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Comparison of Methods for Estimation of Sample Sizes under the Weibull Distribution

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Abstract

Recent technological developments have led to increases in the number and complexity of individual parts of multifunctional systems. Therefore, it is difficult to collect failure data due to time and cost limitations in industrial sites, and only censored failure data can be collected. It is necessary to ensure accuracy of reliability data for small sample sizes to reflect realistic conditions. In this study, we compared the accuracy of reliability measures calculated by conventional statistical and Bayesian methods considering the failure mode, sample size, and censored ratio according to a six-step procedure; a comparison of accuracy of prior information based on the Bayesian method was also performed. Based on the results, guidelines for estimation methods for sample size are provided.

Keywords: Bayesian inference, least squares estimation, maximum likelihood estimation, MCMC, Weibull distribution

INTRODUCTION

The number and complexity of individual parts of multifunctional systems have increased with rapid technological development, leading to new failure modes. In particular, the reliability of systems and equipment must be accurately assessed in key industries (e.g., defense, shipbuilding, etc.) to avoid damage and disasters caused by simple faults or failures [1, 2]. To determine the reliability of the system quantitatively, reliability analysis is widely used to identify the failure distribution by analyzing the failure data of the system. Thus, large amounts of failure data are required; however, it is generally difficult to collect failure data due to time and cost limitations. Usually only censored failure data that cannot precisely identify the failure time are available [3]. To reflect realistic conditions, it is therefore important to ensure accurate reliability for small sample sizes.

Conventional statistical methods, such as maximum likelihood estimation (MLE) and least squares estimation (LSE), are widely used to estimate parameters of the failure distribution. In general, MLE has high accuracy for parameter estimation when the sample size is large. However, for small sample sizes, the Bayesian approach is known to be more accurate [4].

The Bayesian approach estimates the parameters through the posterior distribution using given information (the prior distribution) for the subject, such as the system or parts. The Bayesian estimator, i.e., a posterior distribution, is generally difficult to calculate. It uses various algorithms for this purpose; the Markov chain Monte Carlo (MCMC) method is widely used [5, 6].

Ahmed et al. (2010) and Ibrahim et al. (2012) compared the estimation performance between MLE and the Bayesian method using Jeffrey's distribution in various sample sizes [7, 8]. Kundu and Raqab (2012) used the Bayesian method to calculate the posterior distribution for the shape parameter of the Weibull distribution according to prior distributions that are informative and non-informative [9].

In this research, we assumed a Weibull distribution to represent the initial, accidental, and wear failure. The accuracy of the reliability measure B_{10} calculated by conventional statistical methods and the Bayesian method considering the failure mode, sample size, and censored ratio were compared to justify the use of the Bayesian approach. In addition, the Bayesian method was assessed for accuracy of prior information. Based on our results, we provide estimation method guidelines for the sample size.

PROCEDURES AND METHODS

Study procedure

The study procedure (Fig. 1) consists of six steps for comparison of the estimation accuracy between conventional statistical methods and Bayesian methods. The robustness of the Bayesian method for effectiveness of prior information was also evaluated.

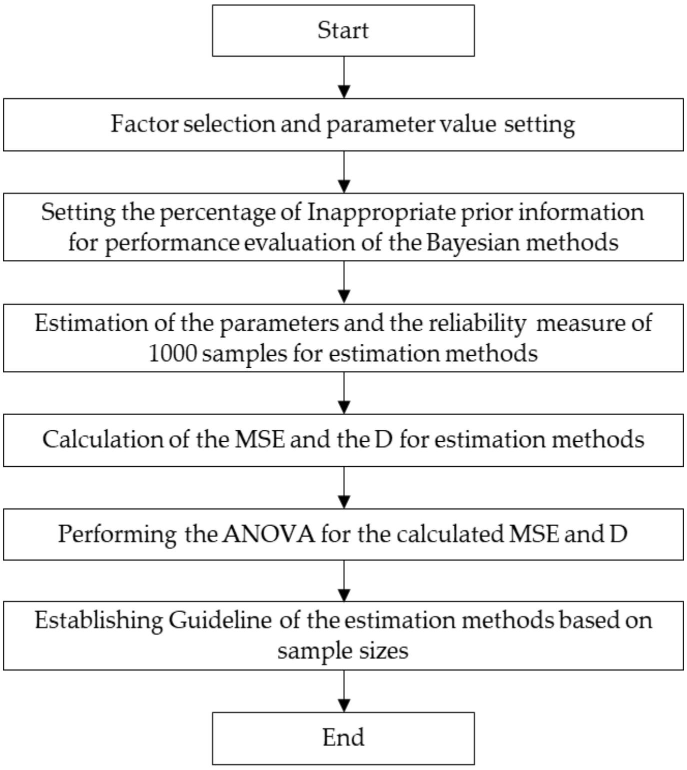

Figure 1: Study procedure for comparison of estimation accuracy.

