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Closed Formed Solution for Thick Plates Resting on Kerr Foundation

Ammar A. Abdul Rahman

PhD Structural Engineering
Faculty Member, Department of Civil Engineering Al Nahrain University, Baghdad, Iraq
Orcid: 0000-0001-5921-2882

Abstract

The soil – raft interaction problems were previously idealized using different models and only in some cases the closed – form solutions were obtained. Due to the difficulties in solving the final differential equations, numerical techniques were used. Herein the exact – closed form stiffness matrices and fixed – end action vectors are derived for rectangular and circular thick plates resting on Kerr foundation model. This was achieved due to the huge development in solution techniques and mathematical packages like MATLAB which supported the researchers in solving most of the complicated differential equations and large matrices effectively. The final stiffness matrices for rectangular plates in cylindrical bending, circular and annular plates resting on Kerr foundation model are formulated and then solved using MATLAB. Examples previously solved using elastic continua and finite element method by other researchers are compared with the results from the closed form solutions formulated in this study. Good agreement in the overall behavior of all cases is recognized. The formulated matrices for the cases considered show how such procedures give exact solutions to problems previously considered difficult and need numerical treatment.

Keyword: Rectangular plate, circular plate, Kerr, MATLAB

INTRODUCTION

A rational approach in the analysis of foundations resting on soil media should take into account both deformational characteristics of the soil medium and the flexibility of the foundation. Foundations very often represent a complex medium. It is often difficult to find suitable analytical models for foundation problems. An acceptable analysis must include behaviour of foundation properly. By using certain assumptions there exist some simplified models to represent the behaviour of foundation. One of the most elementary models is based on the assumption that the foundation behaves elastically. This implies not only that the foundation element return to their original position after removing loads but it is also accepted that their resistance is proportional to the deformation they experience. This assumption can be acceptable if the displacement and pressure underneath foundation are small and approximately linearly related to each other. The analytical treatment of such interaction problems invariably requires a certain amount of mathematical and computational effort. It should be appreciated that a complete analytical treatment of soil-foundation interaction problem should take into account not only the interaction but also the stiffness of the structure which has a considerable influence on the behaviour of soil-foundation system.

AIM OF THE RESEARCH

The analysis of the interaction between structural foundation and supporting media is of fundamental importance in both structural and geotechnical engineering. Results of such analysis provide information, which can be used in the structural design of the foundation and in the analysis of stresses and deformations within the supporting soil medium. This intrinsic interest has generated a variety of analytical and experimental studies in the subject of soil-foundation interaction. Exact closed form stiffness matrices and fixed end actions are derived for rectangular plates in cylindrical bending, circular and annular plates resting on Kerr foundation model. Stiffness matrices have been assembled in MATLAB program to evaluate the deflection, shear force, bending moment and contact pressure in the foundation. Examples are solved and comparisons are made between the formulated models and those previously solved numerically by other researchers.

PLATE BEHAVIOUR ON ELASTIC FOUNDATION

The flexural behaviour of the plates can be adequately described by the Poisson- Kirchhoff thin plate theory. In this theory the effect of transverse shear deformation is neglected. The basic assumptions of the classical theory are often violated in regions of localized loading or in regions of abrupt changes in the flexural stiffness. Similar discrepancies can occur in soil-foundation interaction problems where the supporting soil medium exhibits marked variations in soil properties. In this circumstance it becomes necessary to modify the classical theory by incorporating the effect of shearing force deformations. This soil response can be modelled through several models incorporating shearing force deformations, like the Kerr Foundation Model.

As a generalization of the Pasternak concept, Kerr introduced the three-parameter foundation model which consists of two elastic spring layers interconnected by an elastic shear layer (Figure 1). The response of this:

$$\left(1+\frac{K}{C}\right)p-\frac{G}{C}\nabla^2 p = K\,w - G\,\nabla^2 w \qquad (1)$$

Where C, G, K, are model parameters.
When $C = 3K$, this model is identical with a similar foundation model proposed by Reissner. The main advantages of this model over the Pasternak model are, the constant response does not include concentrated reaction and an additional parameter is available for fitting the theoretical model with experimental results. But the main disadvantage of this model is the fact that instead of a fourth order differential equation the governing equation for a plate on this foundation is of the sixth order. [1]

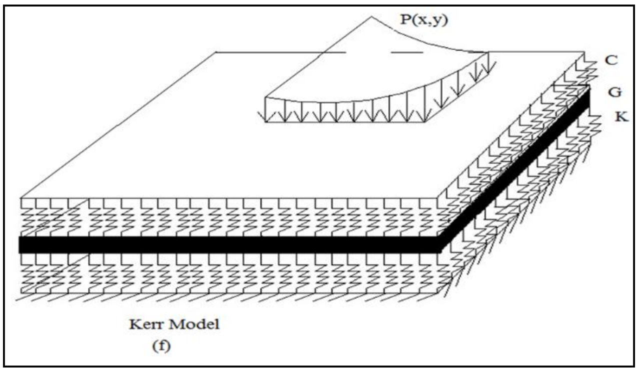


Figure 1: Kerr Foundation Model

ANALYSIS OF PLATES RESTING ON KERR FOUNDATION MODEL

The basic equations governing the flexure of an elemental strip cutting out from the plate may be written as:

$$D\,w^{IV}(x) = q(x) - p(x) \qquad (2)$$
$$p(x) = C(w(x) - \overline{w}(x)) \qquad (3)$$
$$p(x) = K\,\overline{w}(x) - G\overline{w}''(x) \qquad (4)$$

where $w(x)$ is plate deflection ; $\overline{w}(x)$ is shear layer deflection ; $q(x)$ is the distributed load on the plate ; $p(x)$ is the contact pressure ; and C, K, G are Kerr model parameters. Substituting Equation (3) into Equation (2) and Equation (4) gives the governing equations for elemental strip resting on Kerr model.

$$D\,w^{IV}(x) + C\,w(x) - C\,\overline{w}(x) = q(x) \qquad (5)$$
$$-C\,w(x) + (K+C)\overline{w}''(x) = 0 \qquad (6)$$

From Equations (5) and (6), one can find the equations satisfied by $w(x)$ only, and that by $\overline{w}(x)$ only:

$$GD\frac{d^6w}{dx^6} - D\frac{(C+K)d^4w}{dx^4} + G\,C\frac{d^2w}{dx^2} - K\,C\,w =$$
$$G\frac{d^2q}{dx^2} - (C+K)\,q$$
$$GD\frac{d^6\overline{w}}{dx^6} - D\frac{(C+K)d^4\overline{w}}{dx^4} + GC\frac{d^2\overline{w}}{dx^2} - KC\overline{w} = -Cq \qquad (7)$$

The homogeneous solutions of Equation (7) is: [2]

$$w(x) = C_1Z_1(x) + C_2Z_2(x) + C_3Z_3(x) + C_4Z_4(x) + C_5Z_5(x) + C_6Z_6(x)$$
$$\overline{w}(x) = B_1Z_1(x) + B_2Z_2(x) + B_3Z_3(x) + B_4Z_4(x) + B_5Z_5(x) + B_6Z_6(x) \qquad (8)$$

