector space with a scalar multiplication via its first coordinate, i.e., $r\sum_{i=1}^{k} a_i \otimes b_i = \sum_{i=1}^{k} ra_i \otimes b_i$ for $a_i \in A$, $b_i \in B$ and defining a suitable topology, e.g., strongest vector topology such that every embedding of a finite dimensional space into $\mathbb{R}\otimes\mathbb{R}$ is continuous (Example 2.3.3 of [2]). But, one may argue that this is not a desired topology since, for example, the image of the compact set $[0,1]\times[0,1]$ under the obvious map $\mathbb{R}\times\mathbb{R}\to\mathbb{R}\otimes\mathbb{R}$ generates an infinite dimensional subspace as $1\otimes b_1,\ldots,1\otimes b_n$ are linearly independent when $b_1,\ldots,b_n$ are linearly independent over $\mathbb{Q}$.

In Section , we give a definition of a most natural topology on the tensor products of topological abelian groups. This is done by regarding the tensor product as the quotient space of direct limit of the groups of pairs of diagonal matrices with coefficients in the topological abelian groups. A group of diagonal matrices is nothing but the additive group of a power of the topological abelian groups, but we insist to work with matrices to facilitate the comparison of the topology of the tensor power with the topology of the Milnor's K-group in Section 8.

In Section 3, we derive some topological properties on the tensor power of $\mathbb{R}$, $\mathbb{R}^\times$, $\mathbb{C}$ and $\mathbb{C}^\times$. All of these are continuously mapped onto the space of real numbers, but none of them turn out to be locally compact.

In Section 8, we take a look at the algebraic quotient map $T^l(k^\times)\to K_l^M(k)$ and show that it is actually continuous when k is a topological field. In case where k is algebraically closed, this map is shown to be a topological quotient map and thus the topology of $T^l(k^\times)$, which is much easier to define, can be used to determine the topology of $K_l^M(k)$ as in [13].

Our study is not compatible with the work done in [7] and [4] as their tensor product is an object in the category of locally compact abelian groups which has the universal property with respect to bilinear continuous morphisms. Our topology is given to the algebraic tensor product of abelian groups which happen to be topological groups and its purpose is to facilitate the understanding of the algebraic tensor powers and Milnor's K-theory of a topological field.

NATURAL TOPOLOGY ON THE TENSOR PRODUCT OF TOPOLOGICAL ABELIAN GROUPS

For a topological abelian group A, let $M_n(A)$ be the set of $n\times n$ square matrices with entries in A. Then $M_n(A)$ can be naturally regarded as the product of n^2 copies of A and is given the obvious product topology.

Let $D_n(A)$ be the set of $n\times n$ diagonal matrices with entries in A and we consider $D_n(A)$ as a subspace of $M_n(A)$.

For two topological groups A and B, let $M(A)\times M(B)$ be the direct limit $\bigcup_{n\to\infty} M_n(A)\times M_n(B)$ of the topological spaces $M_n(A)\times M_n(B)$ under the standard inclusion $M_n(A)\times M_n(B)\hookrightarrow M_{n+1}(A)\times M_{n+1}(B)$

$$\left(\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}\right) \mapsto \left(\begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & 0 \\ a_{n1} & \cdots & a_{nn} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1n} & 0 \\ \vdots & \ddots & \vdots & 0 \\ b_{n1} & \cdots & b_{nn} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}\right).$$

When A and B are multiplicative groups of topological fields, then it is understood that the right lower entry is the multiplicative identity 1. Now the set $M(A)\times M(B)$ is given with the direct limit topology (a.k.a. inductive limit topology) (c.f., [16]). That is, a subset U of $M(A)\times M(B)$ is open if and only if $U\cap(M_n(A)\times M_n(B))$ is open for every $n\geq 1$.

Using the above inclusion $M_n(A)\hookrightarrow M_{n+1}(A)$, $D_n(A)$ can be also regarded as a subspace of $D_{n+1}(A)$. Consequently, for topological abelian groups A and B, the topological space $D(A)\times D(B)$ is defined as the direct limit $\bigcup_{n\to\infty} D_n(A)\times D_n(B)$ and can be regarded naturally as a subspace of $M(A)\times M(B)$.