We first set the number of factors, such as the values of parameters and sample size. Scale parameters were fixed at 10 and shape parameters were set at 0.5, 1.0, 2.0, and 3.0 to compare the changes with respect to the failure mode. The sample size was divided into 10, 20, 30, 40, and 50. Finally, the proportions of observed failure data (POFD) were 20%, 40% 60%, 80%, and 100%, where POFD of 100% indicates complete failure.

Second, to determine the effectiveness of prior information, deviations of –80%, –40%, +40%, and +80% of the shape parameters were used for the mean value of the prior distribution; for example, given a shape parameter of 1.0, deviation values were set at 0.2, 0.6, 1.4, and 1.8, respectively. These values were not analyzed by conventional statistical methods.

Third, 1000 samples were generated for all test points and the parameters were estimated by LSE, MLE, and the Bayesian method. The reliability measure B_{10} was calculated using the estimated parameters.

The fourth step involves calculation and analysis of the accuracy of B_{10} with respect to the mean and variance. Here, D, is defined as the absolute value of the deviation of the mean of estimated B_{10} with respect to the true value, and variance is the mean squared error (MSE) on B_{10}, as given below:

$$\mathrm{D} = \left|B_{10} - \hat{B}_{10}\right|, \text{where } \hat{B}_{10} = \frac{\sum_{i=1}^{m} \hat{B}_{10i}}{m} \tag{1}$$

$$\mathrm{MSE} = \frac{\sum_{i=1}^{m}\left(B_{10} - \hat{B}_{10i}\right)}{m} \tag{2}$$

Next, analysis of variance (ANOVA) was performed for experimental points of the calculated D and MSE. The four factors tested are given below:

1. Sample size: 10, 20, 30, 40, 50
2. Shape parameter: 0.5, 1.0, 2.0, 3.0
3. Proportion of observed failure data (POFD): 20%, 40%, 60%, 80%, 100%
4. Estimation method: LSE, MLE, NB, IB (−80%), IB (−40%), IB (0%), IB (+40%), IB (+80%)

where NB indicates non-informative Bayesian method, and IB indicates informative Bayesian method.

Finally, following the procedure, we outline guidelines for the estimation method with respect to sample size, based on the ANOVA results. The estimation methods are discussed in the sections that follow.

Conventional statistical methods

We used LSE and MLE as conventional statistical methods. Given a set of random lifetimes $t_1, \dots, t_n$, the probability density function of the Weibull distribution is given below in terms of the scale parameter α and the shape parameter β:

$$f(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right]. \tag{3}$$

LSE estimates unknown parameters by minimizing the sum of squared errors using a linear regression model. To estimate unknown parameters through LSE, the cumulative distribution function (cdf) of the Weibull distribution uses a log transformation:

$$\ln\left[-\ln\left(1 - F(t_i)\right)\right] = \beta\ln(t_i) - \beta\ln(\alpha). \tag{4}$$

Variable transformation is performed as $y_i = \ln\left[-\ln\left(1 - F(t_i)\right)\right]$ and $x_i = \ln(t_i)$. Using the least squares method, the sum of the squared error (SSE) model, *A*, is given by

$$\mathrm{A} = \sum_{i=1}^{n}\left[y_i - \left(\beta x_i - \beta\ln(\alpha)\right)\right]. \tag{5}$$

To minimize the SSE, Eq. (5) is partially differentiated into α and β. The values of α and β were both calculated to be zero. The calculated value is a parameter of the Weibull distribution.

The MLE is a method for finding the parameter that maximizes the likelihood function, which indicates a function of the parameters of a statistical model for the given data. The likelihood function of the Weibull distribution for a given data set is

$$L(t_i|\alpha,\beta) = \prod_{i=1}^{n} f(t_i). \quad (6)$$

For complete failure data, the likelihood function of the Weibull distribution is as follows:

$$L(t_i|\alpha,\beta) = \left(\frac{\beta}{\alpha^n}\right)^n \prod_{i=1}^{n} t_i^{\beta-1} \exp\left[-\left(\frac{t_i}{\alpha}\right)^\beta\right]. \quad (7)$$

