International Journal of Applied Engineering Research ISSN 0973-4562 Volume 12, Number 21 (2017) pp. 11164-11166

To The Question on the Restoring of Second Derivatives from the Analytic Functions Defined in the Upper Semi-Plane

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Abstract

This article is about the linear best method of recovering second derivatives at the point of bounded analytic functions defined in the upper half-plane from their values at a finite number of point. First, the necessary concepts and results from the article written by K. YU. Osipenko are introduced and also required problem of optimal recovery of second derivatives of analytic functions is formulated. Further, the error of the best approximation method is determined. At the end of the article, the coefficients of the linear best method are calculated.

Keywords: optimal recovery, error of the best method, extremal function, linear best method, coefficients of the linear best method.

INTRODUCTION

Statement of the problem.

The optimal recovery of a certain class of analytic functions from information about their values at a finite number of points is used in problems of calibration of elements entering the inertial navigation system. Such systems are used, in particular, in space navigation (see [1]). Let us investigate the following problem of optimal recovery.

Firstly it is necessary to recall some concepts and results from article by K. YU. Osipenko[2].

Let's start with the fact that we denote the unit circle in a complex linear normed space X by W and denote the linear functional by $L, l_1, \ldots, l_n$. If $S(t_1, \ldots t_n)$ is any complex function, then its approximation error by the method S is calculated by the following formula:

$$r_n(S) = \sup_{x \in W} |\, L(x) - S\big(l_1(x), \ldots, l_n(x)\big)|.$$

The method $S_0(t_1, \ldots, t_n)$ is called the best approximation method if

$$r_n(S_0) = \inf_S r_n(S).$$

In [2] it is shown that in this case there exists linear best approximation method $S_0 = \sum_{k=1}^{n} c_k l_k(x)$ (c_k are some complex numbers, hereinafter referred to as the coefficients of the linear best method).It is also established that the error of the best approximation method is calculated by the following formula:

$$r_n(S_0) = \sup_{\substack{x \in W \\ l_1(x) = \cdots = l_n(x) = 0}} |L(x)|. \qquad (1)$$

We denote as $D = \{z: Imz > 0\}$ the upper half-plane. We introduce a family of analytic functions $B^1(\mathrm{D}) = \{f(z): |f(z)| \le 1, \forall z \in \mathrm{D}\}$ in D.Further, let $z_0, z_1, \ldots, z_n$ be a distinct points in D. Suppose that

$$L(f) = f''(z_0), l_1(f) = f(z_1), l_2(f) = f(z_2), \ldots, l_n(f) = f(z_n), l_{n+1}(f) = f(z_0), l_{n+2}(f) = f'(z_0),$$

where f(z) is any bounded analytic function in D.

According to the above, there is a linear best recovery method $\alpha f'(z_0) + \beta f(z_0) + \sum_{k=1}^{n} c_k f(z_k)$ (α, β, c_k are the coefficients of the linear best method), and the error of the best approximation method(we denote it$r_2(z_0, z_1, \ldots, z_n)$) is calculated by the following formula (see (1)):

$$r_2(z_0, z_1, \ldots, z_n) = \sup_{\substack{f(z) \in B^1(D) \\ f(z_1) = f(z_2) = \cdots = f(z_n) = f(z_0) = f'(z_0) = 0}} |f''(z_0)|. \qquad (2)$$

(here$W = B^1(D)$).

We note that the problems of optimal recovery have been considered in many works (see, for example, [2], [3], [4]).

Finally, in the third part, to find the coefficients of the linear best method we need the value of the following integral (see [5]):

$$\int_{-\infty}^{+\infty} \frac{dx}{|x - z_0|^2} = \frac{\pi}{y_0}, \qquad (3)$$

where$z_0 = x_0 + iy_0 (y_0 > 0)$.

FINDING THE ERROR OF THE BEST METHOD

We introduce the following family of analytic functions:$A = \{f(z), f(z) \in B^1(D) : f(z_1) = f(z_2) = \cdots = f(z_n) = f(z_0) = f'(z_0) = 0\}$.Suppose that$f(z) \in A$. We introduce the following notation:

$$g(z)=\frac{f(z)}{\left(\frac{z-z_0}{z-\overline{z_0}}\right)^2 B(z)},$$

where

$B(z)=\prod_{j=1}^{n}\frac{z-z_j}{z-\overline{z_j}}$
(4)
is a finite Blaschke product in the upper half-plane.