Now, we define the topology on $A\otimes B$ as follows.

Definition 1. For topological abelian groups A and B, let $p : D(A) \times D(B) \to A \otimes B$ be the map which sends

$$\left(\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots \end{pmatrix}, \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ddots \end{pmatrix} \right)$$

to $\sum_i a_i \otimes b_i$. We give $A \otimes B$ the quotient topology with respect to the surjective map $p : D(A) \times D(B) \to A \otimes B$.

In the above definition, p can be easily seen to be a surjective map. The quotient topology is the finest topology on $A \otimes B$ such that the map p is continuous (c.f., [11]).

Note 2. In the above construction, $D_n(A)$ and $D_n(B)$ have topological abelian group structures using matrix addition as their group operations. But, $D(A)$ or $D(B)$ is not necessarily a topological group with the direct limit topology and the group operation given by the direct limit of groups. A counterexample is given in [16]). On the other hand, by the same literature, $D(A)$ and $D(B)$ are topological groups when A and B are locally compact abelian groups, e.g., the additive group $\mathbb{R}$ or the multiplicative group $\mathbb{R}^{\times}$.

Even though it is not apparent that continuity of the group operations for A and B are incorporated in Definition 1, there are bunch of commutative diagrams of continuous maps which involve the continuity of group operations when A and B are topological abelian groups. For example,

$$\begin{array}{ccc} (D_n(A) \times D_n(A)) \times D_n(B) & \xrightarrow{(+, id_{D_n(B)})} & D_n(A) \times D_n(B) \\ & {\searrow}^{\gamma_n} \qquad {\swarrow}_{p} & \\ & A \otimes B & \end{array}$$

is a commutative diagram of continuous maps where $\gamma_n(L_1, L_2, M) = p(L_1, M) + p(L_2, M)$. The diagram obtained by passing to the direct limit

$$\begin{array}{ccc} (D(A) \times D(A)) \times D(B) & \xrightarrow{(+, id_{D(B)})} & D(A) \times D(B) \\ & {\searrow}^{\gamma} \qquad {\swarrow}_{p} & \\ & A \otimes B & \end{array}$$

is a commutative diagram of topological quotient maps when A and B are locally compact abelian groups.

Theorem 3. For locally compact second-countable abelian groups A and B, $A \otimes B$ is a topological abelian group with the topology given in Definition 1.

Proof. We define the map $f : (D(A) \times D(B)) \times (D(A) \times D(B)) \to D(A) \times D(B)$ by sending the pair of two elements

$$\left(\begin{pmatrix} a_1 & 0 & \cdots \\ 0 & a_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} b_1 & 0 & \cdots \\ 0 & b_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \right)$$

and

$$\left(\begin{pmatrix} a'_1 & 0 & \cdots \\ 0 & a'_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} b'_1 & 0 & \cdots \\ 0 & b'_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \right)$$

to the element

$$\left(\begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots \\ 0 & a'_1 & 0 & 0 & \cdots \\ 0 & 0 & a_2 & 0 & \cdots \\ 0 & 0 & 0 & a'_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} b_1 & 0 & 0 & 0 & \cdots \\ 0 & b'_1 & 0 & 0 & \cdots \\ 0 & 0 & b_2 & 0 & \cdots \\ 0 & 0 & 0 & b'_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right).$$