The log likelihood function for Eq. (7) is given below:

$$\ln\big(L(t_i|\alpha,\beta)\big) = n\ln\beta + \beta n\ln\left(\frac{1}{\alpha}\right) + (\beta-1)\sum_{i=1}^{n}(\ln t_i) - \left(\frac{1}{\alpha}\right)\sum_{i=1}^{n} t_i^\beta. \quad (8)$$

To minimize the log likelihood function, Eq. (8) is partially differentiated into α and β, respectively, to find a parameter value that becomes zero. The partial differential equations for log likelihood function are described by Eqs. (9) and (10). As it cannot be solved by a simple operation, Eq. (10) is calculated using numerical analysis methods, such as the Newton–Rapson method:

$$\frac{\partial \ln L}{\partial \alpha} = n\beta\alpha + \beta\alpha^{-(\beta+1)}\sum_{i=1}^{n} t_i^\beta = 0, \quad (9)$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + n\ln\left(\frac{1}{\alpha}\right) + \sum_{i=1}^{n}\ln t_i - \sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^\beta \ln\left(\frac{t_i}{\alpha}\right) = 0. \quad (10)$$

Bayesian method

The Bayesian method based on Bayes' theorem estimates the parameters using the observed data and information about the data. The data represent the probability of the parameter through a likelihood function and the information about the data is represented by a prior distribution, which is a probability distribution. The posterior distribution and the prior distribution are as follows:

$$p(\theta|t_i) = \frac{p(t_i|\theta)\pi(\theta)}{p(t_i)} \propto p(t_i|\theta)\pi(\theta), \quad (11)$$

where θ is a parameter of the probability distribution, $p(\theta|t_i)$ is the posterior distribution, $\pi(\theta)$ is the prior distribution, and $p(t_i|\theta)$ is the likelihood function. The posterior distribution for the Weibull distribution calculated using Eq. (11) is as follows:

$$p(\alpha,\beta|t_i) \propto L(t_i|\alpha,\beta)\pi(\alpha,\beta). \quad (12)$$

It is difficult to calculate the posterior distribution by a simple operation as it is usually provided in the form of a double integral. In this study, the Metropolis–Hastings algorithm of MCMC was used to calculate the posterior distribution for the Weibull distribution. The prior distribution of the Bayesian method was divided into informative prior distribution and non-informative prior distribution, according to the presence or absence of information, respectively.

In the absence of prior information about the parameters or past experience, a non-informative prior distribution that performs a minimal role for the prior distribution can be applied. Zaidi et al. (2012) used Jeffrey's distribution as a non-informative prior distribution under the Weibull distribution [10]. Jeffrey's distribution can be expressed as a Fisher information matrix. The Fisher information matrix on the Weibull distribution is as follows:

$$\pi(\alpha,\beta) \propto \sqrt{I(\alpha,\beta)}, \quad (13)$$

$$I(\alpha,\beta) = -E\begin{vmatrix} \frac{\partial^2 \ln f(t|\alpha,\beta)}{\partial\alpha^2} & \frac{\partial^2 \ln f(t|\alpha,\beta)}{\partial\alpha\partial\beta} \\ \frac{\partial^2 \ln f(t|\alpha,\beta)}{\partial\alpha\partial\beta} & \frac{\partial^2 \ln f(t|\alpha,\beta)}{\partial\beta^2} \end{vmatrix} = \left(\frac{1}{\alpha\beta}\right)^2. \quad (14)$$

The non-informative prior distribution on the Weibull distribution is determined according to Eq. (13). The joint posterior distribution using the likelihood function and Jeffrey's distribution is as follows:

$$\pi(\alpha,\beta) \propto \frac{1}{\alpha\beta}, \quad (15)$$

$$\pi(\alpha,\beta|t_i) = \frac{(1/\alpha)^{\beta n+1}\beta^{n-1}\prod_{i=1}^{n} t_i^{\beta-1}\exp[-(t_i/\alpha)^\beta]}{\int_0^\infty\int_0^\infty (1/\alpha)^{\beta n+1}\beta^{n-1}\prod_{i=1}^{n} t_i^{\beta-1}\exp[-(t_i/\alpha)^\beta]\,d\alpha d\beta}. \quad (16)$$

When there is prior information or theoretical knowledge about the parameter to be estimated, the posterior distribution can be determined using the prior distribution. In this study, we assumed that the prior distribution of the shape parameter and the scale parameter on the Weibull distribution is a gamma distribution [11]:

$$\alpha \sim Gamma(a,b), \quad (17)$$

$$\beta \sim Gamma(c,d), \quad (18)$$

where a is a scale parameter on α, b is a shape parameter on α, c is a scale parameter on β, and d is a shape parameter on β. Hyperparameters a, b, c, and d form a set, such that the mean of each gamma distribution is the true value of α and β and the variance of each gamma distribution is 1. We assumed that we have appropriate information if the mean of the prior distribution is the true value of the parameter. Thus, when comparing the performance of the Bayesian method for the accuracy of prior information, we can compare the results according to the mean of the prior distribution by fixing the variance to a value of 1.