Where
$$Z_1(x) = \cosh mx \cos nx$$
$$Z_2(x) = \cosh mx \sin nx$$
$$Z_3(x) = \sinh mx \cos nx$$
$$Z_4(x) = \sinh mx \sin nx$$
$$Z_5(x) = \cosh Px$$
$$Z_6(x) = \sinh Px$$

$$P = \sqrt{\frac{K+C}{3G} + \sqrt[3]{h_2+\sqrt{h_1}} + \sqrt[3]{h_2-\sqrt{h_1}}}$$

$$m = \sqrt{\frac{((\alpha^2+\gamma^2)^{1/2}+\alpha)}{2}}$$
$$n = \sqrt{\frac{((\alpha^2+\gamma^2)^{1/2}-\alpha)}{2}}$$

$$\alpha = \frac{K+C}{3G} - \frac{1}{2}\left(\sqrt[3]{h_2+\sqrt{h_1}} + \sqrt[3]{h_2-\sqrt{h_1}}\right)$$
$$\gamma = \frac{\sqrt{3}}{2} + \left(\sqrt[3]{h_2+\sqrt{h_1}} - \sqrt[3]{h_2-\sqrt{h_1}}\right)$$
$$h_1 = \frac{C}{108D}\left\{\frac{C}{D}\left[4\frac{C}{D} + \frac{1}{G^2}(8\,K^2 - 20KC - C^2)\right] + \frac{4K}{G^2}\,(K+C)^3\right\}$$
$$h_2 = (\frac{C+K}{3G})^3 - \frac{C(C-2K)}{6GD}$$

Then, the homogeneous solution is given in matrix form by the following pair of functions: [3]

$$ww(x) = \lfloor C\rfloor \begin{bmatrix} Z_1(x)\\ Z_2(x)\\ Z_3(x)\\ Z_4(x)\\ Z_5(x)\\ Z_6(x)\end{bmatrix} \;,\; \overline{w} = \lfloor C\rfloor \begin{bmatrix} \rho Z_1(x) - \emptyset Z_4(x)\\ \rho Z_2(x) + \emptyset Z_3(x)\\ \rho Z_3(x) - \emptyset Z_2(x)\\ \rho Z_4(x) + \emptyset Z_1(x)\\ \zeta Z_5(x)\\ \zeta Z_6(x)\end{bmatrix}$$

Or $(x) = \lfloor Z(x)\rfloor\{C\}$, $\overline{w}(x) = \lfloor F(x)\rfloor\,\{C\}$ (9)
Where $\lfloor C\rfloor$ ($= \lfloor C_1C_2C_3C_4C_5C_6\rfloor$) is the unknown coefficients vector;
$\rho = \frac{D}{C}\,(\alpha^2 - \gamma^2) + 1$, $\emptyset = \frac{2D}{C}\,(\alpha\,\gamma)$, $\zeta = \frac{D}{C}\,P^4 + 1.$
From the homogeneous solution, the expressions for displacement derivatives are:

$$w'(x) = \lfloor C \rfloor \begin{bmatrix} mZ_3(x) - nZ_2(x)\\ mZ_4(x) + nZ_1(x)\\ mZ_1(x) - nZ_4(x)\\ mZ_2(x) + nZ_3(x)\\ PZ_6(x)\\ PZ_5(x)\end{bmatrix}$$

$$\bar{w}'(x) = \lfloor C \rfloor \begin{bmatrix} (m\rho - n\emptyset)Z_3(x) - (n\rho + m\emptyset)Z_2(x) \\ (m\rho - n\emptyset)Z_4(x) + (n\rho + m\emptyset)Z_1(x) \\ (m\rho - n\emptyset)Z_1(x) - (n\rho + m\emptyset)Z_4(x) \\ (m\rho - n\emptyset)Z_2(x) + (n\rho + m\emptyset)Z_3(x) \\ P\zeta Z_6(x) \\ P\zeta Z_5(x) \end{bmatrix}$$

$$w''(x) = \lfloor C \rfloor \begin{bmatrix} (m^2 - n^2)Z_1(x) - 2mnZ_4(x) \\ (m^2 - n^2)Z_2(x) + 2mnZ_3(x) \\ (m^2 - n^2)Z_3(x) - 2mnZ_2(x) \\ (m^2 - n^2)Z_4(x) + 2mnZ_1(x) \\ P^2 Z_5(x) \\ P^2 Z_6(x) \end{bmatrix}$$

$$w'''(x) = \lfloor C \rfloor \begin{bmatrix} m(m^2 - 3n^2)Z_3(x) - n(3m^2 - n^2)Z_2(x) \\ m(m^2 - 3n^2)Z_4(x) + n(3m^2 - n^2)Z_1(x) \\ m(m^2 - 3n^2)Z_1(x) - n(3m^2 - n^2)Z_4(x) \\ m(m^2 - 3n^2)Z_2(x) + n(3m^2 - n^2)Z_3(x) \\ P^3 Z_6(x) \\ P^3 Z_5(x) \end{bmatrix}$$

By considering equilibrium and compatibility of an infinitesimal element of the plate, the relations for the shear forces in the plate $V(x)$, shear forces in the shear layer $S(x)$, and bending moments $M(x)$ are:

$V(x) = D\, w'''(x), S(x) = -G\, \bar{w}'(x)$,
$M(x) = D\, w''(x)$ (10)

STIFFNESS MATRIX DERIVATION

The nodal degrees of freedom {e} as shown in Figure (2) can be expressed in terms of the unknown coefficients as follows:

$$\begin{bmatrix} w_1 \\ \theta_1 \\ \bar{w}_1 \\ w_2 \\ \theta_2 \\ \bar{w}_2 \end{bmatrix} = \begin{bmatrix} \lfloor Z(0) \rfloor \\ -\lfloor Z'(0) \rfloor \\ \lfloor F(0) \rfloor \\ \lfloor Z(L) \rfloor \\ -\lfloor Z'(L) \rfloor \\ \lfloor F(L) \rfloor \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix}$$

Or $\{e\} = [R]\{C\}$ where $[R]$ is 6x6 matrix (11)

Equation (11) may be inverted to express the unknown coefficients in terms of the d.o.f., to give:

$\{C\} = [R]^{-1}\{e\}$ (12)

The nodal forces (F) as shown in Figure (2) can be expressed in terms of the unknown coefficients as follows:

$V_1 = D\, w'''(x)$, $M_1 = D\, w''(x)$,
$S_1 = -G\, \bar{w}'(x)$ at x=0
… (13)

$V_2 = -D\, w'''(x)$, $M_2 = -D\, w''(x)$,
$S_2 = G\, \bar{w}'(x)$ at x=L

Where $\lfloor F(L) \rfloor = \lfloor V_1 \;\; M_1 \;\; S_1 \;\; V_2 \;\; M_2 \;\; S_2 \rfloor$; V_1, V_2 are nodal shear forces in the plate; M_1, M_2 are nodal moments; and S_1, S_2 are shear forces in the shear layer.

