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The Calculation of the Diffraction Integral Using Chebyshev Polynomials

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Abstract

The method of calculating the diffraction integral ensuring the required accuracy of the calculation using of Chebyshev polynomials are described in present article. Compare the results of the direct method of calculation of the diffraction integral and the developed in the article method of calculation, based on the use of Chebyshev polynomials. The obtained results can be used for the analysis of propagation light beam with a distorted wavefront through the optical system in the problem of optical diagnostics.

Keywords: diffraction, diffraction integral, Chebyshev polynomials, wave front, distortion, aberrations.

INTRODUCTION

Due to the rapid development of wireless telecommunication systems and informatics [1-2], optical diagnostics tasks [3-7] and computer-based optics elements [8], it is required to analyze the propagation of a light beam with specified spatial characteristics.

As is known, after passage of a light beam through an optical system, a perturbation of the original beam homocentricity takes place. These changes are linked to a perturbation of the wave front sphericity of the original beam and lead to distortions thereof, which are expressed in a real wave front deviation from a reference sphere or plane and are described by wavelength aberrations [9].

Therefore, in the design of modern highly-accurate optical-to-electric converters, it's necessary to specify and take into account different distortions in the process of numerical simulation of propagation or diffraction of a light beam by using different methods based on the calculation of the diffraction integral.

The first solutions of the Fresnel-Kirchhoff integral on a ribbon (a slit) in the form of infinite series on the Mathieu functions and taking into account the Babinet's principle, were obtained in the works [10–12]. However, in view of the properties of elliptical harmonics, these solutions become not very suitable for calculations when the ribbon or slit width is much longer than the in-going wave length. Another method of solving the wave diffraction problem on the ribbon is the method of the integral equation [13–14] which is solved by the successive approximations method. However, as a result of the fast oscillation of sub-integral functions, such a way leads to cumbersome com-putations and does not allow for a visual representation of a scattered field formation. In most cases of practical interest, it is necessary to resort to approximation or asymptotic methods due to mathematical difficulties. Some of the best known methods are the methods of edge waves of P.Ya. Ufimtsev [15], the stationary phase method. In the work [16], a comparative analysis of strict and asymptotic methods is performed.

Therefore, a task of Fresnel-Kirchhoff theory development is relevant with the aim of expanding its application limits and developing new methods of calculation of the diffraction fields [17].

The present work provides a comparative analysis of the direct calculation of the diffraction integral and the calculation based on the Chebyshev polynomials. Indeed, a representation of a polynomial through the Chebyshev polynomial allows lowering its order and approximating by means of a polynomial of lower degree, ensuring thereby a specified approximation error [18, 19].

BASIC CALCULATION RATIOS

In our case, we will consider a distorted wave front in two planes: input and output planes.

The complex field amplitude distribution in the output plane $Q(x_2, y_2, z_2)$ is determined by the diffraction integral [20] of the complex field amplitude distribution in the input plane $Q(x_1, y_1, z_1)$:

$$Q(x_2, y_2, z_2) = \frac{k}{2\pi i} \iint \cos\theta \cdot \frac{\exp ikr_{12}}{r_{12}} \cdot \\ \cdot Q(x_1, y_1, z_1) dx_1 dy_1, \tag{1}$$

where $k = 2\pi/\lambda$ – wave number, λ – incident wavelength on the input radiation plane, $r_{12} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + z^2}$.
(2)

– distance between points of the input and output planes, $z = z_2 - z_1$, $z >> \lambda$, $1/r_{12} \cong 1/z$;

θ – angle between r_{12} and the z-axis, $\cos\theta \cong 1$ when considering the points of the input and output planes that are close to the z-axis.

The question of simplifying the calculation of the output distribution of the field is closely connected to a choice of a method of representation and calculation of the functions which, with the best approximation and desired accuracy, ensure the formation of the analyzable light field. Among various expansions (in a power series, in a trigonometric series, or in a series of other special polynomials or functions), the main approach is to represent the functions in a linear combination of diverse approximating polynomials by using orthogonal polynomials. In most applications, the Chebyshev orthonormal polynomials of the first kind hold a unique position, because they have the least zero-evasion on the $[-1;+1]$ segment compared to any other polynomials of the same degree and provide a more rapid convergence of the expansion of a function in a series [21-22]. We will use the properties of the Chebyshev polynomials to calculate the diffraction integral with a lesser complexity and a preset accuracy.