Then, f is just a shuffling of the coordinates and thus is a homeomorphism. The map f gives $D(A) \times D(B)$ a well-defined continuous binary operation (which is not associative). By the assumption of the theorem, $D_n(A) \times D_n(B)$ is locally compact and second countable and so Lindelöf ([11]). Therefore, the image $X_n = p(D_n(A) \times D_n(B))$ in $(A \otimes B) \times (A \otimes B)$ is a k_ω-space and $X_n \times X_n$ for any positive integers n is a k-space (Corollary 10 of [15]). This proves that $p \times p : (D_n(A) \times D_n(B)) \times (D_n(A) \times D_n(B)) \to (X_n \otimes X_n) \times (X_n \otimes X_n)$ is also a quotient map by Proposition 5.8 in Appendix A of [9]. Now we need to pass to the direct limit, but, in general, but it is not true that $\varinjlim (X_n \times X_n) = \varinjlim X_n \times \varinjlim X_n$ (e.g., [6]). Nevertheless, this equality is essentially shown to be true under that assumption that X_n are k_ω-spaces for all positive integers n in [5]). So, we may pass to the direct limit and conclude that $p \times p : (D(A) \times D(B)) \times (D(A) \times D(B)) \to (A \otimes B) \times (A \otimes B)$ is indeed a quotient map. Furthermore, it is easily seen that $p \circ f = + \circ (p \times p)$, i.e., the following diagram is commutative.

$$\begin{array}{ccc} (D(A) \times D(B)) \times (D(A) \times D(B)) & \xrightarrow{f} & D(A) \times D(B) \\ \downarrow{\scriptstyle p \times p} & & \downarrow{\scriptstyle p} \\ (A \otimes B) \times (A \otimes B) & \xrightarrow{+} & A \otimes B \end{array}$$

By the universal property of a quotient map $p \times p$, we see that the group operation $+$ for $A \otimes B$ is continuous. ■

PROPERTIES OF THE TENSOR POWERS OF GROUPS OF REALS AND COMPLEX NUMBERS

For an abelian group A and an integer $l \geq 1$, the l-th tensor power $T^l(A)$ is defined as the tensor product $\overbrace{A \otimes A \otimes \cdots \otimes A}^{l \text{ times}}$. Since we have an isomorphism $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$ of topological groups whenever A, B, C are abelian groups, we omitted parentheses in the description of l-th tensor power of A as the tensor product of l copies of A. In particular, $T^l(A)$ is a topological abelian group by Theorem 3 if A is a locally compact second-countable abelian group. In the present section, we investigate some properties of the topologies on $T^l(\mathbb{R})$, $T^l(\mathbb{R}^\times)$, $T^l(\mathbb{C})$ and $T^l(\mathbb{C}^\times)$.

Lemma 4. For a homomorphism $\alpha : A \to A'$ of locally compact second-countable abelian groups A and A', we have a natural morphism $T^l(\alpha) : T^l(A) \to T^l(A')$ of topological abelian groups which sends $a_1 \otimes \cdots \otimes a_l$ to $\alpha(a_1) \otimes \cdots \otimes \alpha(a_l)$. $T^l(\alpha)$ is surjective if α is surjective.

Proof. The Lemma follows immediately from the fact that, for homomorphisms $\alpha : A \to A'$ and $\beta : B \to B'$ of locally compact second-countable abelian groups, the group homomorphism $\alpha \otimes \beta : A \otimes B \to A' \otimes B'$ which sends $a \otimes b$ to $\alpha(a) \otimes \beta(b)$ is a morphism of topological abelian groups. This statement can be proved by Definition 1 and the fact that $\alpha \times \beta$ induces a continuous map from $D_n(A) \times D_n(B) \simeq A^n \times B^n$ to $D_n(A') \times D_n(B) \simeq A'^n \times B'^n$ for every integer $n \geq 1$, which then is passed to the direct limit as in the proof of Theorem 3. If α and β are surjective, then $\alpha \otimes \beta$ is surjective since the tensor product by an abelian group is a right exact functor (c.f., [1] or [17]). ■

If a tensor product by an abelian group preserves also "injectivity", then the abelian group is called "flat". It is well known that an abelian group is flat if and only if it is torsion-free, i.e., there is no element of finite order except the identity ([17]). If A and A' are flat and $\alpha : A \to A'$ is one-to-one, then $T^l(\alpha) : T^l(A) \to T^l(A')$ is also one-to-one.