It is obvious that the function $g(z)\in B^1(D)$. Consequently

$$f''(z_0)=-\frac{1}{2{y_0}^2}B(z_0)\cdot g(z_0).$$

Then the error of the best approximation method has the following form (see (2)):

$r_2(z_0,z_1,\dots,z_n)=\frac{1}{2{y_0}^2}\cdot|B(z_0)|\cdot\sup_{g(z)\in B^1(D)}|g(z_0)|=\frac{|B(z_0)|}{2{y_0}^2}$.
(5)

Consequence. The extremal function $f^*(z)$ of problem (2) is unique up to a factor $e^{i\delta}$ ($\delta\in R$; $\delta-constantnumber$), and has the following form:

$$f^*(z)=e^{i\delta}\left(\frac{z-z_0}{z-\overline{z_0}}\right)^2 B(z)$$

CALCULATION OF THE COEFFICIENTS OF THE LINEAR BEST APPROXIMATION METHOD

It follows from [4] that the linear best approximation method is unique.

Suppose that the function $f(z)\in B^1(D)$. We introduce the following integral:

$-\frac{B(z_0)}{2\pi y_0}\cdot\int_{-\infty}^{\infty}\frac{x-\overline{z_0}}{B(x)(x-z_0)^3}f(x)dx.$ (6)

We estimate the integral modulo from above. From the formula (3) we obtain the following estimate:

$\left|-\frac{B(z_0)}{2\pi y_0}\cdot\int_{-\infty}^{\infty}\frac{(x-\overline{z_0})}{B(x)(x-z_0)^3}f(x)dx\right|=\frac{1}{2\pi y_0}\cdot\left|\int_{-\infty}^{\infty}\frac{B(z_0)}{\left(\frac{x-z_0}{x-\overline{z_0}}\right)^2 B(x)(x-\overline{z_0})(x-z_0)}f(x)dx\right|\le\frac{1}{2\pi y_0}\cdot\int_{-\infty}^{\infty}\frac{|B(z_0)|}{|x-\overline{z_0}|\cdot|x-z_0|}dx=\frac{|B(z_0)|}{2\pi y_0}\cdot\int_{-\infty}^{\infty}\frac{1}{|x-z_0|^2}dx=\frac{|B(z_0)|}{2\pi y_0}\cdot\frac{\pi}{y_0}=\frac{|B(z_0)|}{2{y_0}^2}$.(7)

Thereafter, we calculate the integral (6) via the residue theorem. We introduce the following notation:

$Q(z)=\frac{z-\overline{z_0}}{B(z)(z-z_0)^3}$.(8)

We decompose Q(z) into Laurent's series at a point z_0:

$Q(z)=\frac{C_{-3}}{(z-z_0)^3}+\frac{C_{-2}}{(z-z_0)^2}+\frac{C_{-1}}{z-z_0}+\sum_{n=0}^{\infty}C_n(z-z_0)^n$ (9)
(z_0 is a third-order pole).

Let us find the coefficients C_{-3},C_{-2},C_{-1} of this series. Since (see(9))

$(z-z_0)^3Q(z)=C_{-3}+C_{-2}(z-z_0)+C_{-1}(z-z_0)^2+\sum_{n=0}^{\infty}C_n(z-z_0)^{n+3}$, (10)
then

$C_{-3}=\lim_{z\to z_0}(z-z_0)^3Q(z)=\lim_{z\to z_0}\frac{z-\overline{z_0}}{B(z)}=\frac{2y_0 i}{B(z_0)}$. (11)

Now from the formula (10) we find the coefficient C_{-2}:

$C_{-2}=\lim_{z\to z_0}((z-z_0)^3Q(z)))'=\lim_{z\to z_0}\frac{B(z)-(z-\overline{z_0})\cdot B'(z)}{B^2(z)}=\frac{B(z_0)-2y_0 i\cdot B'(z_0)}{B^2(z_0)}$. (12)

