International Journal of Applied Engineering Research ISSN 0973-4562 Volume 12, Number 22 (2017) pp. 12425-12430

Decomposition of a Singularly Perturbed Model of a Two-time-scale Nonlinear System with Multiplicative Connections

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Abstract

A decomposition of singularly perturbed models of nonlinear systems with multiplicative interaction of processes is considered. The paper highlights an important class of such models in which the fast subsystem doesn't include nonlinear interactions of characteristic variables with the other fast variables. The features of the decomposition of such model class on the basis of the singular perturbance theory are shown.

Keywords: singular perturbation; singularly perturbed model of dynamical system; two-time-scale system; model order reduction; nonlinear system with multiplicative connections

INTRODUCTION

Modern automatic control systems are complex and in general are described by nonlinear mathematical models of high order. If small parameters (in relation to the other parameters), for instance, small time constants, are taken into account, that leads to the model order increase and to the description of the processes with essentially different speeds of free components of motions. Because of that the initial mathematical model must be simplified by means of its order decrease or decomposition into the models of smaller dimensions within the permissible accuracy of the processes description. Singularly perturbed models are of great practical interest for the problem solution.

The singularly perturbed models which are in the form of a system of differential equations in the Cauchy normal form with small parameters at the derivatives in some equations are studied most well. The description of the multytime-scale system dynamics can be presented in the form of such models. When the small parameters become zero the order of the singularly perturbed models decreases and under certain conditions the simplified model adequately describes the processes in the initial system. The mathematical justification of the possibility to get a simplified model is the theory of singular perturbances. It is based on *A.N. Tikhonov's theorem on the passage to the limit* (see [1, 2] or [3, 4] in English). It gives the sufficient conditions under which the difference between the processes described by the initial and the simplified models converges to zero as the small parameters tend to zero. It may be difficult to check these conditions in practice because it is necessary to investigate the stability properties of some subsystems of equations of the initial model which are nonlinear in general.

This paper is dedicated to the decomposition of a class of nonlinear systems with multiplicative interaction of the processes on the basis of the Tikhonov's theorem. Models of the given class are widely used in the description of the processes in the various objects. Examples include a model of a three-phase induction motor [5], the Rikitake model of the processes in the Earth's magnetic field [6], the Lorenz-Stenflo system for acoustic-gravity waves in the atmosphere [7], a description of the propagation of a wave in a plasma during the lightning discharge [8], the Lotka-Volterra model of interspecific competition [9], an epidemic model [10] and a description of a childhood illness called “Hand, foot and mouth disease” (HFMD) [11] in medicine, mathematical models in economics [12]. A type of models which is widely used in practice is considered in the paper, the feature of this type is that the fast subsystem doesn't include nonlinear interactions of characteristic variables with the other fast variables (for example, as in [5], [6], [8], etc).

CONDITIONS OF DECOMPOSITION AND THEIR CHECK

In general the model of the nonlinear dynamical system with multiplicative connections may be written in the form:

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij} y_j + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} b_{ijk} y_j y_k, \qquad i = \overline{1, n}.$$

Let us consider the case of two-time-scale systems in which a part of the processes has fast proper motions in comparison with the proper motions of other processes. Their mathematical description can be represented as a singularly perturbed model of the following form:

$$\begin{cases} \dfrac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij} y_j + \sum_{k=1}^{m} c_{ik} z_k + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} b_{ijk} y_j y_k + \\ \qquad + \sum_{j=1}^{n} \sum_{k=1}^{m} d_{ijk} y_j z_k + \sum_{j=1}^{m-1} \sum_{k=j+1}^{m} e_{ijk} z_j z_k, \quad i = \overline{1,\mathrm{n}}, \\ \mu \dfrac{dz_l}{dt} = \sum_{j=1}^{n} a_{n+l,j} y_j + \sum_{k=1}^{m} c_{\mathrm{n}+l,k} z_k + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} b_{n+l,j,k} y_j y_k + \\ \qquad + \sum_{j=1}^{n} \sum_{k=1}^{m} d_{n+l,j,k} y_j z_k + \sum_{j=1}^{m-1} \sum_{\substack{k=j+1 \\ j,k \neq l}}^{m} e_{n+l,j,k} z_j z_k, \; l = \overline{1,\mathrm{m}}. \end{cases} \tag{1}$$

The model (1) consists of two subsystems, one of them has slow variables y_i and the other has fast variables z_l in the sense of free components of their motions. Here μ is a small parameter, the order of the system is s = n + m.

