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Construction of a Dynamic Finite Element Model for Vibration Analysis of Reticulate Systems

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Abstract

In the present work a model of dynamic finite element for vibration analysis of reticulate systems is proposed. It is proposed a method of construction of dynamic stiffness matrices and inertia matrices for the cases of bending, traction and torsion in free and forced vibrations. From exact analytical solutions of vibration equations it is established the dynamic shape functions allowing obtaining the coefficients of dynamic stiffness matrices and inertia matrices. These coefficients depend on frequency of free vibrations of the system. This dynamic finite element model allows obtaining an exact solution for reticulate systems in classical approach of the dynamic analysis of structures.

Keywords: Dynamic finite element; dynamic shape functions; reticulate systems.

INTRODUCTION

The study on dynamics of reticulate systems in recent years knows a renewed interest due to the increasing number of industrial projects with design of large span structures subject to solicitations of dynamic origin (sport and commercial building covers, metal bridges in seismic zones, naval and aeronautical constructions). Many works related to the design of such structures are available in the technical literature [1-5]. There are many methods to improve reticulated bar models, for example, focus on the consideration of shear deformations, Saint-Venant's solutions and variational asymptotic solutions. Some works focused on the use of shear correction coefficients to improve the accuracy of calculations [6-9]. These correction coefficients often used in static analysis of structures are not always effective in case of dynamic analysis. It is shown by [10] that these coefficients may depend on own frequencies of bar vibrations. The disadvantage is that there is not generalized approach for the determination of shear correction coefficient [11-12]. Another approach for analysis of dynamic behavior of reticulated structures is the method of homogenization of discrete periodic media [13] which allows a more thorough description of the physical functioning of the discrete element [14-15]. Homogenization of periodic reticulate systems proceeds in two steps [16]: discretization of the dynamic balance then actual homogenization, leading to the continuous model developed from the discreet representation. Another method known for vibration analysis of reticulate systems is the dynamic stiffness method developed by [17-23]. This method involves a dynamic stiffness matrix established in a frequency domain using dynamic shape functions obtained from the exact solutions of governing differential equations [24]. The main disadvantage of this method is that it does show its effectiveness in the case of free vibrations. In case of forced vibrations it seems hardly applicable. Another method developed by [24] is the method of spectral elements which consists in coupling spectral analysis method with dynamic stiffness method. In first step will be a discretization of the structure then it sets the dynamic stiffness matrix. The transformation of this matrix is done according to the algorithm of fast Fourier transformation. After solving the resulting system of equations it performs a reverse transformation where one gets a time-dependent solution therefore a spectral analysis method.

In this work it is proposed a dynamic finite element model in free and forced vibrations built using shape functions obtained from exact analytical solutions of vibration equations for the cases of traction, bending and torsional deformations.

MATERIAL AND METHOD

For the problem in dynamics we will consider that the discretization concerns the rigidity and inertia parameters of the system. On the other hand the displacement of any point of the element will be according to nodal displacements q_i :

$$u(z,t) = \sum_i q_i(t) f_i(z) , \tag{1}$$

$f_i(z)$ - Selected shape functions such as displacements of any point are continuing both within the element and in limits of neighboring elements.

The dynamic problem solving in structural mechanics by finite element method, is in most cases to find solutions to very complicated elliptic problems. Establishing basic or shape functions is decisive for the sought accuracy level [25]. These shape functions can be presented as gradients of harmonic functions in Hilbert spaces and are determined by collocation method [26-27]:

$$N_i(z) = \sum_{j=1}^{m} a_j^{(i)} f_{g(j)}(z) \tag{2}$$

$a_j^{(i)}$ - Unknown coefficients, g(j) - index function, *j=1,…m.*

Functions $f_{g(j)}(z)$ are chosen in following forms:

$$p_{n+1,k+1}(z) = \sigma_{nk}(\frac{r}{r_0})^n \cos(k\varphi) P_n^k(cos\theta) \quad (3)$$

$$q_{n+1,k+1}(z) = \sigma_{nk}(\frac{r}{r_0})^n \sin(k\varphi) P_n^k(cos\theta) \quad (4)$$

For z = (r, θ, φ) we have: $\sigma_{nk} = \frac{(2n+1)(n-k)!}{(n+k)!}$ which are calculated by the recurrent formulae that appear in the general solution of Dirichlet problem for a sphere interior radius r_0 [28].