In order to facilitate the calculation of the integral (1) in accordance with the above assumptions $|x_1 - x_2| << z$ and $|y_1 - y_2| << z$, the values of Taylor series expansion r_{12} (2) are often used in the exponential term of the sub- integral expression of the diffraction integral (1)

$$r_{12} = z\left[1 + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{z^2}\right]^{\frac{1}{2}} = z[1+\rho]^{\frac{1}{2}} \cong z\left[1 + \frac{1}{2}\rho - \frac{1}{8}\rho^2 + \frac{1}{16}\rho^3 - \frac{5}{128}\rho^4 + \frac{7}{256}\rho^5\right], \quad (3)$$

where

$$\rho = \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{z^2} = \frac{u^2}{z^2},$$

the variable u includes the apertures of the input and output planes, and the ratio $u/z < 1$ corresponds to the tangent of the angle at which the cumulative aperture is visible from the output plane point.

Since r_{12} is part of the exponent together with k, when evaluating the calculation accuracy and the neglected terms, it is necessary to analyze the expansion in series of the product kr_{12}:

$$kr_{12} \cong kz\left[1 + \frac{1}{2}\rho - \frac{1}{8}\rho^2 + \frac{1}{16}\rho^3 - \frac{5}{128}\rho^4 + \frac{7}{256}\rho^5\right] = kz + 4\pi N\frac{u^2}{u_{\max}^2} \times$$
$$\times\left(1 - \frac{1}{4}\rho + \frac{1}{8}\rho^2 - \frac{5}{64}\rho^3 + \frac{7}{128}\rho^4\right) = kz + r(\rho), \quad (4)$$

In (4), an important system parameter N is entered:

$$N = \frac{ku_{\max}^2}{8\pi z} = \frac{u_{\max}^2/4}{\lambda z}; \quad (5)$$

– a modified Fresnel number (under the condition $u_{\max}^2/4 << (z/(u_{\max}/2)^2)$, which is practically always fulfilled) approximately equal to the number of Fresnel zones on the maximum possible cumulative aperture $u_{\max}$, defined by the apertures of the input and output planes $(u/u_{\max} < 1)$.

The function $r(\rho)$ is presented as the following expansion:

$$r(\rho) = \sum_{m=0}^{M} \tilde{a}\rho^m = 4\pi N\frac{u^2}{u_{\max}^2}\sum_{m=0}^{M} a\rho^m. \quad (6)$$

The introduction of the parameter N and the relative cumulative aperture $u/u_{\max}$ lowers the degree of expansion in the series $r(\rho)$ compared to the initial series r_{12} (see (4)).

To further simplify the diffraction integral (1), we will express ρ^m in the expression (6) in a linear combination of the Chebyshev polynomials. If we consider the points on the optical axis in the input plane $(x_1 = 0, y_1 = 0)$ and in the output plane $(x_2 = 0, y_2 = 0)$, it is obvious that $\rho = 0$ (see (3)). On the other hand, since the apertures in the input and output planes are small compared to the distance between the planes z, then $\rho < 1$, so that $0 \le \rho < 1$. So we will use the shifted Chebyshev polynomials of the first kind $T_m^*(\rho)$, which are deviating least from zero on the $[0;+1]$ segment.

For calculation of the polynomials $T_m^*(\rho)$, one may make use of the recurrence formula

$$T_m^*(\rho) = (4\rho - 2)T_{m-1}^*(\rho) - T_{m-2}^*(\rho), \quad (7)$$

and the first two terms are determined simply:

$T_0^*(\rho) = 1$ and $T_1^*(\rho) = 2\rho - 1$.