The linearity of the fast subsystem of the model (1) mentioned above means the absence of a certain component in the right part of the equation for $\mu \frac{dz_l}{dt}$ in the model (1), it is carried out by means of limitation on sum indexes j and k in the last sum of the second equation in (1). The above mentioned component contains the multiplication of z_l and another "fast" variable.

In order to solve the Cauchy problem to find the process in the dynamical system the equations (1) are supplemented by the initial conditions

$$y_i(+0) = y_{i0}, \; i = \overline{1,\mathrm{n}}; \; z_l(+0) = z_{l0}, \; l = \overline{1,\mathrm{m}}. \tag{2}$$

Let us follow the theorem [1] and consider models of reduced and associated systems.

The reduced system may be obtained from (1) if we set $\mu = 0$ in it. The order of the reduced system is lower than the order of the initial model (by the value m) and it is a set of systems of n differential and m algebraic equations. It is the reduced system of differential equations that may be used as a model of the simplified system after the decomposition.

The associated system may be obtained from the subsystem of the fast variables of the model (1), if a new independent variable $\tau = t/\mu$ is introduced into it and variables y_i are considered as parameters.

The associated system for this case is

$$\frac{d\tilde{z}_l}{d\tau} = \sum_{j=1}^{n} a_{n+l,j} y_j + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} b_{n+l,j,k} y_j y_k + \sum_{k=1}^{m} c_{\mathrm{n}+l,k} \tilde{z}_k + \\ + \sum_{j=1}^{n} \sum_{k=1}^{m} d_{n+l,j,k} y_j \tilde{z}_k + \sum_{j=1}^{m-1} \sum_{\substack{k=j+1 \\ j,k \neq l}}^{m} e_{n+l,j,k} \tilde{z}_j \tilde{z}_k, \; l = \overline{1,\mathrm{m}}, \tag{3}$$

where y_i - const, $i = \overline{1,\mathrm{n}}$.

Let us represent the associated system (3) in a more compact form:

$$\frac{d\tilde{z}_l}{d\tau} = \alpha_l(\vec{y}) + \sum_{k=1}^{m} \beta_{lk}(\vec{y}) \tilde{z}_k + f_l(\vec{\tilde{z}}), \; l = \overline{1,\mathrm{m}}, \tag{4}$$

where $\alpha_l(\vec{y}) = \sum_{j=1}^{n} a_{n+l,j} y_j + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} b_{n+l,j,k} y_j y_k$,

$\beta_{lk}(\vec{y}) = c_{\mathrm{n}+l,k} + \sum_{j=1}^{n} d_{n+l,j,k} y_j$,

$f_l(\vec{\tilde{z}}) = \sum_{j=1}^{m-1} \sum_{\substack{k=j+1 \\ j,k \neq l}}^{m} e_{n+l,j,k} \tilde{z}_j \tilde{z}_k$

($f_l(\vec{\tilde{z}})$ can also be represented in the following form:

$f_l(\vec{\tilde{z}}) = \frac{1}{2} \sum_{j=1}^{m} \sum_{\substack{k=1 \\ j,k \neq l \\ j \neq k}}^{m} h_{n+l,j,k} \tilde{z}_j \tilde{z}_k$,

where $h_{n+l,j,k} = h_{n+l,k,j} = e_{n+l,j,k}$),

or in a matrix form:

$$\frac{d\vec{\tilde{z}}}{d\tau} = \vec{\alpha}(\vec{y}) + L(\vec{y}) \vec{\tilde{z}} + \vec{f}(\vec{\tilde{z}}), \tag{5}$$

where $\vec{\alpha}(\vec{y}) = (\alpha_1(\vec{y}), \ldots, \alpha_m(\vec{y}))^T$,

$L(\vec{y}) = [\beta_{lk}]$ - a square matrix, $l,k = \overline{1,\mathrm{m}}$,

$\vec{f}(\vec{\tilde{z}}) = (f_1(\vec{\tilde{z}}), \ldots, f_m(\vec{\tilde{z}}))^T$,

$\vec{y} = (y_1, \ldots, y_n)^T$, $\vec{\tilde{z}} = (\tilde{z}_1, \ldots, \tilde{z}_m)^T$.

The equilibrium point $\vec{\tilde{z}}^0$ of the associated system (5) is a root of the equation

$$0 = \vec{\alpha}(\vec{y}) + L(\vec{y}) \vec{\tilde{z}} + \vec{f}(\vec{\tilde{z}}). \tag{6}$$

It is obvious that this root is also the root $\vec{z}^0$ of the algebraic equations subsystem of the reduced model. The substitution of this root into the differential equations subsystem under certain conditions can give us a full description of the reduced system as a simplified model of the initial model (1).