Unknown coefficients can be calculated by solving following system:

$$\sum_{j=1}^{m} a_j^{(i)} f_{g(j)}(z_i) = \delta(z_i, z_j) \text{ , j=1,…m} \quad (5)$$

Index function g (j) is selected such that the system is soluble [28]. Approximation accuracy using functions $N_i(z)$ is determined by the degree of harmonic terms of function $f_{g(j)}$, *(j=1 ;…m)* for which approximation (1) is exact.

RESULTS AND DISCUSSION

Construction of Dynamic Stiffness Matrices

Axial vibrations of the bar in traction

The bar finite element in traction is shown in figure 1.

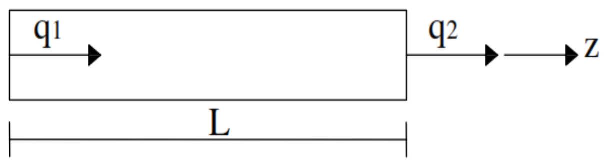

Figure 1. Traction bar finite element

Governing equation is given by:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial z^2} = 0 \text{ ,} \quad (6)$$

Sought solution for equation (6) is harmonic type [29]:

$$u(z,t) = U(z)\cos(kt + \varphi) \quad (7)$$

U (z) is amplitude function. Amplitude function for equation (6) has the following expression:

$$U(z) = c_1 \cos \alpha z + c_2 sin\alpha z \quad (8)$$

Here $\propto = \frac{k}{a}$, *k* - own frequency, *a* - wave propagation speed;

$a = \sqrt{\frac{E}{\rho}}$, E and ρ are respectively the elastic modulus and the density of the material.

The coefficients c_1 and c_2 determine shape functions $f_1(z)$ and $f_2(z)$ and can be expressed by the nodal displacements q_1 and q_2 :

$$q_1 = U(0) = c_1 \quad (9)$$

$$q_2 = U(l) = q_1 cos \propto l + c_2 sin \propto l \quad (10)$$

$$c_2 = \frac{q_2 - q_1 cos\propto l}{sin\propto l} \quad (11)$$

In our case we considered that the origin of the local coordinate system is q_1 .

By replacing expressions (9) and (10) in (8) we get:

$$U(z) = q_1 cos \propto z + (q_2 - q_1 cos \propto l)\frac{1}{sin\propto l} sin \propto z =$$
$$= q_1(cos \propto z - ctg \propto lsin \propto z) + q_2 \frac{sin\propto z}{sin\propto l} . \quad (12)$$

Thus finite element shape functions of bar in traction will have following expressions:

$$f_1(z) = cos \propto z - ctg \propto\propto lsin \propto z \quad (13)$$

$$f_2(z) = \frac{sin\propto\propto z}{sin\propto l} \quad (14)$$

Finite element stiffness matrix for figure 1 have dimension 2 x 2:

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad (15\ )$$

Coefficients of this matrix are determined by following relationship:

$$c_{ij} = \frac{\partial^2 \Pi}{\partial q_i \partial q_j} \quad (16)$$

Π - Potential energy of the system [30]. Particularly for the bar we have:

$$c_{ij} = EF \int_0^l f_i'(z) f_j'(z) dz \quad (17)$$

F - The bar section

From expression (17) it is clear that the bar element stiffness matrix is a symmetric matrix.

By deriving expressions (13) and (14) we get:

$$f_1'(z) = -\alpha sin\alpha z - ctg\alpha l cos\alpha z \quad (18)$$

$$f_2'(z) = \frac{\alpha cos\alpha z}{sin\alpha l} \quad (19)$$

By replacing expressions (18) and (19) into (17) we get:

$$c_{11} = c_{22} = \frac{EF\alpha}{4sin^2\alpha l}(2\alpha l + sin2\alpha l) \quad (20)$$

$$c_{12} = C_{21} = -\frac{EF\alpha}{2sin^2\alpha l}(sin\alpha l + \alpha l cos\alpha l) \quad (21)$$

Finally finite element stiffness matrix in traction will be:

$$C(\alpha) = \begin{bmatrix} \frac{EF\alpha}{4sin^2\alpha l}(2\alpha l + sin2l & -\frac{EF\alpha}{2sin^2\alpha l}(sin\alpha l + \alpha l cos\alpha l) \\ -\frac{EF\alpha}{2sin^2\alpha l}(sin\alpha l + \alpha l cos\alpha l & \frac{EF\alpha}{4sin^2\alpha l}(2\alpha l + sin2\alpha l \end{bmatrix} \quad (22)$$

As shown in expression (22) stiffness matrix coefficients depend on the frequency of free vibrations.

Free vibrations in bending

The bar finite element under bending is shown in figure 2. Each node is associated with two freedom degrees: a vertical displacement and rotation.