Equation (13) can be rewritten in matrix form as:

$$\begin{bmatrix} V_1 \\ M_1 \\ S_1 \\ V_2 \\ M_2 \\ S_2 \end{bmatrix} = \begin{bmatrix} D\,\lfloor Z'''(0) \rfloor \\ D\,\lfloor Z''(0) \rfloor \\ -G\,\lfloor F'(0) \rfloor \\ -D\,\lfloor Z'''(L) \rfloor \\ -D\lfloor Z''(L) \rfloor \\ G\,\lfloor F'(L) \rfloor \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} \quad \ldots (14)$$

Substituting Equation (2) into Equation (14) gives

$\{F\} = [K]\{e\}$ … (15)

Where [K] is the required stiffness matrix, the stiffness terms in its upper triangle are:

$K_{11} = -D(F_1T_1 + F_2T_2 + F_3T_3)$

$K_{12} = D(F_1T_4 + F_2T_5 + F_3T_6)$

$K_{13} = G(F_1T_7 + F_2T_8 + F_3T_9)$

$K_{14} = D\,\{F_2[n(\alpha - P^2) + m\gamma] + F_3[n(\alpha - P^2) - n\gamma]\}$

$K_{15} = -DF_1(\zeta\,\gamma - P^2 + \alpha)$

$K_{16} = -G\{F_2[n(\rho - \zeta) + m\emptyset] + F_3[m(\rho - \zeta) - n\emptyset]\}$

$K_{22} = -D(F_4T_4 + F_5T_5 + F_6T_6)$

$K_{23} = D(F_7T_4 + F_8T_5 + F_9T_6)$

$K_{24} = -DF_1(\zeta\,\gamma - P^2 + \alpha)$

$K_{25} = DF_4(\zeta\,\gamma - P^2 + \alpha)$

$K_{26} = DF_7(\zeta\gamma - P^2 + \alpha)$

$K_{33} = G(F_7T_7 + F_8T_8 + F_9T_9)$

$K_{34} = -G\{F_2[n(\rho - \zeta) + m\emptyset] + F_3[m(\rho - \zeta) - n\emptyset]\}$

$K_{35} = -DF_7(\zeta\,\gamma - P^2 + \alpha)$

$K_{36} = -G\{F_8[n(\rho - \zeta) + m\emptyset] + F_9[m(\rho - \zeta) - n\emptyset]\}$

$K_{44} = -D(F_1T_1 + F_2T_2 + F_3T_3)$

$K_{45} = -D(F_1T_4 + F_2T_5 + F_3T_6)$

$K_{46} = G(F_1T_7 + F_2T_8 + F_3T_9)$

$$K_{56} = -D(F_7T_4 + F_8T_5 + F_9T_6)$$

$$K_{66} = G(F_7T_7 + F_8T_8 + F_9T_9)$$

$$\begin{bmatrix} F_1 & F_2 & F_3 \\ F_4 & F_5 & F_6 \\ F_7 & F_8 & F_9 \end{bmatrix} = \frac{1}{P_O}\begin{bmatrix} P_2P_6 - P_3P5 & P_3P_4 - P_1P_6 & P_1\,P_5 - P_2P_4 \\ P_2P_9 - P_3P_8 & P_3P_7 - P_1P_9 & P_1P_8 - P_2P_7 \\ P_5P_9 - P_6P_8 & P_6P_7 - P_4P_9 & P_4P_8 - P_5P_7 \end{bmatrix}$$

$$P_O = P_1(P_5P_9 - P_6P_8) + P_2(P_6P_7 - P_4P_9) + P_3(P_4P_8 - P_5P_7)$$

$$\zeta = (\zeta - \rho)/\emptyset$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{bmatrix} = \begin{bmatrix} (\zeta\rho - \emptyset)A_4 + \zeta(A_1 - A_5) \\ \rho A_2 + \emptyset A_3 - \frac{n}{P}\zeta\, A_6 \\ \rho A_3 - \emptyset A_2 - \frac{m}{P}\zeta\, A_6 \\ (\zeta n + m)A_3 + (\zeta m - n)A_2 - P\, A_6 \\ m\,A_4 + n\,A_1 - n\,A_5 \\ m\,A_1 - n\,A_4 - mA_5 \\ A_1 + \zeta\,A_4 - A_5 \\ A_2 - \frac{n}{P}\,A_6 \\ A_3 - \frac{m}{P}\,A_6 \end{bmatrix},$$

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix} = \begin{bmatrix} \cosh mL \cos nL \\ \cosh mL \sin nL \\ \sinh mL \cos nL \\ \sinh mL \sin nL \\ \cosh PL \\ \sinh PL \end{bmatrix}$$

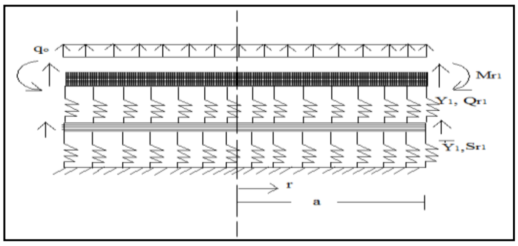


Figure 2: Rectangular Plate Strip on Kerr Model

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} = \begin{bmatrix} (\zeta\alpha - \gamma)(mA_2 + nA_3) + (\zeta\gamma + \alpha)(mA_3 - nA_2) - P^3A_6 \\ (m\alpha - n\gamma)A_4 + (n\alpha + m\gamma)A_1 - nP^2A_5 \\ (m\alpha - n\gamma)A_1 - (n\alpha + m\gamma)A_4 - mP^2A_5 \\ (\zeta\gamma - \alpha)A_1 + (\zeta\alpha - \gamma)A_4 - P^2A_5 \\ \alpha A_2 + \gamma\, A_3 - nP\, A_6 \\ \alpha A_3 - \gamma A_2 - mP\, A_6 \\ (\zeta\rho - \emptyset)(m\,A_2 + n\,A_3) + \zeta(m\,A_3 - n\,A_2) - P\zeta\,A_6 \\ (m\rho - n\emptyset)A_4 + (n\rho + m\emptyset)A_1 - n\,\zeta\,A_5 \\ (m\rho - n\emptyset)A_1 - (n\rho + m\emptyset)A_4 - m\zeta\,A_5 \end{bmatrix}$$

CIRCULAR PLATE

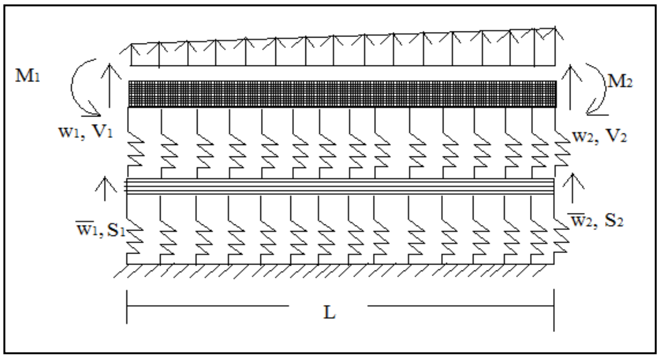
The basic equations governing the flexure of a thin circular plate which is subjected to axisymmetric external loading are [2]:

$$D\,\nabla^4\,w(r) = q(r) - p(r) \qquad \ldots (16)$$

$$p(r) = C\,w(r) - C\,\overline{w}(r) \qquad \ldots (17)$$

$$p(r) = -G\,\nabla^2\,\overline{w}(r) + K\,\overline{w}(r) \qquad \ldots (18)$$

Substituting Equation (17) into Equation (16) and Equation



(18) gives the governing equations for circular plate resting on Kerr model.

$$D\,\nabla^4\,w(r) + Cw(r) - C\,\overline{w}(r) = q(r) \qquad \ldots(19)$$

$$-C\,w(r) + (K + C)\overline{w}(r) - G\,\nabla^2\,\overline{w}(r) = 0 \qquad \ldots(20)$$

From Equation (19) and Equation (20), one can find the equations satisfied by $w(r)$ only, and that by $\overline{w}(r)$ only.