Let us consider the conditions of the theorem on the passage to the limit [1]. According to this theorem, in order to make the solution of the initial problem (1), (2) tend towards the solution of the reduced system, as $\mu \to 0$, the fulfillment of two conditions is sufficient:

1) the root $\vec{\tilde{z}}^0$ that determines the form of the reduced model is a stable root of the associated system;

2) the initial conditions $\tilde{z}_0$ belong to the domain of influence of the root $\vec{\tilde{z}}^0$ when $\vec{y} = \vec{y}_0$.

Here it is assumed that the root $\vec{\tilde{z}}^0$ is isolated.

The solutions of the initial and the reduced systems are close to each other on the whole interval $[0, \mathrm{T}]$ for $\vec{y}$ and outside a small neighborhood of t = 0 for $\tilde{z}$ (this neighborhood is called *a boundary layer*).

Let us consider the fulfillment of these conditions for the initial model (1) when the orders m of the associated systems are various.

Let us assume that $m = 1$, i.e. only one equation of the system (1) contains a small parameter. The associated system is of the form (4) when $l = 1$, $f_l\left(\vec{\tilde{z}}\right) = 0$.

This system is a linear system with constant coefficients. The only (and thus isolated) equilibrium point z_1^0 of this system may be determined from (6):

$$z_1^0 = -\frac{\alpha_1(\vec{y})}{\beta_{11}(\vec{y})} = -\frac{\sum_{j=1}^{n} a_{n+1,j} y_j + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} b_{n+1,j,k} y_j y_k}{c_{n+1} + \sum_{j=1}^{n} d_{n+1,j} y_j},$$

($\beta_{11}(\vec{y}) \neq 0$).

The necessary and sufficient condition of its asymptotical stability according to Lyapunov when $\tau \to \infty$ uniformly in respect to $\vec{y} \in \bar{D}$ (where $\bar{D}$ is a certain closed limited domain of admissible values of $\vec{y}$) is the negative value of the real part of the root of the associated system characteristic equation. According to (4) the equation is

$$\lambda - \beta_{11}(\vec{y}) = 0.$$

This equation has the only root $\lambda = \beta_{11}(\vec{y})$ and with regard to (4) it leads to the requirement that the following inequality must hold

$$\left(c_{n+1} + \sum_{j=1}^{n} d_{n+1,j} y_j\right) < 0 \tag{7}$$

for all values of $\vec{y} \in \bar{D}$. Thus the check of the fulfillment of the first condition of the theorem reduces to the estimation of the sign of the coefficient $\beta_{11}(\vec{y})$ that, on the one hand, is not difficult and, on the other hand, it becomes complicated because it must be done in the whole domain $\bar{D}$ of the parameters $\vec{y}$.

In order to check the fulfillment of the second condition of the theorem the associated system is considered when the values of the parameters $\vec{y}$ are equal to their initial values given in (2), i.e. when $\vec{y} = \vec{y}_0$, with the initial condition for $\tilde{z}_1 = z_{10}$ from (2).

Because the associated system is linear, the equilibrium point z_1^0 is globally stable when (7) is fulfilled, i.e. with any initial conditions among them $z_1(+0) = z_{10}$. Thus, if (7) is fulfilled, it guarantees the fulfillment of the second condition of the theorem.

Let us consider the case when two equations of the model (1) contain the small parameter. When $m = 2$ the associated system is in the form (4) with $l = 1, 2$. Functions $f_1\left(\vec{\tilde{z}}\right)$ and $f_2\left(\vec{\tilde{z}}\right)$ as it was in the previous case are equal to zero because according to (4) there are no multiplicative components in the form of $\tilde{z}_1 \cdot \tilde{z}_2$ in the model because it is linear with regard to its characteristic variable:

$$f_1\left(\vec{\tilde{z}}\right) = 0,\ f_2\left(\vec{\tilde{z}}\right) = 0. \tag{8}$$

So, the associated system is linear when $m = 2$.

The equilibrium point $\vec{\tilde{z}}^0$ of the associated system (5) is the root of the equation

$$0 = \vec{\alpha}(\vec{y}) + L(\vec{y}) \cdot \vec{\tilde{z}}.$$

and if $\det L \neq 0$ it is in the following form

$$\vec{\tilde{z}}^0 = \varphi(\vec{y}) = -L^{-1}(\vec{y}) \cdot \vec{\alpha}(\vec{y}). \tag{9}$$

The linearity of the system determines that this root is isolated because it is unique. The condition of the stability of the root (9), i.e. the equilibrium point of the associated system, is the requirement that the real parts of its characteristic equation roots must be negative. According to (5), the characteristic equation is the following

$$\begin{vmatrix} \beta_{11} - \lambda & \beta_{12} \\ \beta_{21} & \beta_{22} - \lambda \end{vmatrix} = 0. \tag{10}$$