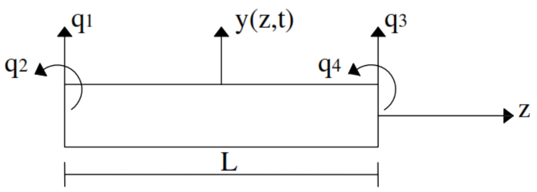

Figure 2. Bar finite element in bending

The free vibration governing equation in bending is given by:

$$EI\frac{\partial^4 y}{\partial z^4} + m_0\frac{\partial^2 y}{\partial t^2} = 0 \quad (23)$$

I - bar inertia moment in bending

m_0 - The mass of unit length.

Sought solution for equation (23) have following form [29]:

$$y(z,t) = Y(z)\cos(kt + \varphi) \quad (24)$$

Y (z) is amplitude function satisfying conditions at the ends;

k - Free vibrations frequency.

Equation (23) amplitude function has following form:

$$Y(z) = A_1 ch \propto z + A_2 sh \propto z + A_3 cos \propto z + A_4 sin \propto z \quad (25)$$

Here $\propto = \sqrt{\frac{m_0 k^2}{EI}}$.

The dynamic shape functions are determined from four nodal conditions:

First condition:

$$q_1 = 1, q_2 = q_3 = q_4 = 0 \quad (26)$$

Taking into account expression (25), we have:

$q_1 = Y(0) = A_1 + A_3 = 1$,

$$A_3 = 1 - A_1 \quad (27)$$

$q_2 = Y'(0) = \propto (A_2 + A_4) = 0$,

$$A_4 = -A_2 \quad (28)$$

$$q_3 = Y(l) = A_1 ch \propto l + A_2 sh \propto l + A_3 cos \propto l + A_4 sin \propto l = 0 \quad (29)$$

$$q_4 = Y'(l) = \propto (A_1 sh \propto l + A_2 ch \propto l - A_3 sin \propto l + A_4 cos \propto l) = 0 \quad (30)$$

Replacing expressions (27) and (28) respectively in (29) and (30) we will have:

$$A_1(ch \propto l - cos \propto l) + A_2(sh \propto l - sin \propto l) = -cos \propto l \quad (31)$$

$$A_1(sh \propto l + sin \propto l) + A_2(ch \propto l - cos \propto l) = sin \propto l \quad (32)$$

Solving the system of equations (31) and (32) and taking into account (27) and (28) we get expressions for the coefficients of first shape function. If we designate these coefficients by

A_{11}, A_{12}, A_{13} et A_{14}, the first shape function will have following expression:

$$f_{b1}(z) = A_{11} ch \propto z + A_{12} sh \propto z + A_{13} cos \propto z + A_{14} sin \propto z \quad , \quad (33)$$

and the coefficients of first shape function will be:

$$A_{11} = \frac{1 - ch\propto l cosl - sin\propto l sh\propto l}{2(1 - ch\propto l cos\propto l)} \quad (34)$$

$$A_{12} = \frac{ch\propto l sin\propto l + cos\propto l sh\propto l}{2(1 - ch\propto l cos\propto l)} \quad (35)$$

$$A_{13} = \frac{1 - ch\propto l cos\propto l + sin\propto l sh\propto l}{2(1 - ch\propto l cos\propto l)} \quad (36)$$

$$A_{14} = \frac{-ch\propto l sin\propto l - cos\propto l sh\propto l}{2(1 - ch\propto l cos\propto l)} \quad (37)$$

Second condition:

$$q_2 = 1 , q_1 = q_3 = q_4 = 0. \quad (38)$$

Taking into account expression (25), we have:

$q_1 = Y(0) = A_1 + A_3 = 0$,

$$A_3 = -A_1 \quad (39)$$

$q_2 = Y'(0) = \propto (A_2 + A_4) = 1$,

$$A_4 = \frac{1}{\propto} - A_2 \quad (40)$$

Expressions q_3 and q_4 will remain unchanged in the forms (22) and (23).