Now we turn our attention to the tensor power of groups of real or complex numbers. Remark 2.3 of [14] gives a surjective continuous map $g_{\mathbb{R}} : T^l(\mathbb{R}) \to \mathbb{R}$ which sends $a_1 \otimes \cdots \otimes a_l$ to $a_1 a_2 \cdots a_l$. Similarly, we have a continuous surjective map $g_{\mathbb{R}^\times} : T^l(\mathbb{R}^\times) \to \mathbb{R}$ which sends $a_1 \otimes \cdots \otimes a_l$ to $\log(|a_1|)\log(|a_2|)\cdots\log(|a_l|)$. This shows that $T^l(\mathbb{R})$ and $T^l(\mathbb{R}^\times)$ are far from trivial as topological spaces. Similarly, as the map $g_{\mathbb{C}} : T^l(\mathbb{C}) \to \mathbb{C}$, which sends $a_1 \otimes \cdots \otimes a_l$ to $a_1 a_2 \cdots a_l$, and the map $g_{\mathbb{C}^\times} : T^l(\mathbb{C}^\times) \to \mathbb{R}$, which sends $a_1 \otimes \cdots \otimes a_l$ to $\log(|a_1|)\log(|a_2|)\cdots\log(|a_l|)$, are surjective continuous maps, we also see that both $T^l(\mathbb{C})$ and $T^l(\mathbb{C}^\times)$ are far from trivial. This is a big departure from the topologies on the Milnor's K-groups which turn out to be pretty primitive as $K_l^M(\mathbb{R})$ is a disjoint union of two indiscrete open sets and $K_l^M(\mathbb{C})$ has a trivial topology.

Proposition 5. If A is a connected topological abelian group, then $T^l(A)$ is connected for every $l \geq 1$.

Proof. Since $D_n(A)^l$ is connected for every $n \geq 1$, $D(A)^l$ is connected. So, its continuous image $T^l(A)$ under the quotient map $p : D(A)^l \to T^l(A)$ is also connected. ■

For example, $T^l(\mathbb{R})$, $T^l(\mathbb{C})$ and $T^l(\mathbb{C}^\times)$ are all connected topological abelian groups.

Proposition 6. $T^l(\mathbb{R}^\times)$ has two connected components for $l \geq 1$.

Proof. For any positive integer n, $D_n(\mathbb{R}^\times)^l$ has 2^{nl} connected components according to the signs of nl coordinates $a_{j,i}$ ($i = 1, \ldots, n$, $j = 1, \ldots, l$) of its elements

$$a = \left(\begin{pmatrix} a_{1,1} & & & \\ & a_{1,2} & & \\ & & \ddots & \\ & & & a_{1,n} \end{pmatrix}, \ldots, \begin{pmatrix} a_{l,1} & & & \\ & a_{l,2} & & \\ & & \ddots & \\ & & & a_{l,n} \end{pmatrix} \right).$$

Now, to each element a of $D_n(\mathbb{R}^\times)^l$, we assign the value

$$R(a) = \prod_{i=1}^{n} (a_{1,i}, a_{2,i}, \ldots, a_{l,i})_{\mathbb{R}}$$

where $(a_1, a_2, \ldots, a_n)_{\mathbb{R}}$ is the Hilbert symbol of nonzero real numbers $a_1, a_2, \ldots, a_n$. The Hilbert symbol $(a_1, a_2, \ldots, a_n)_{\mathbb{R}}$ is 1 if at least one of the coordinate is positive and is

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-1 if all of the coordinates are negative. Then by continuity, the value of R is constant on each connected component of $D_n(\mathbb{R}^\times)^l$. Furthermore, if the value of R is 1 on a connected component of R, then the image of the connected component under p contains the identity element of $T^l(\mathbb{R}^\times)$. Meanwhile, if the value of R is -1 on a connected component of $D_n(\mathbb{R}^\times)^l$, then the image of the connected component contains $(-1)\otimes(-1)\otimes\cdots\otimes(-1)\in T^l(\mathbb{R}^\times)$. Therefore, $T^l(\mathbb{R}^\times)$ can have at most two connected component.