Hence we obtain

$$\lambda^2 - (\beta_{11} + \beta_{22})\lambda - \beta_{12}\beta_{21} = 0. \tag{11}$$

According to the Routh-Hurwitz stability criterion [13] for the real parts of the roots of the equation (11) to be negative it is necessary and sufficient that all coefficients of the polynom in the left part of (11) have the same sign, i.e.:

$$(\beta_{11} + \beta_{22}) < 0,\ \beta_{12}\beta_{21} < 0. \tag{12}$$

Let us finally write the requirements for the stability (12) in the expanded form taken (4) into consideration:

$$c_{n+1,1} + c_{n+2,2} + \sum_{j=1}^{n} \left(d_{n+1,j,1} + d_{n+2,j,2} \right) y_j < 0, \tag{13}$$

$$c_{n+1,2} \cdot c_{n+2,1} + \sum_{j=1}^{n} \left(c_{n+1,2} \cdot d_{n+2,j,1} + c_{n+2,1} \cdot d_{n+1,j,2} \right) y_j +$$

$$+ \sum_{r=1}^{n} \sum_{\nu=1}^{n} d_{n+2,r,1} \cdot d_{n+1,\nu,2} y_r y_\nu < 0.$$

The fulfillment of the inequations (13) guarantees the stability of the equilibrium point, i.e. the fulfillment of the first condition of the theorem. Because the associated system is linear, the stability takes place with any initial conditions among them $\tilde{z}_{10} = z_{10}$, $\tilde{z}_{20} = z_{20}$, that means the fulfillment of the second condition. Thus, when the initial model (1) has two small parameters, the check of the possibility to decrease its order is also not difficult. The complexity of the procedure is determined by the necessity to check the fulfillment of the inequations (13) for any values of $\vec{y} \in \bar{D}$. If the quantity of the small parameters in the system (1) is more than two (m>2), then the associated system (5) is nonlinear, because $\vec{f}\left(\vec{\tilde{z}}\right) \neq 0$ and there appear multiplicative components in the form of $\tilde{z}_j \tilde{z}_k$ that do not contain the characteristic variable. The check of the first condition of the theorem is carried out on the basis of the first Lyapunov method of the determination of the equilibrium point stability when the deviations from it are small. The equilibrium point $\vec{\tilde{z}}^0 = \left(\tilde{z}_1^0, \ldots, \tilde{z}_m^0\right)$ is determined by the value of the root (6). Let us present the system (4) in the form of the variables deviations from the equilibrium point $\Delta\tilde{z}_i = \tilde{z}_i - \tilde{z}_i^0$, $i = \overline{1, \text{m}}$, by means of expanding nonlinear functions $f_i\left(\vec{\tilde{z}}\right)$ into the Taylor series and taking into consideration only linear components. After rather easy transformations we can get:

$$\frac{d\Delta\tilde{z}_l}{d\tau} = \sum_{k=1}^{m} \beta_{lk}\left(\vec{y}\right) \Delta\tilde{z}_k + \sum_{j=1}^{m} \sum_{\substack{k=1 \\ j,k \neq l \\ j \neq k}}^{m} h_{n+l,j,k} \tilde{z}_k^0 \cdot \Delta\tilde{z}_j, \; l = \overline{1, \text{m}}. \tag{14}$$

The characteristic equation of the linear system (14) is

$$\left|\Gamma - \lambda \text{I}\right| = 0, \tag{15}$$

where $\Gamma = \left(\gamma_{lk}\right)$ - a square matrix $m \times m$, I – an identity matrix.

In order to fulfill the first condition of the theorem on the passage to the limit it is necessary and sufficient that the real parts of all the roots λ_i ($i = \overline{1, \text{m}}$) of the equation (15) are negative. The check of this requirement for an arbitrary value of m may be carried out by means of algebraic stability criteria for the linear systems and it is not difficult. As in the case when $m \leq 2$ the check must be carried out for any values of $\vec{y} \in \bar{D}$ and that determines the complexity of this procedure. Difficulties arise in checking the second condition of the theorem (in checking the fact that the initial conditions $\vec{\tilde{z}}(+0) = \vec{z}_0$ belong to the domain of influence of the equilibrium point $\vec{\tilde{z}}^0$ of a nonlinear system). The problem of the stability domains determination for the equilibrium states of nonlinear systems is well-known. Let us consider two possible ways of the problem solution in the case under consideration. The first one is based on the determination of the stability domain by means of the second Lyapunov method. The main difficulty of this method application is that there is no procedure of the Lyapunov function construction for a certain type of the nonlinear system. If the first condition of the theorem is fulfilled, it is possible to find the Lyapunov function for the system (5) in the class of the functions of the quadratic form. The reasons for it are the proximity of the nonlinear model (4) and linear model (14) in some region of the phase space in relation to the equilibrium point and the existence of the Lyapunov function of the class mentioned for stable linear systems.