Replacing expressions (39) and (40) respectively in (29) and (30), we get following system of equations:

$$A_1(ch \propto l - cos \propto l) + A_2(sh \propto l - sin \propto l) = -\frac{1}{\propto} sin \propto l \quad (41)$$

$A_1(sh \propto l + sin \propto l) + A_2(ch \propto l - cos \propto l) = -\frac{1}{\propto} cos \propto l$ (42)

Solving the system of equations (41) and (42) and taking into account (39) and (40), we get expressions for the second shape function coefficients. Designating these coefficients by A_{21}, A_{22}, A_{23} et A_{24}, the second shape function will be as following expression:

$f_{b2}(z) = A_{21}ch \propto z + A_{22}sh \propto z + A_{23}cos \propto z + A_{24}sin \propto z$, (43)

and the second shape function coefficients will be:

$$A_{21} = \frac{sin\propto lch\propto l - cos\propto lsin\propto l}{2\propto(1 - ch\propto lcos\propto l)} \quad (44)$$

$$A_{22} = \frac{1 + sh\propto lsin\propto l - ch\propto lcos\propto l}{2\propto(1 - ch\propto lcos\propto l)} \quad (45)$$

$$A_{23} = \frac{cos\propto lsin\propto l - sin\propto lch\propto l}{2\propto(1 - ch\propto lcosl)} \quad (46)$$

$$A_{24} = \frac{1 - sh\propto lsin\propto l - ch\propto lcos\propto l}{2\propto(1 - ch\propto lcos\propto l)} \quad (47)$$

Third condition:

$$q_3 = 1, q_1 = q_2 = q_4 = 0 \quad (48)$$

Taking into account expression (25), we have:

$q_3 = Y(l) = A_1 ch \propto l + A_2 sh \propto l + A_3 cos \propto l + A_4 sin \propto l = 1$ (49)

Expressions q_1, q_2 and q_4 will not change forms (39), (28) and (30) respectively.

Replacing (39) and (28) in (49) and (30) we get following system of equations:

$A_1(ch \propto l - cos \propto l) + A_2(sh \propto l - sin \propto l) = 1$ (50)

$A_1(sh \propto l + sin \propto l) + A_2(ch \propto l - cos \propto l) = 0$ (51)

Solving this system of equations and taking into account (39) and (28), we get expressions of the third shape function coefficients A_{31}, A_{32}, A_{33} et A_{34}. So the third shape function will be written:

$f_{b3}(z) = A_{31} = ch \propto z + A_{32}sh \propto z + A_{33}cos \propto z + A_{34}sin \propto z$. (52)

The third shape function coefficients will be:

$$A_{31} = \frac{ch\propto l - cos\propto l}{2(1 - ch\propto lcos\propto l)} \quad (53)$$

$$A_{32} = \frac{-sh\propto l - sin\propto l}{2(1 - ch\propto lcos\propto l)} \quad (54)$$

$$A_{33} = \frac{cos\propto l - ch\propto l}{2(1 - ch\propto lcos\propto l)} \quad (55)$$

$$A_{34} = \frac{sh\propto l + sin\propto l}{2(1 - ch\propto lcos\propto l)} \quad (56)$$

Fourth condition:

$$q_4 = 1, q_1 = q_2 = q_3 = 0 . \quad (57)$$

Taking into account the expression (25), we have:

$q_4 = Y'(l) = \propto (A_1 sh \propto l + A_2 ch \propto l - A_3 sin \propto l + A_4 cos \propto l) = 1$. (58)

Expressions q_1, q_2 and q_3 are unchanged under forms (39), (28) and (29) respectively. Replacing (39) and (28) in (58) and (29) we get following system of equations:

$A_1(ch \propto l - cos \propto l) + A_2(sh \propto l - sin \propto\propto l) = 0$ (59)

$A_1(sh \propto l + sin \propto l) + A_2(ch \propto l - cos \propto l) = \frac{1}{\propto}$. (60)

Solving this system of equations and taking into account (39) and (28) we get expressions for the fourth shape function coefficients A_{41}, A_{42}, A_{43} and A_{44} .

The fourth shape function will have following expression:

$f_{b4}(z) = A_{41}ch \propto z + A_{42}sh \propto l + A_{43}cos \propto z + A_{44}sin \propto z$, (61)

and the fourth shape function coefficients will be:

$$A_{41} = \frac{sin\propto l - sh\propto l}{2\propto(1 - ch\propto lcos\propto l)} \quad (62)$$

$$A_{42} = \frac{ch\propto l - cos\propto l}{2\propto(1 - ch\propto lcos\propto l)} \quad (63)$$

$$A_{43} = \frac{sh\propto l - sin\propto\propto l}{2\propto(1 - ch\propto lcos\propto\propto l)} \quad (64)$$

$$A_{44} = \frac{cos\propto l - ch\propto l}{2\propto(1 - ch\propto lcos\propto l)} \quad (65)$$

Let's introduce following notations:

$R_1 = sh \propto lch \propto l$, $R_4 = sin \propto lcos \propto l,$ $R_7 = ch \propto l - cos \propto l,$

$R_2 = sh \propto lcos \propto l$, $R_5 = sh \propto lsin \propto l,$ $R_8 = sh \propto l + sin \propto l,$