On the other hand, we may define a map $R' : T^l(\mathbb{R}^\times) \to \{\pm 1\}$ by

$$R'\left(\sum_i a_{1i}\otimes a_{2i}\otimes\cdots\otimes a_{li}\right) = \prod_i (a_{1i}, a_{2i}, \ldots, a_{li})_{\mathbb{R}}.$$

R' is easily checked to be well-defined on $T^l(\mathbb{R}^\times)$ and R' takes value 1 for $1\otimes 1\otimes\cdots\otimes 1$ and -1 for $(-1)\otimes(-1)\otimes\cdots\otimes(-1)$. This shows that $T^l(\mathbb{R}^\times)$ is not connected and we are done with the proof. ■

Proposition 7. $T^l(\mathbb{R})$ is compactly generated, but not locally compact when $l \geq 2$.

Proof. The fact that $T^l(\mathbb{R})$ is compactly generated in the sense of [11]), i.e., that it is a k-space holds since it is the topological quotient image of a k-space $D(\mathbb{R})^l$ as in the proof of Theorem 3. Let $X_n = p(D_n(\mathbb{R})^l)$ $(n = 1, 2, \ldots)$ where $p : D_n(\mathbb{R})^l \to T^l(\mathbb{R})$ is as in Definition 1. Then each X_n is closed in $T^l(\mathbb{R})$ and we have $\bigcup_{n=1}^{\infty} X_n = T^l(\mathbb{R})$. Now, let U be any open neighborhood of 0 in $T^l(\mathbb{R})$. Then it can be seen that $U \cap (X_n - X_{n-1})$ is nonempty for every integer $n \geq 2$. One way to see this is to construct an infinite tower of subfields of $\mathbb{R}$

$$\mathbb{Q} \subsetneq k_1 \subsetneq k_2 \subsetneq k_3 \subsetneq \cdots$$

and choose a sequence $a_1, a_2, \ldots$ such that $a_i \in k_i - k_{i-1}$ and $x_n = \sum_{i=1}^{n} \overbrace{a_i\otimes a_i\otimes\cdots\otimes a_i}^{n \text{ times}} \in U$. Then $x_n \in U \cap (X_n - X_{n-1})$ for every integer $n \geq 2$. But, then the set $X = \{x_n | n = 2, 3, \ldots\}$ is a discrete closed subset of $T^l(\mathbb{R})$ and has no limit point in $T^l(\mathbb{R})$. So, the closure of U cannot be compact in $T^l(\mathbb{R})$ (Theorem 28.1 of [11]) and consequently $T^l(\mathbb{R})$ is not locally compact. ■

The following corollary can be shown to be true with a similar proof which we omit.

Corollary 8. $T^l(\mathbb{R}^\times)$, $T^l(\mathbb{C})$ and $T^l(\mathbb{C}^\times)$ are compactly generated, but not locally compact.

TOPOLOGY ON THE MILNOR'S *K*-THEORY OF A TOPOLOGICAL FIELD

For a field k and an integer $l \geq 1$, the l-th Milnor's K-group $K_l^M(k)$ is defined as the quotient group of $\overbrace{k^\times\otimes k^\times\otimes\cdots\otimes k^\times}^{l \text{ times}}$ by the subgroup generated by the elements of the form $a_1\otimes a_2\otimes\cdots\otimes a_l$ where $a_i + a_j = 1$ for some $1 \leq i < j \leq l$ (c.f., [10])). When k is a topological field, a natural topology is introduced in [13] and is explicitly described when k is the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. To define the topology on $K_l^M(k)$, a difficult isomorphism $\phi_l : GW_l(k) \to K_l^M(k)$ of [12] was utilized. In this section, we study the case where the obvious algebraic quotient map $q : T^l(k^\times) \to K_l^M(k)$ is actually a topological quotient map and thus the topology on the Milnor's K-group may be obtained in much easier way.

Theorem 9. For any topological field k, the algebraic quotient map $q : T^l(k^\times) \to K_l^M(k)$ is continuous. Furthermore, q is a topological quotient map if k is algebraically closed.