RESULTS AND DISCUSSION

ANOVA was performed for each of MSE and D for a total of 800 experimental points, and the interaction was considered as the second interaction. The ANOVA tables for this are shown in Tables 1 and 2. The MSE can be affected by all of the factors, and the influence of the sample size and the shape parameter is significant based on the F value. In the main effect, the MSE decreased as the sample size and POFD increased, and the shape parameter was small. The MSEs of Bayesian methods were generally smaller than the others, and the negative deviation of informative Bayesian methods had a lower MSE than the positive deviation. The interaction of the sample size and the shape parameter was the greatest effect in the interactions.

The main effect of D was similar to the results of the MSE, and all factors were significant. In addition, the sample size and shape parameters had the strongest influence, and the interaction between the shape parameter and estimation method was the greatest among the interactions. This was because as the shape parameter increased, the D for the informative Bayesian method (–80%) was about twice those of the other methods. The non-informative Bayesian method is preferred when the POFD exceeds 40%.

Table 1: ANOVA results for the MSE

Factor	Df	Adj SS	Adj MS	F	*P*
Sample size	4	76.943	19.2358	1152.35	0.000
Shape parameter	3	52.600	17.5334	1050.36	0.000
POFD	4	6.453	1.6134	96.65	0.000
Method	7	16.991	2.4273	145.41	0.000
Sample size*Shape parameter	12	13.820	1.1517	68.99	0.000
Sample size*Censor ratio	16	3.165	0.1978	11.85	0.000
Sample size*Method	28	18.262	0.6522	39.07	0.000
Shape parameter*Censor ratio	12	1.661	0.1384	8.29	0.000
Shape parameter*Method	21	7.292	0.3472	20.80	0.000
Censor ratio*Method	28	4.110	0.1468	8.79	0.000
Error	664	11.084	0.0167		
Total	799	212.383			

Table 2: ANOVA results for D

Factor	Df	Adj SS	Adj MS	F	*P*
Sample size	4	7.8145	1.9536	436.66	0.000
Shape parameter	3	5.6624	1.8874	421.87	0.000
POFD	4	1.2375	0.3093	69.15	0.000
Method	7	9.8408	1.4058	314.22	0.000
Sample size*Shape parameter	12	0.5963	0.0496	11.11	0.000
Sample size*Censor ratio	16	0.3648	0.0228	5.10	0.000
Sample size*Method	28	1.2543	0.04479	10.01	0.000
Shape parameter *Censor ratio	12	0.1911	0.0159	3.56	0.000
Shape parameter *Method	21	9.4493	0.4499	100.57	0.000
Censor ratio*Method	28	2.5885	0.0924	20.66	0.000
Error	664	2.9708	0.0044		
Total	799	41.9703			

The MSE and D according to the estimation method including all sample sizes are shown in Fig. 2. The solid line represents the mean value and the dashed line represents the error range of 20% on the mean value. As shown in Fig. 2, the informative Bayesian method showed better performance with more accurate prior information. However, it is recommended to underestimate the value compared to the true parameter when prior information is uncertain. The LSE, MLE, and non-informative Bayesian methods did not show better performance than the informative Bayesian method.

Finally, ANOVA was performed again for the MSE and D using the shape parameter, estimation method, and POFD, for the sample size that was the most significant. For a quantitative index, the rank of the guidelines is based on the mean value of the MSE and D given by the ANOVA results (the given mean). The MSE and D were reanalyzed for sample sizes of 10, 20, and 30, as shown in Fig. 3.