The other way is based on the fulfillment of the sufficient conditions of the approaching to the equilibrium point in every coordinate $\vec{\tilde{z}}$. For it, it is sufficient that the deviation from the equilibrium point $\Delta\tilde{z}_j = \tilde{z}_j - \tilde{z}_j^0$ and the derivative value for this deviation have different signs. These conditions may be written as a set of inequations:

$$\left.\frac{d\tilde{z}_l}{d\tau}\right|_{\tilde{z}_l = \tilde{z}_l^0 + \Delta\tilde{z}_l} \cdot \Delta\tilde{z}_l < 0, \; l = \overline{1, \text{m}}. \tag{16}$$

The conditions (16) after some transformations and when the formulae (4) are taken into account may be presented in the expanded form through the initial model parameters in the following way:

$$\left(g_l + \sum_{k=1}^{m} \beta_{lk}\left(\vec{y}\right) \Delta\tilde{z}_k + \sum_{j=1}^{m} \sum_{\substack{k=1 \\ j,k \neq l \\ j \neq k}}^{m} h_{n+l,j,k} \tilde{z}_k^0 \Delta\tilde{z}_j + \frac{1}{2} \sum_{j=1}^{m} \sum_{\substack{k=1 \\ j,k \neq l \\ j \neq k}}^{m} h_{n+l,j,k} \Delta\tilde{z}_j \Delta\tilde{z}_k \right) \cdot \Delta\tilde{z}_l < 0, \; l = \overline{1, \text{m}}, \tag{17}$$

where $g_l = \alpha_l\left(\vec{y}\right) + \sum_{k=1}^{m} \beta_{lk}\left(\vec{y}\right) \tilde{z}_k^0 + \frac{1}{2} \sum_{j=1}^{m} \sum_{\substack{k=1 \\ j,k \neq l \\ j \neq k}}^{m} h_{n+l,j,k} \tilde{z}_j^0 \tilde{z}_k^0$,

the equations for $\alpha_l\left(\vec{y}\right)$ and $\beta_{lk}\left(\vec{y}\right)$ are given in (4).

Let Q be the domain of values of $\Delta\vec{\tilde{z}}$ that fulfill the inequalities (17). In this case the second condition of the theorem is fulfilled, if the initial conditions deviations

$\vec{\tilde{z}}(+0)=\vec{z}_0$ from the equilibrium point $\vec{\tilde{z}}^0$ belong to this region:

$$\left(z_{0l}-\tilde{z}_l^0\right)\in Q,\ l=\overline{1,\text{m}}.$$

CONCLUSION

Let us summarize the main results of the paper. The main problem of the nonlinear dynamical system models decomposition on the basis of the theory of singular perturbances is the check of the sufficient conditions of the procedure correctness. They are established by A.N. Tikhonov's theorem on the passage to the limit. The difficulty is that it is necessary to investigate the stability domain of the equilibrium point of the associated system which is nonlinear. The paper considers the dynamical systems which have nonlinear features determined by the multiplicative interaction of the characteristic variables. A wide-spread class of the two-time-scale multiplicative systems which doesn't include nonlinear interactions of characteristic variables with the other fast variables is taken for research. The stability conditions for the equilibrium point in the analytical form for the associated systems of the 1st and 2nd order are obtained in the paper. For a higher order it is shown that the investigation of the stability properties of the equilibrium point of the associated system may be carried out on the basis of the first Lyapunov method using the equation of the first approximation. The general form of the characteristic equation was determined through the initial model parameters. The conclusion about the stability may be obtained on the basis of the linear systems algebraic stability criteria using the coefficients of the equation mentioned above.

The check if the initial conditions of the Cauchy problem for the associated system belong to the stability domain of the equilibrium point can be divided into two cases. For the associated systems of the first and the second order the equilibrium point stability leads to the positive conclusion about the fulfillment of the condition. There are no procedures of this condition check for the systems of higher order. Two approaches for this problem solution are considered in this paper. In the first approach the construction of the Lyapunov function in the class of the quadratic form is justified on the basis of the second Lyapunov method. The other variant is based on the fulfillment of the sufficient condition of the approaching to the equilibrium point in every coordinate. A set of the inequations that determine the stability domain in this case are obtained in the paper.

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