Proof. It is easily seen that the obvious inclusion $D_n(k^\times)\times\cdots\times D_n(k^\times) \to Comm_l(k)$ followed by the topological quotient map $\Phi_l : Comm_l(k) \to K_l^M(k)$ in [13] is equal to the map $D_n(k^\times)\times\cdots\times D_n(k^\times) \to T^l(k^\times)$ followed by the algebraic quotient map $q : T^l(k^\times) \to K_l^M(k)$. By passing to the direct limit and by the universal property of quotient map $D(k^\times)^l \to T^l(k^\times)$, we see that the algebraic quotient map $T^l(k^\times) \to K_l^M(k)$ is continuous. So the following diagram of continuous maps are commutative.

$$\begin{array}{ccc} D(k^\times)^l & \xrightarrow{\;j\;} & Comm_l(k) \\ \downarrow p & & \downarrow \Phi \\ T^l(k^\times) & \xrightarrow{\;q\;} & K_l^M(k) \end{array}$$

The inclusion map $j : D(k^\times)^l \to Comm_l(k)$ is a closed map since $Comm_l(k)$ was given the subspace topology of $GL(k)^l$. Now suppose that k is an algebraically closed field. We let the symmetric group S_n act on $D_n(k^\times)^l$ as follows. If σ is a permutation on $\{1,\dots,n\}$, then σ sends $\left([a_1^1,\dots,a_n^1],\dots,[a_1^l,\dots,a_n^l]\right)$ to $\left([a_{\sigma(1)}^1,\dots,a_{\sigma(n)}^1],\dots,[a_{\sigma(1)}^l,\dots,a_{\sigma(1)}^l]\right)$, where the symbol $[a_1,\dots,a_n]$ temporarily denotes the diagonal matrix with entries $a_1,\dots,a_n$ in that order. Then p gives rise to quotient maps from the quotient spaces $D_n(k^\times)^l/S_n$ to $T^l(k^\times)$ for all positive integer n which, after passing to the direct limit, produce a quotient map $D(k^\times)^l/Sym(\mathbb{N}) \to T^l(k^\times)$ where $Sym(\mathbb{N})$ is the symmetric group on the set of positive integers.

When k is an algebraically closed field, it is well known that every l tuples of commuting $n\times n$ matrices $(A_1,\dots,A_l)$ is simultaneously triangularizable (e.g., [3]). It then gives rise to n joint eigenvalues which are l-tuples of diagonal entries of the triangular matrices and which in turn produce an well-defined element of $D(k^\times)^l/Sym(\mathbb{N})$. This process is clearly continuous and gives rise to a map $r: Comm_l(k) \to D(k^\times)^l/Sym(\mathbb{N})$ such that the composite $D(k^\times)^l \to Comm_l(k) \to D(k^\times)^l/Sym(\mathbb{N})$ is the canonical quotient map. Consequently, r is a topological quotient map. To see this, let U be a subset of $D(k^\times)^l/Sym(\mathbb{N})$ such that $r^{-1}(U)$ is open in $Comm_l(k)$. Then $j^{-1}(r^{-1}(U))$ is open in $D(k^\times)^l$ and thus U which is the image of $j^{-1}(r^{-1}(U))$ under the canonical quotient map $D(k^\times)^l \to Comm_l(k) \to D(k^\times)^l/Sym(\mathbb{N})$ must be open.

$$\begin{array}{ccc} D(k^\times)^l & \xrightarrow{\ j\ } & Comm_l(k) \\ \downarrow & \swarrow_{r} & \\ D(k^\times)^l/Sym(\mathbb{N}) & & \end{array}$$

As the action of $Sym(\mathbb{N})$ on each element of $D(k^\times)^l$ leaves its image under p invariant, the quotient map $p: D(k^\times)^l \to T^l(k^\times)$ factors as the composite of two quotient maps $D(k^\times)^l \to D(k^\times)^l/Sym(\mathbb{N}) \to T^l(k^\times)$. Therefore, $Comm_l(k) \to D(k^\times)^l/Sym(\mathbb{N}) \to T^l(k^\times)$ is a topological quotient map and so q is a topological quotient map by Theorem 22.2 of [11]. ■

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