$$G\ D\ \nabla^6\ w(r) - D(C+K)\nabla^4\ w(r) + G\ D\ \nabla^2\ w(r) - C\ K\ w(r) = G\ \nabla^2 q(r) - (C+K)\ q(r) \quad \dots (21)$$

$$G\ D\ \nabla^6\ \overline{w}(r) - D\ (C+K)\nabla^4 \overline{w}(r) + G\ D\ \nabla^2 \overline{w}(r) - C\ K\ \overline{w}(r) = -C\ q(r)$$

The differential operator of Equation (21) can be split into a product of three operators of second – order, in the form [3]:

$$(\nabla^2 - P^2)(\nabla^2 + \gamma 1)(\nabla^2 + \gamma 2)\ w(r) = \frac{1}{GD}\ [G\ \nabla^2\ q(r) - (C+K)q(r)] \quad \dots (22)$$

$$(\nabla^2 - P^2)(\nabla^2 + \gamma 1)(\nabla^2 + \gamma 2)\overline{w}(r) = -C\ q(r)/GD$$

Where

$$P^2 = \frac{C+K}{3G} + \sqrt[3]{h_2 + \sqrt{h_1}} + \sqrt[3]{h_2 - \sqrt{h_1}}$$
$$\gamma 1 = -\alpha - i\gamma = \sqrt{(\alpha^2 + \gamma^2)}\ e^{-2\emptyset i} = l^2\ e^{-2\emptyset i}$$

$$\gamma 2 = -\alpha + i\gamma = \sqrt{(\alpha^2 + \gamma^2)}\ e^{2\emptyset i} = l^2\ e^{2\emptyset i}$$

$$\cos\emptyset = -\frac{\alpha}{\sqrt{\alpha^2 + \gamma^2}}$$

$$\emptyset = cos^{-1}\left(-\frac{\alpha}{\sqrt{\alpha^2 + \gamma^2}}\right)$$

$$\alpha = \frac{C+K}{3G} - \frac{1}{2}\sqrt[3]{h_2 + \sqrt{h_1}} + \sqrt[3]{h_2 - \sqrt{h_1}}$$

$$\gamma = \frac{\sqrt{3}}{2}\left(\sqrt[3]{h_2 + \sqrt{h_1}} - \sqrt[3]{h_2 - \sqrt{h_1}}\right)$$

$$h_1 = \frac{C}{108D}\left\{\frac{C}{D\left[\frac{4C}{D} + \frac{1}{G^2(8\ K^2 - 20KC - C^2)}\right]} + \frac{4C}{G^4}\ (K+C)^3\right\}$$

$$h_2 = (\frac{C+K}{3G})^3 - \frac{C(C-2K)}{6GD}$$

The homogeneous solution of Equation (22) is [3]:

$$w(r) = C_1 u_1(r) + C_2 u_2(r) + C_3 u_3(r) + C_4 u_4(r) + C_5 u_5(r) + C_6 u_6(r)$$
$$\overline{w}(r) = B_1 u_1(r) + B_2 u_2(r) + B_3 u_3(r) + B_4 u_4(r) + B_5 u_5(r) + B_6 u_6(r) \quad \dots (23)$$

The u function can be written in terms of power series as follows:

$$u_1(r) = \sum_{n=0}^{n=5} \frac{-1^n (\frac{r}{2})^{2n} l^{2n}}{(n!)^2} \cos n\emptyset$$

$$u_2(r) = \sum_{n=0}^{n=5} \frac{-1^n (\frac{r}{2})^{2n} l^{2n}}{(n!)^2} \sin n\emptyset$$

$$u_3(r) = \left[1 - \frac{\emptyset}{\pi}\right] u_1(r) - \frac{2}{\pi}\left[u_2(r) \log\left(l\frac{\gamma^* r}{2}\right) - R_1\right]$$

$$u_4(r) = \left[1 - \frac{\emptyset}{\pi}\right] u_2(r) + \frac{2}{\pi}\left[u_1(r) \log\left(l\frac{\gamma^* r}{2}\right) - R_2\right]$$

Where

$$R_1 = \sum_{n=0}^{n=5} \frac{-1^n (\frac{r}{2})^{2n} l^{2n}}{(n!)^2} \sin n\emptyset\ \ \rho_n$$

$$R_2 = \sum_{n=0}^{n=5} \frac{-1^n (\frac{r}{2})^{2n} l^{2n}}{(n!)^2} \cos n\emptyset\ \ \rho_n$$

$\rho_o = 0, \rho_n = \sum_{t=1}^{n}(\frac{1}{t}), \gamma = \ln\gamma^* = 0.5772157$ Euler's constant

$$u_5(r) = \sum_{n=0}^{n=5} \frac{(P\ r)^{2n}}{(n!)^2}$$

$$u_6(r) = -u_5(r)\log\left(\gamma^* P\frac{r}{2}\right) + \overline{R1}$$

$$\overline{R_1} = \sum_{n=0}^{n=5} \frac{(\frac{Pr}{2})^{2n}}{(n!)^2}\ \rho_n$$

Substituting Equation (23) into Equation (19), the homogeneous solution may be given in matrix form by the following pair of functions [3]:

$$w(r) = \lfloor C \rfloor \begin{bmatrix} u_1(r) \\ u_2(r) \\ u_3(r) \\ u_4(r) \\ u_5(r) \\ u_6(r) \end{bmatrix}$$

$$\bar{w}(r) = \lfloor C \rfloor \begin{bmatrix} (\frac{D}{C} l^4 \cos 2\emptyset + 1)u_1(r) - (\frac{D}{C} l^4 \sin 2\emptyset)u_2(r) \\ (\frac{D}{C} l^4 \cos 2\emptyset + 1)u_2(r) + (\frac{D}{C} l^4 \sin 2\emptyset)\ u_1(r) \\ (\frac{D}{C} l^4 \cos 2\emptyset + 1)u_3(r) - (\frac{D}{C} l^4 \sin 2\emptyset)\ u_4(r) \\ (\frac{D}{C} l^4 \cos 2\emptyset + 1)u_4(r) + (\frac{D}{C} l^4 \sin 2\emptyset)\ u_3(r) \\ \left(\frac{D}{C} P^4 + 1\right) u_5(r) \\ \left(\frac{P}{D} P^4 + 1\right) u_6(r) \end{bmatrix}$$

Or $w(r) = \lfloor C \rfloor \{u(r)$, $\bar{w}(r) = \lfloor C \rfloor \{F(r)\}$ … (24)
Where $\lfloor C \rfloor (= \lfloor C_1\ C_2\ C_3\ C_4\ C_5\ C_6 \rfloor)$ is the unknown coefficients vector.
From the homogeneous solution, the expression for the displacement derivative is

$$w'(r) = -C_1 u_1'(r) - C_2 u_2'(r) - C_3 u_3'(r) - C_4 u_4'(r) - C_5 u_5'(r) - C_6 u_6'(r)$$