For our example (4) $M = 4$. We will make use of the representations $\rho^0 .. \rho^4$ through $T_m^*(\rho)$:

$$\rho^0 = T_0^*(\rho),$$

$$\rho^1 = \frac{1}{2}\left[T_1^*(\rho) + T_0^*(\rho)\right],$$

$$\rho^2 = \frac{1}{8}\left[T_2^*(\rho) + 4T_1^*(\rho) + 3T_0^*(\rho)\right],$$

$$\rho^3 = \frac{1}{32}\left[T_3^*(\rho) + 6T_2^*(\rho) + 15T_1^*(\rho) + 10T_0^*(\rho)\right],$$

$$\rho^4 = \frac{1}{128}\left[T_4^*(\rho) + 8T_3^*(\rho) + 28T_2^*(\rho) + 56T_1^*(\rho) + 35T_0^*(\rho)\right]$$

After grouping on the Chebyshev polynomials, we will obtain $r(\rho)$ as follows:

$$r(\rho) = \sum_{m=0}^{M} b_m T_m^*(\rho), \tag{8}$$

where

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = 4\pi N \frac{u^2}{u_{\max}^2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{8} & \frac{5}{16} & \frac{35}{128} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{15}{32} & \frac{7}{16} \\ 0 & 0 & \frac{1}{8} & \frac{3}{16} & \frac{7}{32} \\ 0 & 0 & 0 & \frac{1}{32} & \frac{1}{16} \\ 0 & 0 & 0 & 0 & \frac{1}{128} \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{4} \\ \frac{1}{8} \\ -\frac{5}{64} \\ \frac{7}{128} \end{pmatrix}.$$

After the conversions, we obtain

$$r(\rho) = 4\pi N \frac{u^2}{u_{\max}^2}\left[\frac{14949}{16384}T_0^*(\rho) + \left(-\frac{77}{1024}\right)T_1^*(\rho) + \frac{53}{4096}T_2^*(\rho) + \left(-\frac{1}{1024}\right)T_3^*(\rho) + \frac{7}{16384}T_4^*\right]. \tag{9}$$

The advantage of expansion (9) is that the value of each of the Chebyshev polynomials is not greater than 1 by modulus just as $\frac{u^2}{u_{\max}^2} \le 1$, which allows to reduce the complexity of calculation of the integral (1) by way of economization in the series (9). Thereby, to maintain the series calculation accuracy not worse than ε, it is necessary to find a series $\tilde{r}(\rho)$ with $m < M$:

$$\tilde{r}(\rho) = \sum_{m=0}^{m<M} b_m T_m^*(\rho) \tag{10}$$

such as

$$\left|\tilde{r}(\rho) - r(\rho)\right| < \varepsilon. \tag{11}$$

Since $\left|T_m^*(\rho)\right| \le 1$, then $\tilde{r}(\rho)$ (10) will satisfy the required accuracy (11) if

$$\sum_{m=m+1}^{M} \left|b_m\right| < \varepsilon. \tag{12}$$

Let $m = 2$ in our example and the maximum phase change brought in by the higher-order terms (with m = 3 and m = 4), does not exceed $\varepsilon = \pi/2$, i.e. the required accuracy of calculation is $r_{12}\ \delta = \lambda/4$. Then according to (12)

$$4\pi N\left(\frac{1}{1024} + \frac{7}{16384}\right) \le \frac{\pi}{2},$$

and the requirement for ε is fulfilled at $N \le 89$.

Table 1 shows the cases of economization in the series (9) with truncation of four, three, two and one expansion terms $(m = 0 \ldots 3)$, as well as the restrictions to the parameter N to ensure a different accuracy of these series $\varepsilon_1 = \pi/2,\ \varepsilon_2 = \pi/5$, $(\delta_1 = \lambda/4,\ \delta_2 = \lambda/10)$.