$R_3 = ch \propto lsin \propto l$, $R_6 = ch \propto lcos \propto l,$ $R_9 = sh \propto l - sin \propto l$. (66))

Then, taking into account the expressions (66), the shape functions become:

$f_{b1}(z) = \frac{1}{2(1-R_6)}[(1 - R_5 - R_6)cos \propto z + (R_2 + R_3)sh \propto z + (1 + R_5 - R_6)cos \propto l - (R_2 + R_3)sin \propto z]$ (67)

$f_{b2}(z) = \frac{1}{2\propto(1-R_6)}[(R_2 - R_3)ch \propto z + (1 + R_5 + R_6)sh \propto z + (R_3 - R_2)cos \propto z - (1 - R_5 - R_6)sin \propto z]$ (68)

$f_{b3}(z) = \frac{1}{2(1-R_6)}(R_7 ch \propto z - R_8 sh \propto z - R_7 cos \propto z + R_8 sin \propto z)$ (69)

$f_{b4}(z) = \frac{1}{2\propto(1-R_6}(-R_9 ch \propto z - R_7 sh \propto z - R_9 cos \propto z + R_7 sin \propto z)$ (70)

The stiffness matrix of finite element shown in figure 2 is 4 x 4 size:

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} \quad (71)$$

The stiffness matrix coefficients are determined by relationship (16). Particularly in the case of bending we have:

$C_{ij} = EI \int_0^l f_i''(z) f_j''(z) dz$ (72)

It is obvious that from the expression (72), element stiffness matrix in bending is a symmetric matrix.

By substituting the second order derivatives of expressions (67), (68), (69) and (70) in (72) we get after mathematical transformations, expressions of the stiffness matrix coefficients.

Let's introduce following notations:

$H_1 = R_1 + 2R_2 + 2R_3 + R_4 + 2 \propto l$ (73)

$H_2 = R_1 - 2R_2 + 2R_3 - R_4$ (74)

$H_3 = (sh \propto l + sin \propto l)^2$ (75)

$H_4 = R_2 + R_3 + R_4 + \propto l$ (76)

$H_5 = R_5 + R_6 - cos^2 \propto l$ (77)

$H_6 = R_5 - R_6 + 1 + sin^2 \propto l$ (78)

$R_7 = R_3 - R_2 - R_4 + \propto l$. (79)

Finally the stiffness matrix coefficients will be:

$c_{11} = \frac{EI\propto^3}{2}[A_{11}^2 H_1 + A_{12}^2 H_2 + 2A_{11}A_{12}H_3 - 2A_{11}H_4 - 2A_{12}H_5 + \propto l + R_4]$ (80)

$c_{22} = \frac{EI\propto^3}{2}\left[A_{21}^2 H_1 + A_{22}^2 H_2 + 2A_{21}A_{22}H_3 - \frac{2}{\propto}(A_{21}H_6 + A_{22}H_7) + \frac{1}{\propto^2}(\propto l - R_4)\right]$ (81)

$c_{33} = \frac{EI\propto^3}{2}[A_{31}^2 H_1 + A_{32}^2 H_2 + 2A_{31}A_{32}H_3]$ (82)

$c_{44} = \frac{EI\propto^3}{2}[A_{41}^2 H_1 + A_{42}^2 H_2 + 2A_{41}A_{42}H_3]$ (83)

$c_{12} = c_{21} = \frac{EI\propto^3}{2}\left[A_{11}A_{21}H_1 + A_{12}A_{22}H_2 + (A_{11}A_{22} + A_{12}A_{21})H_3 - \frac{1}{\propto}(A_{11}H_6 + A_{12}H_7 - sin^2 \propto l) - A_{21}H_4 - A_{22}H_5\right]$ (84)

$c_{13} = c_{31} = \frac{EI\propto^3}{2}[A_{11}A_{31}H_1 + A_{12}A_{32}H_2 + (A_{11}A_{32} + A_{12}A_{31})H_3 - A_{31}H_4 - A_{32}H_5]$ (85)

$c_{14} = c_{41} = \frac{EI\propto^3}{2}[A_{11}A_{41}H_1 + A_{12}A_{42}H_2 + (A_{11}A_{42} + A_{12}A_{41})H_3 - A_{41}H_4 - A_{42}H_5]$ (86)

$c_{23} = c_{32} = \frac{EI\propto^3}{2}\left[A_{21}A_{31}H_1 + A_{22}A_{32}H_2 + (A_{21}A_{32} + A_{22}A_{31})H_3 - \frac{1}{\propto}(A_{31}H_6 + A_{32}H_7)\right]$ (87)

$c_{24} = c_{42} = \frac{EI\propto^3}{2}\left[A_{21}A_{41}H_1 + A_{22}A_{42}H_2 + (A_{21}A_{42} + A_{22}A_{41})H_3 - \frac{1}{\propto}(A_{41}H_6 + A_{42}H_7)\right]$ (88)

$c_{34} = c_{43} = \frac{EI\propto^3}{2}[A_{31}A_{41}H_1 + A_{32}A_{42}H_2 + (A_{31}A_{42} + A_{32}A_{41})H_3]$ (89)

Here $\propto = \sqrt[4]{\frac{m_0 k^2}{EI}}$, k – free vibration frequency.