Or $w'(r) = -\lfloor C \rfloor \{u'(r)\}$ … (25)
Where

$$u_1'(r) = \sum_{n=0}^{n=5} \frac{-1^n (\frac{r}{2})^{2n-1} l^{2n}}{n!^2} \cos n\emptyset$$

$$u_2'(r) = \sum_{n=0}^{n=5} \frac{-1^n (\frac{r}{2})^{2n-1} l^{2n}}{n!^2} \sin n\emptyset$$

$$u_3'(r) = \left[1 - \frac{\emptyset}{\pi}\right] u_1'(r) - \frac{2}{\pi} [u_2'(r) \log\left(\frac{l\gamma^* r}{2}\right) - R_1']$$

$$u_4'(r) = \left[1 - \frac{\emptyset}{\pi}\right] u_2'(r) - \frac{2}{\pi} [\frac{-1}{r + u_1'(r)} \log\left(l \frac{\gamma^* r}{2}\right) - R_2']$$

And

$$R_1' = \frac{1}{2} \sum_{n=0}^{n=5} \frac{-1^n \left(\frac{r}{2}\right)^{2n-1} l^{2n}}{n!^2} (\rho_n + \rho_{n+1}) l^{2n} \sin n\emptyset$$

$$R_2' = \frac{1}{2} \sum_{n=0}^{n=5} \frac{-1^n \left(\frac{r}{2}\right)^{2n-1} l^{2n}}{n!^2} (\rho_n + \rho_{n+1}) l^{2n} \cos n\emptyset$$

$$u_5'(r) = -\sum_{n=0}^{n=5} \frac{P(\frac{Pr}{2})^{2n-1}}{n!^2}$$

$$u_6'(r) = -u_5'(r) \log\left(\frac{\gamma^* \mathrm{Pr}}{2}\right) - \bar{\mathrm{R}}_2 + \frac{1}{r}$$

$$\bar{R}_2 = \frac{1}{2}\ u5'(r) = -\sum_{n=0}^{n=5} \frac{P\left(\frac{Pr}{2}\right)^{2n-1}}{n!^2} (\rho_n + \rho_{n+1})$$

$$\rho_{n+1} = \sum_{t=1}^{n+1} \frac{1}{t}$$

CIRCULAR PLATE STIFFNESS MATRIX

By making use of the conditions of zero slope and zero shearing forces at the centre of the plate, the following is obtained $C_3 = C_4 = C_6 = 0$, thus, Equation (24) and Equation (25) become:

$$w(r) = C_1 u_1(r) + C_2 u_2(r) + C_5 u_5(r)$$

Or

$$W(r) = \lfloor u(r) \rfloor \{C\} \qquad \text{… (26)}$$

$$\theta(r) = C_1 u_1'(r) + C_2 u_2'(r) + C_5 u_5'(r)$$

Or

$$\theta(r) = \lfloor u'(r) \rfloor \{C\}$$

Where $\lfloor C \rfloor = (\lfloor C_1\ \ C_2\ \ C_5 \rfloor)$ is the unknown coefficients vector.
The nodal d.o.f. {e} as shown in Figure (3) can be expressed in terms of unknown coefficients as follows:

$$\begin{bmatrix} y_1 \\ \theta_1 \\ \overline{y_1} \end{bmatrix} = \begin{bmatrix} \lfloor u(a) \rfloor \\ \lfloor u'(a) \rfloor \\ \lfloor F(a) \rfloor \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_5 \end{bmatrix}$$

Or $\{e\} = [R]\{C\}$ where [R] is 3x3 matrix … (27)

Equation (27) may be inverted to be expressed the unknown coefficients in terms of the d.o.f. to give

$$\{C\} = [R]^{-1} \{e\} \qquad \text{… (28)}$$

The nodal forces {F} as shown in Figure (3) can be expressed in terms of the unknown coefficients as follows:

$$Qr_1 = -D \frac{d}{dr} \nabla^2 w \quad , \quad Mr_1 = -D \left[\frac{d^2 w}{dr^2} + \frac{?_p}{r} \frac{dw}{dr}\right] \ ,$$

$$Sr_1 = G \frac{d\bar{w}}{dr} \qquad \text{at r=a} \qquad \text{… (29)}$$

Where $\lfloor F \rfloor = \lfloor Qr_1\ \ Mr_1\ Sr_1 \rfloor$
Using Equation (29) can be rewritten in matrix form as:

$$\begin{bmatrix} Qr_1 \\ Mr_1 \\ Sr_1 \end{bmatrix} = \begin{bmatrix} 0 & -Dl^2 \cos\emptyset & 0 \\ D\ l^2 \cos\emptyset & -D \frac{1-?_p}{a} & 0 \\ 0 & -G\left(\frac{D}{C} l^4 \cos 2\emptyset + 1\right) & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \\ \overline{y_1} \end{bmatrix} + \begin{bmatrix} A_1 V_2 & -A_1 V_1 & A_2 V_5 \\ -A_1 w_2 & A_1 w_1 & -A_2 W5 \\ A_3 V_2 & -A_3 V_1 & -A_1 V_5 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_5 \end{bmatrix}$$

(30)

$$\begin{bmatrix} w_1 V_1 \\ w_2 V_2 \\ w_3 V_3 \end{bmatrix} = \begin{bmatrix} u_1(a) & u_1{}'(a) \\ u_2(a) & u_2{}'(a) \\ u_5(a) & u_5{}'(a) \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} D\, l^2 \sin \emptyset \\ D(P^2 + l^2 \cos \emptyset) \\ \dfrac{GD}{C}\, l^4 \sin 2\emptyset \\ \dfrac{GD}{C}(P^4 - l^4 \sin 2\emptyset) \end{bmatrix}$$

,

Substituting Equation (28) into Equation (30) gives:

$$\{F\} = [K]\{e\} \qquad \ldots (31)$$

Where [K] is the required stiffness matrix.
Figure 3: Circular Plate on Kerr Model

APPLICATIONS USING MATLAB

In this section, the results of solving the final analytical matrices derived previously by using MATLAB software are given for various problems to illustrate the range of practical applications of the interactive methods presented in the previously for plates on elastic foundation analysis and to put the structural analysis of foundations on a more rational basis.

MATLAB is a powerful software package that has seen massive uptake and popularity among the science and engineering communities. In addition to its ease of use, MATLAB can perform complex matrix, mathematical, and graphics operations, while requiring minimal programing effort from the user. It can also be used to build complete stand- alone applications using its graphical user – interface capabilities.

Infinitely Long Plate of Finite Width with Free Ends Resting On Kerr Model and Subjected to Concentrated Load at the Edges

A rectangular plate with free ends resting on Kerr foundation Model and subjected to concentrated load along the edges will be solved in this application. The concentrated load will be taken equal to (3.649 kN/m) and the length L =3.048 m, thickness h = 0.3048 m. The material and the foundation properties are:

E_p, □$_p$: Plate young's modulus & Poisson's ratio (20690 MPa ,0.20)

C, K, G :
Kerr model parameter $(407.28 \frac{MN}{m^3}, 7.47 \frac{MN}{m^3}, 245.24 \frac{MN}{m})$ [3].