Table 1: Examples of economization in the series

Точность $\varepsilon_1 = \pi/2\ (\delta_1 = \lambda/4)$		
$m = 0$	$N \le 1{,}4$	$r(\rho) = 4\pi N \frac{u^2}{u_{\max}^2}\left[\frac{14949}{16384}T_0^*(\rho)\right]$
$m = 1$	$N \le 8{,}7$	$r(\rho) = 4\pi N \frac{u^2}{u_{\max}^2}\left[\frac{14949}{16384}T_0^*(\rho) + \left(-\frac{77}{1024}\right)T_1^*(\rho)\right]$
$m = 2$	$N \le 89$	$r(\rho) = 4\pi N \frac{u^2}{u_{\max}^2}\left[\frac{14949}{16384}T_0^*(\rho) + \left(-\frac{77}{1024}\right)T_1^*(\rho) + \frac{53}{4096}T_2^*(\rho)\right]$
$m = 3$	$N \le 292{,}6$	$r(\rho) = 4\pi N \frac{u^2}{u_{\max}^2}\left[\frac{14949}{16384}T_0^*(\rho) + \left(-\frac{77}{1024}\right)T_1^*(\rho) + \frac{53}{4096}T_2^*(\rho) + \left(-\frac{1}{1024}\right)T_3^*(\rho)\right]$
Точность $\varepsilon_2 = \pi/5\ (\delta_2 = \lambda/10)$		
$m = 0$	$N \le 0{,}6$	$r(\rho) = 4\pi N \frac{u^2}{u_{\max}^2}\left[\frac{14949}{16384}T_0^*(\rho)\right]$

$m=1$	$N \le 3{,}5$	$r(\rho)=4\pi N\frac{u^2}{u_{\max}^2}\left[\frac{14949}{16384}T_0^*(\rho)+\left(-\frac{77}{1024}\right)T_1^*(\rho)\right]$
$m=2$	$N \le 35{,}6$	$r(\rho)=4\pi N\frac{u^2}{u_{\max}^2}\left[\frac{14949}{16384}T_0^*(\rho)+\left(-\frac{77}{1024}\right)T_1^*(\rho)+\frac{53}{4096}T_2^*(\rho)\right]$
$m=3$	$N \le 117$	$r(\rho)=4\pi N\frac{u^2}{u_{\max}^2}\left[\frac{14949}{16384}T_0^*(\rho)+\left(-\frac{77}{1024}\right)T_1^*(\rho)+\frac{53}{4096}T_2^*(\rho)+\left(-\frac{1}{1024}\right)T_3^*(\rho)\right]$

SIMULATION AND ANALYSIS

We will consider a distorted wave front with a defocusing value λ, which is falling on a slit or circular diaphragm and is recorded in the output plane distanced from the diaphragm at a certain distance. The following input parameters were specified in the simulation:

- radiation wavelength 532 nm,
- slit diaphragm size 1 mm by 100 µm,
- circular diaphragm diameter 1 mm,
- distance to the diffraction pattern recording plane 10 mm,
- analysis plane size 300 µm and 100 µm for the slit and circular diaphragms, respectively.

In the case of a paraxial approximation and a diaphragm size smallness in relation to the distance to the recording plane, as well as in the case of illumination of the recording plane by a plane wave falling on the observation plane perpendicularly or almost perpendicularly [23–24], it is possible to introduce the following restrictions during the integral calculations (1):

$\cos\theta \approx 1$,

$r_{12} \cong z$, in the denominator.

Figure 1 shows the results of the direct method of calculating the diffraction integral (1) in the LabVIEW program in the form of diffraction patterns and intensity distribution graphs.

In the simulation by using the Chebyshev polynomials expansion, a modified Fresnel number N was calculated to evaluate the possible economization in the series (9). Thus, for the slit diaphragm $N = 21{,}7$, and for the circular diaphragm $N = 56{,}9$.

According to Table 1, it is possible to use in both cases an expansion in three, four, or five polynomials at such Fresnel numbers.

The intensity distribution graphs and the relative intensity distribution error graphs are shown in Figures 2 and 3. Since the intensity graphs are virtually merged due to the small error values, the relative error numerical values in arbitrary sections with an approximation of 3 polynomials for the slit diaphragm are shown in Table 2 for illustrative purposes.