So we can see that stiffness matrix coefficients in case of bending depend on free vibrations frequency.

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Torsional free vibrations

Torsional finite element is shown in figure 3.

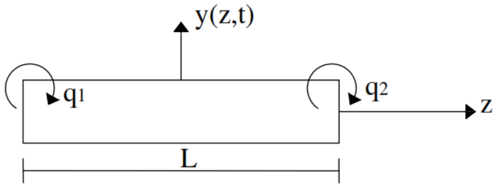

Figure 3. Torsional finite element

Torsional free vibrations are described by following governing equation:

$$\frac{\partial^2\varphi}{\partial t^2} - \beta^2\frac{\partial^2\varphi}{\partial z^2} = 0 \tag{90}$$

Sought solution for this equation is the same type for longitudinal vibrations [29]. So amplitude function for equation (90) can be written in following form:

$$\emptyset(z) = c_1 cos\beta z + c_2 sin\beta z \tag{91}$$

Here $\beta = \frac{k}{b}$, k - own frequency, $b = \sqrt{\frac{G}{\rho}}$, G and ρ, shear modulus and density of the material respectively.

Coefficients c_1 and c_2 determine the shape functions $f_{k1}(z)$ and $f_{k2}(z)$ and are functions of nodal displacements q_1 and q_1 :

$$q_1 = \emptyset(0) = c_1 \tag{92}$$

$$q_2 = \emptyset(l) = q_1 cos\beta l + c_2 sin\beta l \tag{93}$$

$$c_2 = \frac{q_2 - q_1 cos\beta l}{sin\beta l} \tag{94}$$

By substituting expressions (92) and (93) in (91) we will have:

$$\emptyset(z) = q_1 cos\beta z + (q_2 - q_1 cos\beta l)\frac{1}{sin\beta l} sin\beta z = q_1(cos\beta z - ctg\beta l sin\beta z) + q_2\frac{sin\beta z}{sin\beta l} \quad . \tag{95}$$

So the shape functions of finite element in torsion will have following expressions:

$$f_{k1}(z) = cos\beta z - ctg\beta l sin\beta z \tag{96}$$

$$f_{k2}(z) = \frac{sin\beta z}{sin\beta l} \tag{97}$$

In the general case, the stiffness matrix coefficients are determined by relationship (16). For the particular case in torsion we have:

$$c_{kij} = GI_k\int_0^l f'_{ki}(z) f'_{kj}(z)dz \tag{98}$$

Where G and I_k are the shear modulus and torsional inertia moment respectively.

It is obvious that obtained shape functions (96) and (97) are similar to the shape functions (13) and (14) for a bar in traction (figure 1). So the stiffness matrix coefficients will be similar to the stiffness matrix coefficients (20) and (21) of the bar finite element in traction:

$$c_{k11} = c_{k12} = \frac{GI_k\beta}{4sin^2\beta l}(2\beta l + sin2\beta l) \tag{99}$$

$$c_{k12} = c_{k21} = -\frac{GI_k\beta}{4sin^2\beta l}(2sin\beta l + 2\beta l cos\beta l) \tag{100}$$

So finite element stiffness matrix in torsion will have following form:

$$C_k(\beta) = \begin{bmatrix} \frac{GI_k\beta}{4\sin^2\beta l}(2\beta l + \sin 2\beta l) & -\frac{GI_k\beta}{4\sin^2\beta l}(2\sin\beta l + 2\beta l\cos\beta l) \\ -\frac{GI_k\beta}{4\sin^2\beta l}(2\sin\beta l + 2\beta l\cos\beta l) & \frac{GI_k\beta}{4\sin^2\beta l}(2\beta l + \sin 2\beta l) \end{bmatrix} = \frac{GI_k\beta}{4\sin^2\beta l}\begin{bmatrix} (2\beta l + \sin 2\beta l) & -(2\sin\beta l + 2\beta l\cos\beta l) \\ -(2\sin\beta l + 2\beta l\cos\beta l) & (2\beta l + \sin 2\beta l) \end{bmatrix} \tag{101}$$

Here β = k/b, k - the natural frequency of the structure.