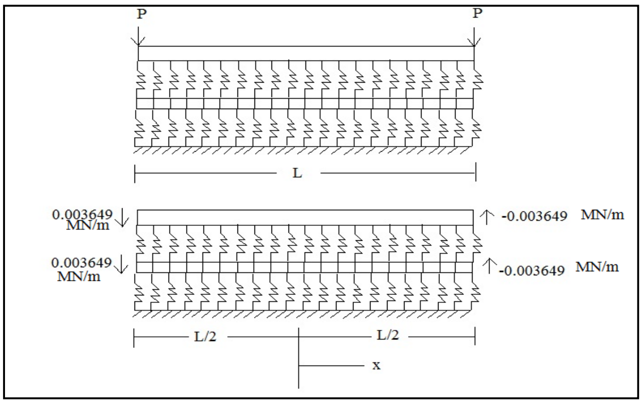


Figure 4: Retangular plate of Finte Width with Free Ends Resing on Kerr Model and Subjected to Concentrated Load at The Edges

The displacement function at the plate and the displacement function at the shear layer governing the problem are Equation (8):

$$w = C_1 Z_1 + C_2 Z_2 + C_3 Z_3 + C_4 Z_4 + C_5 Z_5 + C_6 Z_6$$

$$w = C_1 F_1 + C_2 F_2 + C_3 F_3 + C_4 F_4 + C_5 F_5 + C_6 F_6$$

$Z_1 = \cosh(mx)\cos(nx)$, $Z_2 = \cosh(mx)\sin(nx)$
$Z_3 = \sinh(mx)\cos(nx)$, $Z_4 = \sinh(mx)\sin(nx)$
$Z_5 = \cosh(Px)$, $Z_6 = \sinh(Px)$
From MATLAB the values of m, n, P described in section (4) are:
$m = 1.14424$
$n = 0.88792$ … (32)
$P = 0.80587$
At the center of the plate
The slope $\theta = \frac{dw}{dx} = 0$ and $w''' = 0$ $\overline{w}' = 0$ will give:

$$C_2 = C_3 = C_6 = 0 \qquad \ldots (33)$$

Applying the shear force and bending moment condition at the edge of the plate will give:

$$C_1 = 1.90803E - 15$$
$$C_4 = 3.80739E - 16$$
$$C_5 = 5.39997E - 13$$
… (34)

Then the bending moment, shear force and contact pressure at the centre and the edges of the plate are calculated and used in the MATLAB formulation.

The distribution of deflection, bending moment, shear force and the contact pressure are plotted as shown in Figures (5) to (8) and compared to previous analytical solutions.

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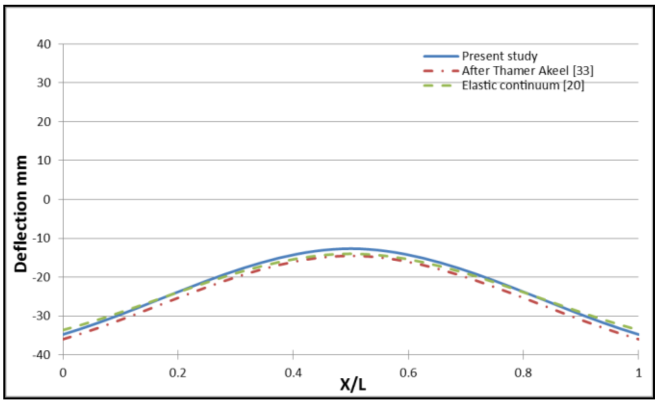

Figure 5: Deflection of an Infinitely Long Plate of Finite Width Resting on Kerr Model and Subjected to Concentrated Forces along the Edges.

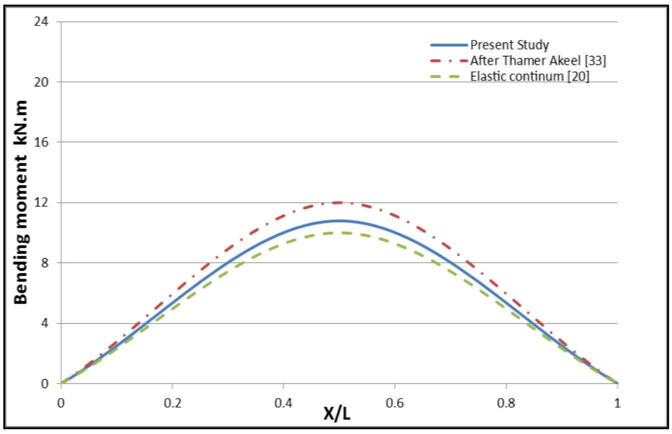

Figure 6: Bending moment of an Infinitely Long Plate of Finite Width Resting on Kerr Model and Subjected to Concentrated Forces along the Edges.

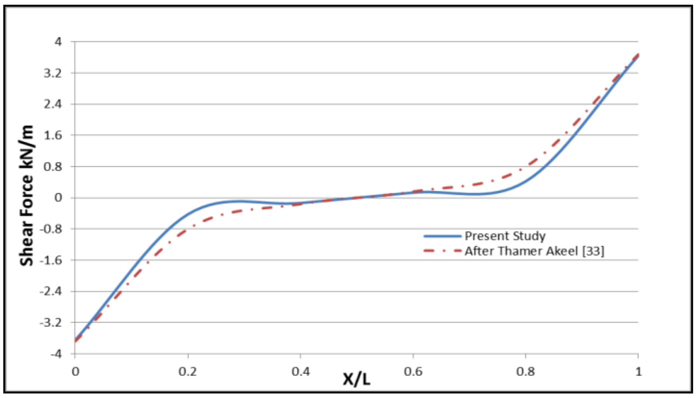

Figure 7: Shear Force of an Infinitely Long Plate of Finite Width Resting on Kerr Model and Subjected to Concentrated Forces along the Edges.

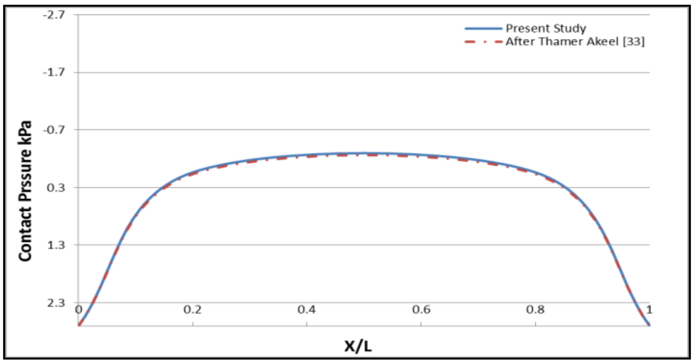

Figure 8: Contact Pressure of an Infinitely Long Plate of Finite Width Resting on Kerr Model and Subjected to Concentrated Forces along the Edges.

Infinitely Long Plate of Finite Width with Free Ends Resting on Kerr Model and Subjected to Uniform Distributed Load

A rectangular plate with finite width and free ends resting on Kerr Model and subjected to uniformly distributed load will be solved in this application where the load (q) will be taken equal to (0.239 MPa) with plate length(L=3.048 m) and thickness (h = 0.3048 m). The material and the foundation properties are:

E_p, □$_p$: Plate young's modulus & poisson's ratio (20690 MPa ,0.20)

C, K, G :

$Kerr\ model\ parameter\ (407.28\frac{MN}{m^3}, 7.47\frac{MN}{m^3}, 245.24\frac{MN}{m})$ [3].

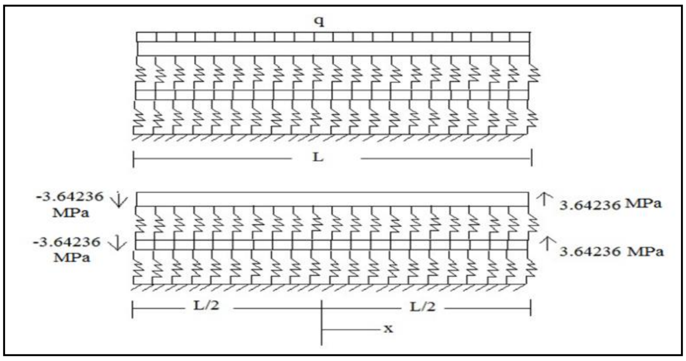

Figure 9: Retangular Plate of Finte Width with Free Ends Resing on Kerr Model And Subjected to Uniform Distrbuted Load.