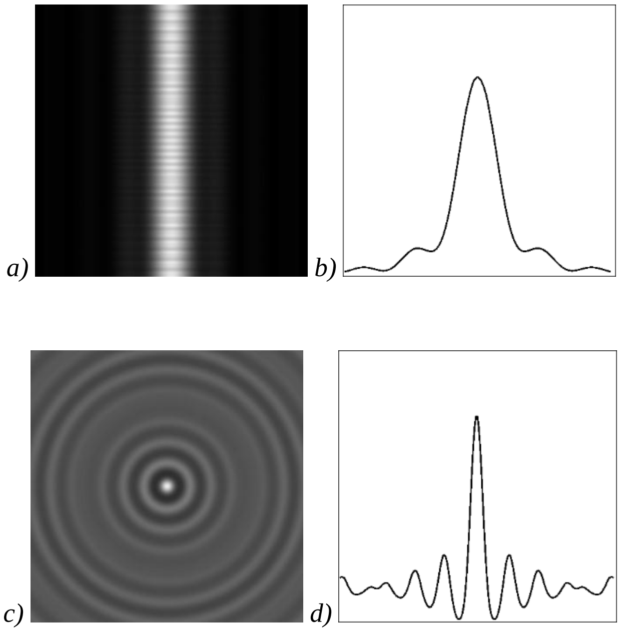

Figure 1: Diffraction patterns and intensity distributions in the case of diffraction on the slit diaphragm (a, b) and the circular diaphragm (c, d)

CONCLUSION

It has been demonstrated that introducing the parameter N allows to reduce the initial expansion dimension, and using the properties of the Chebyshev polynomials allows to reduce the series dimension while maintaining the required accuracy. Thereby, it is sufficient to comply with the upper restrictions per the modified Fresnel number N or per the summarized aperture value at a known distance between the planes.

During the reconstruction and analysis of the output distribution of the field specified not by an analytic expression, but by a set of points obtained from a photo-detector array experiment, it is also possible to use the Chebyshev polynomials as basis functions to form the light distribution under study.

As a result of the distorted wave front diffraction simulation by using the Chebyshev polynomials, it has been demonstrated that the results of the expansion in five Chebyshev polynomials have a virtually zero error with reference to the direct calculation of the diffraction integral. The further economization in the series, up to three polynomials, does not lead to strong deviations, and the error remains within the permissible limits.