We can see that the torsional stiffness matrix coefficients are function of free vibrations frequency.

Construction of Inertia Matrices

Bar traction (compression)

Inertia matrix of the bar finite element shown in figure 1 is 2 x 2 size:

$$M=\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \quad (102)$$

Inertia matrix coefficients in general case are determined by following expression [29]:

$$m_{ij}=\frac{\partial^2 T}{\partial q_i \partial q_j} \quad (103)$$

T - Kinetic energy of the system.

For particular case in traction we have:

$$m_{ij}=\int_0^l m_0 f_i(x) f_j(x) dx \quad (104)$$

m_0 - Mass of length unit of the bar.

From expression (104) it is obvious that finite element inertia matrix in traction (or compression) is a symmetric matrix.

By substituting (13) and (14) in (104) we get expressions for determination the inertia matrix coefficients:

$$m_{11}=m_{22}=\frac{m_0}{2\propto sin^2 \propto l}\left(\propto l-\frac{1}{2}sin2\propto l\right) \quad (105)$$

$$m_{12}=m_{21}=\frac{m_0}{2\propto sin^2 \propto l}(sin \propto l-\propto l cos \propto l) \quad (106)$$

Finally, inertia matrix of the bar finite element in traction or compression will be:

$$M(\alpha)=\frac{m_0}{2\alpha \sin^2 \alpha l}\begin{vmatrix} \alpha l-\frac{1}{2}\sin 2\alpha l & \sin \alpha l-\alpha l \cos \alpha l \\ \sin \alpha l-\alpha l \cos \alpha l & \alpha l-\frac{1}{2}\sin 2\alpha l \end{vmatrix} \quad (107)$$

$\propto$ = k/a, k - Natural frequency of the structure.

From expression (107) it is obvious that inertia matrix coefficients depend on free vibrations frequency.

Case of bending

Inertia matrix in case of bending of the bar finite element shown in figure 2 is 4 x 4 size:

$$M=\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \quad (108)$$

In general case the inertia matrix coefficients are determined by expression (103).

For the particular case of bending the inertia matrix coefficients have form (104).Inertia matrix for this case is also a symmetric matrix.

By substituting the shape functions (67), (68), (69) and (70) in (104) and performing the corresponding mathematical transformations we get the inertia matrix coefficients.

Let's introduce the following notations:

$$H_{m1}=R_1-2R_2-2R_3+R_4+2\propto l \quad (109)$$

$$H_{m2}=R_1+2R_2-2R_3-R_4 \quad (110)$$

$$H_{m3}=(sh \propto l-sin \propto l)^2 \quad (111)$$

$$H_{m4}=R_2+R_3-R_4-\propto l \quad (112)$$

$$H_{m5}=R_3+R_4-R_2-\propto l \quad (113)$$

$$H_{m6}=R_5-R_6+cos^2 \propto l \quad (114)$$

$$H_{m7}=R_6+R_5-sin^2 \propto l-1 . \quad (115)$$

Then the inertia matrix coefficients will be:

$$m_{11}=\frac{m_0}{2\propto}[A_{11}^2 H_{m1}+A_{12}^2 H_{m2}+2A_{11}A_{12}H_{m3}+2A_{11}H_{m4}+2A_{12}H_{m7}+\propto l+R_4] \quad (116)$$

$$m_{22}=\frac{m_0}{2\propto}\left[A_{21}^2 H_{m1}+A_{22}^2 H_{m2}+2A_{21}A_{22}H_{m3}+\frac{2}{\propto}\left(A_{21}H_{m6}+A_{22}H_{m5}+\frac{1}{\propto^2}(\propto l-R_4)\right)\right] \quad (117)$$

$$m_{33}=\frac{m_0}{2\propto}[A_{31}^2 H_{m1}+A_{32}^2 H_{m2}+2A_{31}A_{32}H_{m3}] \quad (118)$$

$$m_{44}=\frac{m_0}{2\propto}[A_{41}^2 H_{m1}+A_{42}^2 H_{m2}+2A_{41}A_{42}H_{m3}] \quad (119)$$

$$m_{12}=m_{21}=\frac{m_0}{2\propto}\left[A_{11}A_{21}H_{m1}+A_{12}A_{22}H_{m2}+(A_{11}A_{22}+A_{12}A_{21})H_{m3}+\frac{1}{\propto}(A_{11}H_{m6}+A_{12}H_{m5}+A_{12}H_{m5}+A_{21}H_{m4})+sin^2 \propto l\right] \quad (120)$$