The displacement function for the Plate and for the shear layer governing the problem is Equation (9):

$$w(x) = C_1Z_1 + C_2Z_2 + C_3Z_3 + C_4Z_4 + C_5Z_5 + C_6Z_6 + w_p$$

$$\overline{w} = C_1F_1 + C_2F_2 + C_3F_3 + C_4F_4 + C_5F_5 + C_6F_6 + \overline{w}_p$$

Where $w_p = \frac{C+K}{C\,K}\left(\frac{q_0x}{L}\right)$, $\overline{w}_p = \frac{q_0x}{LK}$

Applying boundary condition at the center of the plate give:

$$C_2 = 4.31895E - 04$$
$$C_3 = -2.22154E - 04$$
$$C_6 = -0.00281$$

Then Applying boundary conditions at the edge of the plate will yield:

$$C_1 = 2.22154E - 04$$
$$C_4 = -4.31895E - 04$$
$$C_5 = 0.00281$$

Then the bending moment, shear force and contact pressure at the centre and the edges of the plate are calculated and used in the MATLAB formulation.
The distribution of deflection, bending moment, shear force and the contact pressure are plotted as shown in Figures (10) to (13) and compared to previous analytical solutions.

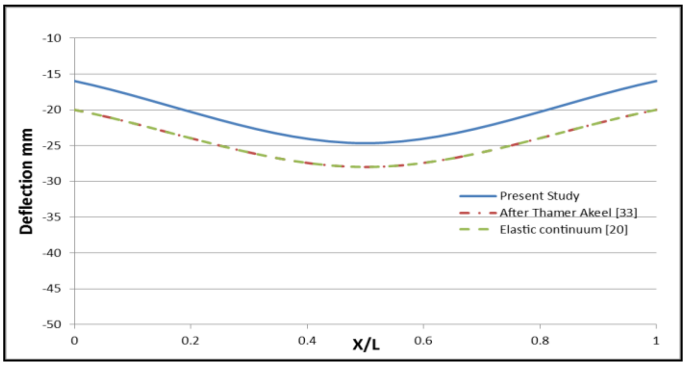

Figure 10: Deflection of an Infinitely Long Plate of Finite Width Resting on Kerr Model and Subjected to Uniformly Distributed load.

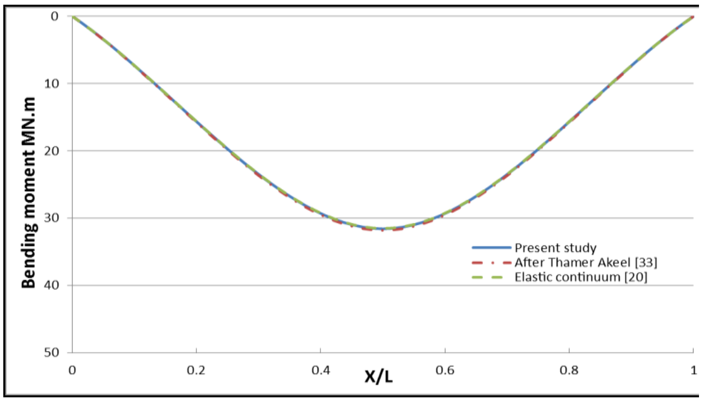

Figure 11: Bending Moment of an Infinitely Long Plate of Finite Width Resting on Kerr Model and Subjected to Uniformly Distributed Load.

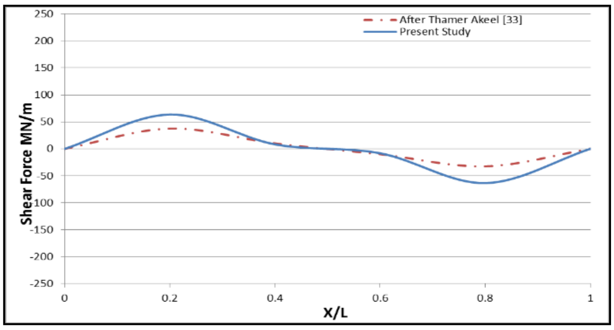

Figure 12: Shear force of an Infinitely Long Plate of Finite Width Resting on Kerr Model and Subjected to Uniformly Distributed Load.

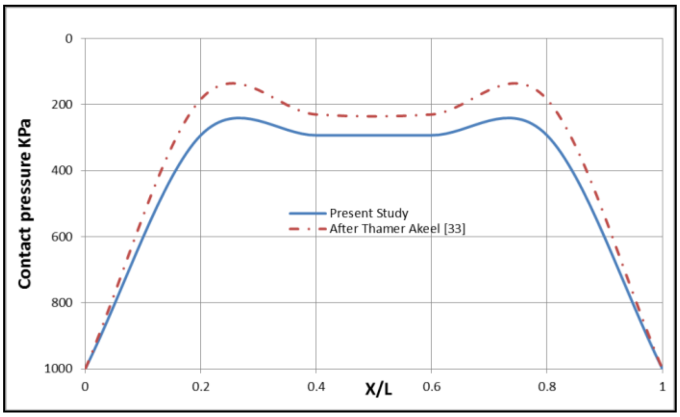

Figure 13: Contact Pressure of an Infinitely Long Plate of Finite Width Resting on Kerr Model and Subjected to Uniformly Distributed Load

Axisymmetric Loading Circular Plate with Free Ends Resting on Kerr Model

In this application, a circular plate loaded by a uniform load distributed over a circular centre region resting on Kerr model will be solved , where the uniform load will be taken equal to (0.0981 MPa), the plate radius equal to (0.01 m), load radius (0.001579 m) and thickness of the plate (h=0.002m), the material and foundation properties are :

$E_p, \nu_p: plate\ young's\ modulus\ , possion's\ ratio$ $(206010\ MPa, 0.3)$
$C, K, G: Kerr\ Model\ parameters$ $\left(4905000\frac{MN}{m^3}, 208463\frac{MN}{m^3}, 88.29\ MN/m\right)$[3]

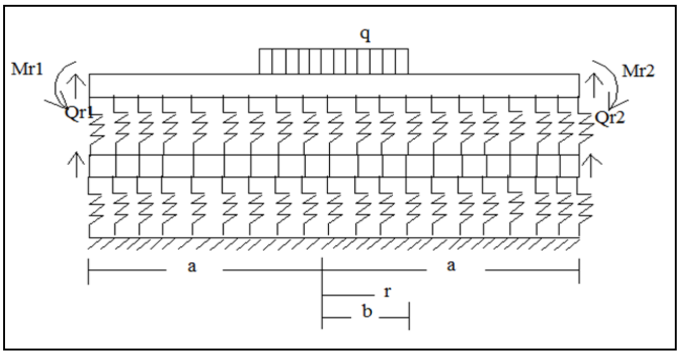

Figure 14: Axisymmetric Loading Circular Plate with Free Ends Resting On Kerr Model