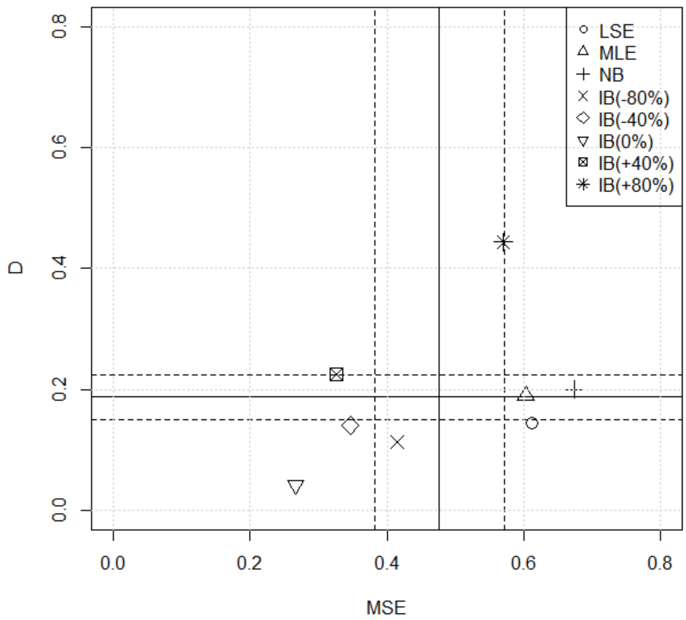

Figure 2: MSE and D for the estimation methods.

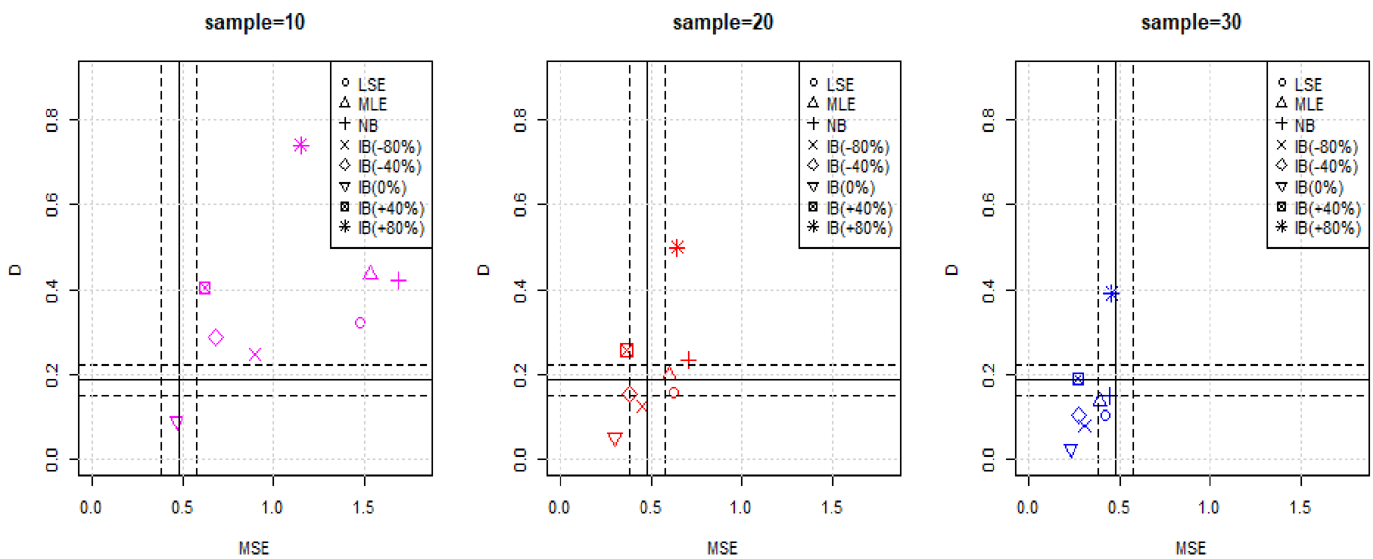

Figure 3: MSE and D for sample sizes of 10, 20, and 30.

Table 3: Guidelines for the estimation methods

Sample size	Classical methods			Bayesian methods				
	LSE	MLE	NB	IB (–80%)	IB (–40%)	IB (0%)	IB (+40%)	IB (+80%)
10					△	○		
20	△	△	△	○	○	◎	○	
30	○	◎	○	◎	◎	◎	○	△

CONCLUSION

In this paper, we compared the accuracy of estimation methods using various factors, including sample size, shape parameter, and POFD, from the viewpoint of both the mean and variance. It was confirmed that the accuracy of the estimation increased as the sample size increased. For a sample size greater than 30, the performance of the estimation methods was similar. Thus, guidelines for the estimation methods were established based on sample sizes of 10, 20, and 30, as shown in Table 3.

For a sample size of 10, the results showed that it is better to use the informative Bayesian method. If there is no information, then the LSE is recommended. For a sample size of 20, the informative Bayesian method showed the best performance. If no information is available, one of LSE, MLE, or NB can be used. For a sample size of 30, the informative Bayesian method was the best and MLE was the second best.

When the informative Bayesian method is used, the accuracy of the estimation improves with greater accuracy of the prior information. If the prior information is uncertain, it is better to underestimate the value than to use the true value of the parameter.

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