$$m_{13}=m_{31}=\frac{m_0}{2\propto}[A_{11}A_{31}H_{m1}+A_{12}A_{32}H_{m2}+(A_{11}A_{32}+A_{12}A_{31})H_{m3}+A_{31}H_{m4}+A_{32}H_{m7}] \quad (121)$$

$$m_{14}=m_{41}=\frac{m_0}{2\propto}[A_{11}A_{41}H_{m1}+A_{12}A_{42}H_{m2}+(A_{11}A_{42}+A_{12}A_{41})H_{m3}+A_{41}H_{m4}+A_{42}H_{m7}] \quad (122)$$

$$m_{23}=m_{32}=\frac{m_0}{2\propto}\left[A_{21}A_{31}H_{m1}+A_{22}A_{32}H_{m2}+(A_{21}A_{32}+A_{22}A_{31})H_{m3}+\frac{1}{\propto}(A_{31}H_{m6}+A_{32}H_{m5})\right] \quad (123)$$

$$m_{24}=m_{42}=\frac{m_0}{2\propto}\left[A_{21}A_{41}H_{m1}+A_{22}A_{42}H_{m2}+(A_{21}A_{42}+A_{22}A_{41})H_{m3}+\frac{1}{\propto}(A_{41}H_{m6}+A_{42}H_{m5})\right] \quad (124)$$

$$m_{34}=m_{43}=\frac{m_0}{2\propto}[A_{31}A_{41}H_{m1}+A_{32}A_{42}H_{m2}+(A_{31}A_{42}+A_{32}A_{41})H_{m3}] \quad (125)$$

$\propto = \sqrt[4]{\frac{m_0 k^2}{EI}}$, k - Free vibration frequency.

We can see that for the case of bending the inertia matrix coefficients depend on free vibrations frequency.

Case of torsion

The inertia matrix coefficients in general case are determined by expression (103). For particular case of torsion these coefficients have form (104):

So inertia matrix in torsion will be:

$$m_{ij} = \int_0^l \theta_0 f_{ki}(z) f_{kj}(z) dz \qquad (126)$$

θ_0- Unit angle of torsion,

$\theta_0 = \rho I_k$.

Similarly to the stiffness matrix for the case of traction, the inertia matrix coefficients will be:

$$m_{11} = m_{22} = \frac{\theta_0}{2\beta sin^2\beta l}\left(\beta l - \frac{1}{2} sin2\beta l\right) \qquad (128)$$

$$m_{12} = m_{21} = \frac{\theta_0}{4\beta sin^2\beta l}(2sin\beta l - 2\beta l cos\beta l) \qquad (129)$$

$$M_k(\beta) = \frac{\theta_0}{2\beta \sin^2 \beta l} \begin{Vmatrix} \beta l - \frac{1}{2}\sin 2\beta l & \frac{1}{2\beta}(2\sin\beta l - 2\beta l \cos\beta l) \\ \frac{1}{2\beta}(2\sin\beta l - 2\beta l \cos\beta l) & \beta l - \frac{1}{2}\sin 2\beta l \end{Vmatrix}$$

(130)

In expression (130) parameter β is equal to β = k/b; k - natural frequency of the structure.

We can see that the inertia matrix coefficients depend on free vibrations frequency.

The proposed dynamic finite element model is developed from the shape functions built on basis of the solutions of differential equations of vibrations. Equation of axial vibrations (6), established on basis of the flat section assumptions has an approximate solution because the inertia forces are not taken into account. Finite element obtained from this equation takes into account this aspect. We can expect an error in assessment of vibration for very high levels of frequencies.

About the torsion, finite element obtained from equation (90) will have the same limitations as the equation. However we can expect great accuracy in the calculation of a bar with circular section. This finite element in torsion will be unusable for flat profiles because torsional vibration equation solving does not take into account axial displacements of the sections.

As vibration equation in bending (23), its solving does not take into account the rotating inertia of bar sections this is why we can expect significant errors for evaluation of vibrations of short length bars.

CONCLUSION

From this work we can hold the following:

For vibration analysis of the bar systems it is proposed a dynamic finite element model where the shape functions are built from analytical exact solutions of governing equations of bar vibrations.

1. It's got the stiffness matrices and inertia matrices of bar finite elements in free vibrations for cases of bending, traction and torsion.
2. It was established that the stiffness matrix coefficients and inertia matrix coefficients are based on free vibration frequencies of the system.
3. This dynamic finite element model allows getting exact solutions for minimum number of finite elements of bar structures.

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