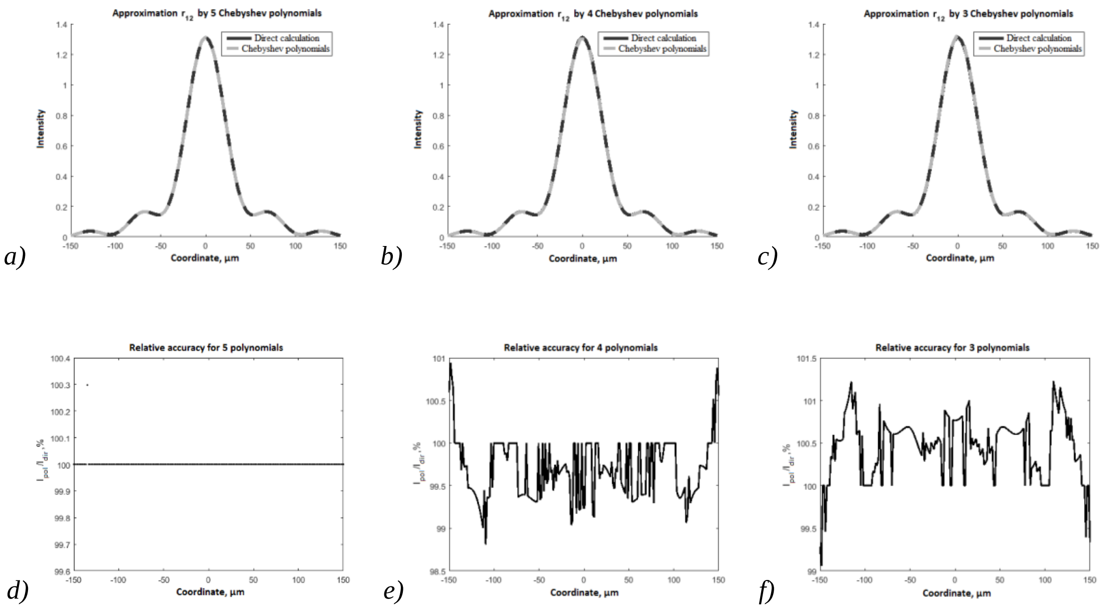

Figure 2: Intensity distributions of the direct calculation method and the calculation by using the Chebyshev polynomials in the case of diffraction on the slit diaphragm (a, b, c) and relative accuracy of the intensity distribution (d, e, f) calculations.

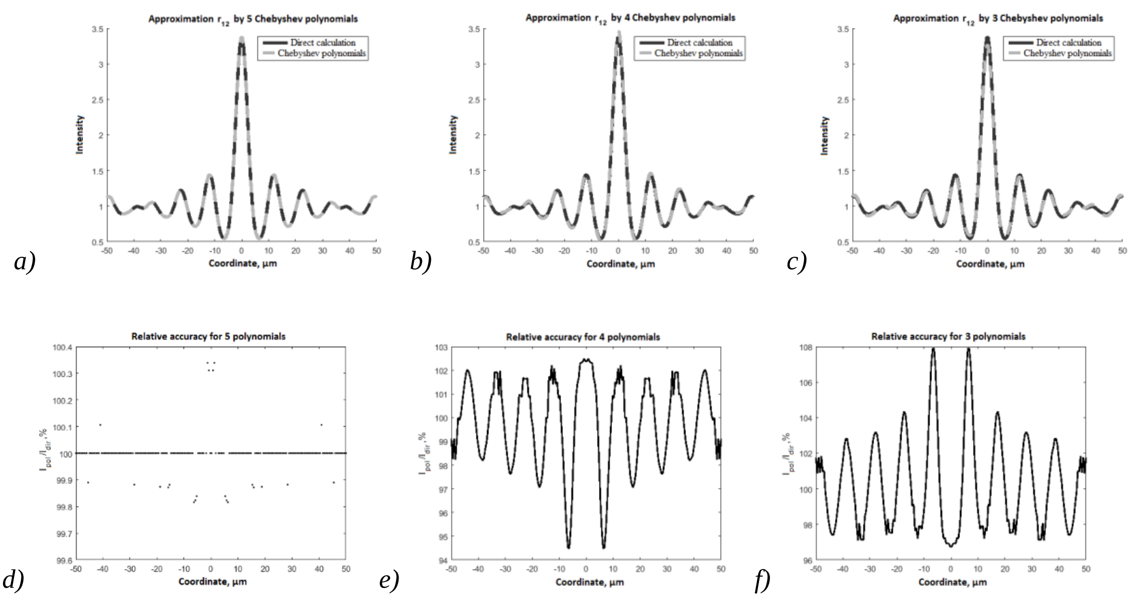

Figure 3: Intensity distributions of the direct calculation method and the calculation by using the Chebyshev polynomials in the case of diffraction on the circular diaphragm (a, b, c) and relative accuracy of the intensity distribution (d, e, f) calculations.

Table 2: Relative accuracy of the intensity distribution calculations for the slit diaphragm in the case of the diffraction integral calculation by using 4 Chebyshev polynomials on arbitrary sections

Coordinate, μm	**Intensity**		
	Direct calculation method	Calculation by using the Chebyshev polynomials	Relative accuracy
-130.40	0.0063252	0.0060729	0.9601
-70.10	0.075175	0.074768	0.9946
0.75	1.1354	1.1228	0.9889
29.40	0.58978	0.58391	0.9900

The minimum deviation from the direct calculation method in the case of expansion economization (9) is observed at diffraction on the slit diaphragm. The maximum relative error did not exceed 1.5% in case of expansion in three polynomials. In the case of diffraction on a circular diaphragm, the relative error was no more than 8%, which is also acceptable.

The given calculations show the applicability of Chebyshev polynomials in the simulation of light beams diffraction in view of their further analysis.

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