The displacement function for shear layer element Kerr model governing the problem is Equation (24):

$$w(r) = C_1\, u_1(r) + C_2\, u_2(r) + C_3 u_3(r) + C_4 u_4(r) + C_5 u_5(r) + C_6 u_6(r) \quad \dots (35)$$

$$\overline{w} = C_1 F_1(r) + C_2 F_2(r) + C_3 F_3(r) + C_4 F_4(r) + C_5 F_5(r) + C_6 F_6(r)$$

By making use of the conditions of zero slopes and zero shearing force at the centre of the plate($C_3 = C_4 = C_6 = 0$), thus Equation (35) becomes:

$$w(r) = C_1 u_1(r) + C_2 u_2(r) + C_5 u_5(r) \quad \dots (36)$$

$$\overline{w}(r) = C_1 F_1(r) + C_2 F_2(r) + C_5 F_5(r)$$

Now, we find constants C_1, C_2, C_5 in order to find the radial Bending moment and shear force for the plate and shear force in the shear layer at the edge
*From Timoshenko [31], at the centre of the plate $at\ (r = 0)$,

$$M_r = \frac{(1-\Box)P(a^2-b^2)}{8\pi a^2} - \frac{(1+\Box)P\ log\ (\frac{h}{a})}{4\pi}$$

$$M_r = \frac{(1-0.3)*0.0981*(0.01^2-0.001579^2)}{8*\pi*0.001579^2} - \frac{(1+0.3)*0.0981*log(\frac{0.002}{0.01})}{4*\pi}$$

$$\therefore\ M_r = 0.01079\ MN.m$$

At the centre of the plate

$$Mr_1 = D[\frac{d^2w}{dr^2} + \frac{\Box p}{r}\frac{dw}{dr} = 0.01079 \quad \text{Will give:}$$

$$C_5 = 58.57658 * C_1 + 74.97941 + 0.06036 \quad \dots (37)$$

At the edge of the plate

$$Mr_2 = -D\left[\frac{d^2w}{dr^2} + \frac{\Box p}{r}\frac{dw}{dr}\right] = 0 \quad \text{Will give:}$$

$$-\ 39.48753 * C_1 - 21.09439 * C_2 - 0.25034 * C_5 = 0 \quad \dots (38)$$

Solving these equations yield:

$$C_1 = -2.09722E - 04$$
$$C_{\ 2} = -9.42020E - 05$$
$$C_5 = 0.04101$$

Now, we can find the radial bending moment and shear force at the edges of the plate. Then they will be used in the MATLAB formulation.
The distribution of deflection, bending moment, shear force and the contact pressure are plotted as shown in Figures (15) to (18). And compared to previous analytical solutions.

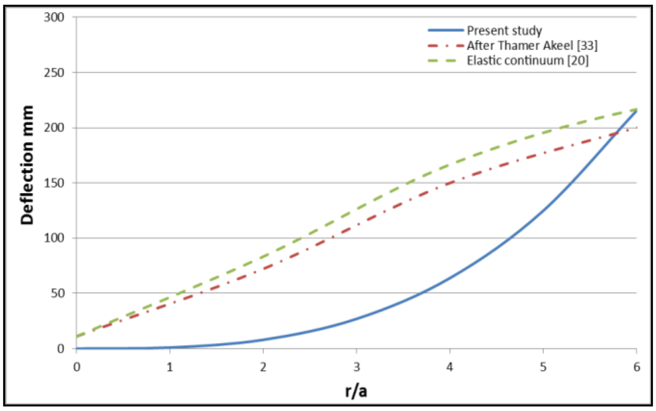

Figure 15: Deflection of Axisymmetric Loading Circular Plate with Free Ends resting on Kerr Model and Subjected to Uniform Distributed Load over a Circular Centre Region
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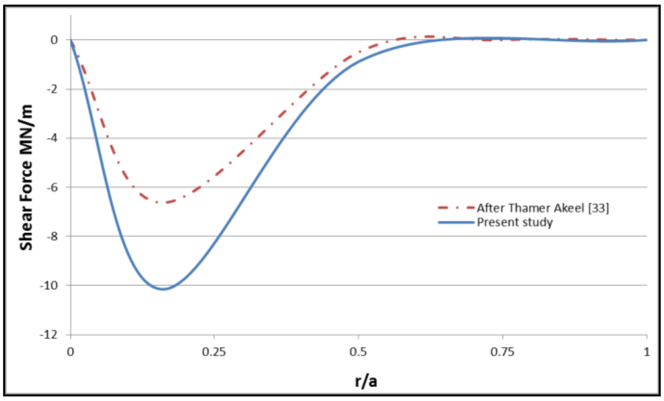

Figure 16: Shear force of Axisymmetric Loading Circular Plate with Free Ends resting on Kerr Model and Subjected to Uniform Distributed Load over a Circular Centre Region

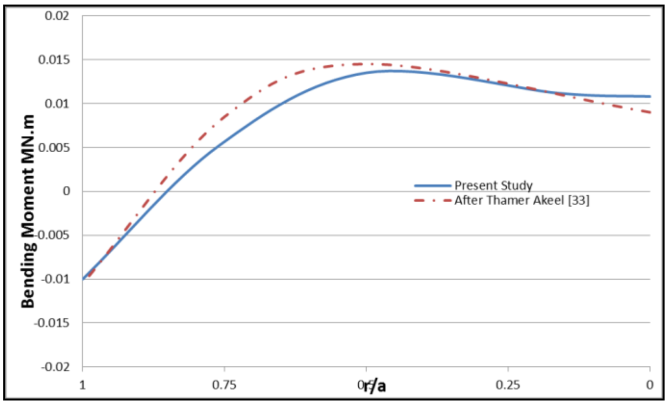

Figure 17: Radial Bending Moment of Axisymmetric Loading Circular Plate with Free Ends Resting on Kerr Model and Subjected to Uniform Distributed load over a Circular Centre Region.

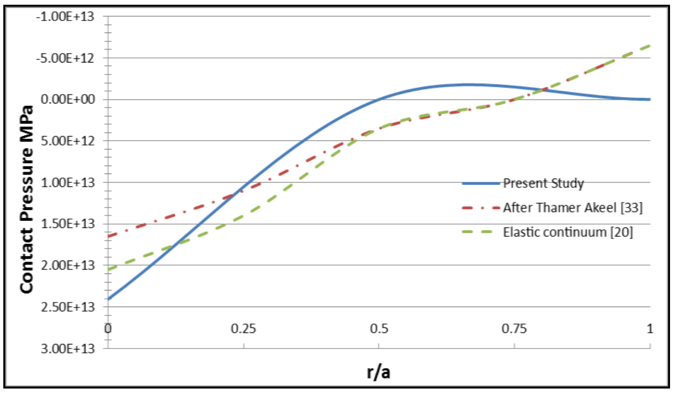

Figure 18: Contact Pressure of Axisymmetric Loading Circular Plate with Free Ends Resting on Kerr Model and Subjected to Uniform Distributed Load over a Circular Centre Region.

CONCLUSIONS

1- The exact closed – form stiffness matrices and fixed – end actions for rectangular and circular plates resting on Kerr foundation model are formulated based on a direct analytical solution. The efficiency and accuracy are demonstrated through examples appearing in the literature. This procedure can be easily implemented into a computer program and solution may be obtained on any small size microcomputer.

2- The formulated programs using MATLAB for the solution of rectangular and circular plates resting on Kerr model showed the ease in programing and solving such complicated mathematical forms. The programs were simple, short in listing and gave accurate results that represent exact solution to the problems solved.

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