The residue C_{-1} can be found by the following formula:

$$C_{-1}=\frac{1}{2}\left[-\frac{B'(z_0)}{B^2(z_0)}-\frac{\left(B'(z_0)+2y_0 i\cdot B''(z_0)\right)B(z_0)-4y_0 i\cdot\left(B'(z_0)\right)^2}{B^3(z_0)}\right].\quad(13)$$

We introduce a circle γ centered at a point z_0 lying entirely in the upper half-plane. This circle has a positive orientation and does not contain singular points $z_1,\dots,z_n$ inside itself (and on the circumference itself). Then (see(11) and (12)):

$$\operatorname*{Res}_{z=z_0}Q(z)f(z)=\frac{1}{2\pi i}\int_\gamma\left(\frac{C_{-3}}{(z-z_0)^3}+\frac{C_{-2}}{(z-z_0)^2}+\frac{C_{-1}}{(z-z_0)}+\sum_{n=0}^{\infty}C_n(z-z_0)^n\right)f(z)dz=\frac{C_{-3}}{2\pi i}\cdot\int_\gamma\frac{f(z)}{(z-z_0)^3}dz+\frac{C_{-2}}{2\pi i}\cdot\int_\gamma\frac{f(z)}{(z-z_0)^2}dz+\frac{C_{-1}}{2\pi i}\cdot\int_\gamma\frac{f(z)}{z-z_0}dz=\frac{1}{2\pi i}\cdot$$

$\frac{2y_0 i}{B(z_0)}\int_\gamma\frac{f(z)}{(z-z_0)^3}dz+\frac{1}{2\pi i}\cdot\frac{B(z_0)-2y_0 i\cdot B'(z_0)}{B^2(z_0)}\int_\gamma\frac{f(z)}{(z-z_0)^2}dz+\frac{1}{2\pi i}\cdot C_{-1}\int_\gamma\frac{f(z)}{z-z_0}dz=\frac{y_0 i}{B(z_0)}f''(z_0)+\frac{B(z_0)-2y_0 i\cdot B'(z_0)}{B^2(z_0)}\cdot f'(z_0)+C_{-1}f(z_0)$.(14)

We introduce the following notation: $p_k=\operatorname*{res}_{z=z_k}Q(z)(k=1,\dots,n)$. Then (see (6) and (14))

$-\frac{B(z_0)}{2\pi y_0}\cdot\int_{-\infty}^{\infty}\frac{(x-\overline{z_0})}{B(x)(z-z_0)^3}f(x)dx=f''(z_0)-\frac{2y_0\cdot B'(z_0)+iB(z_0)}{y_0B(z_0)}\cdot f'(z_0)-\frac{i\cdot B(z_0)\cdot C_{-1}}{y_0}\cdot f(z_0)-\frac{iB(z_0)}{y_0}\cdot\sum_{k=1}^{n}p_k f(z_k)$.(15)

Consequently (see (7) and (5))

$$\left| f''(z_0) - \frac{2y_0 \cdot B'(z_0) + iB(z_0)}{y_0 B(z_0)} \cdot f'(z_0) - \frac{i \cdot B(z_0) \cdot C_{-1}}{y_0} \cdot f(z_0) - \frac{iB(z_0)}{y_0} \cdot \sum_{k=1}^{n} p_k f(z_k) \right| \leq$$

$$\leq r_2(z_0, z_1, \dots, z_n).$$

From this inequality it follows that

$$\alpha \cdot f'(z_0) + \beta \cdot f(z_0) + \frac{iB(z_0)}{y_0} \cdot \sum_{k=1}^{n} p_k f(z_k) \quad - \qquad (16)$$

is the linear best method, where (see (15) and (13))

$$\alpha = \frac{2y_0 B'(z_0) + iB(z_0)}{y_0 B(z_0)},$$

and

$$\beta = \frac{y_0 B(z_0) B''(z_0) - 2y_0 \left(B'(z_0)\right)^2 - iB(z_0) B'(z_0)}{y_0 B^2(z_0)}.$$

Since(see (16))

$$c_k = \frac{iB(z_0)}{y_0} p_k \ \ (k = 1, \dots, n)$$

and (see (8) and (4))

$$p_k = \frac{2(z_k - \overline{z_0}) y_k i}{(z_k - z_0)^3 \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{z_k - z_j}{z_k - \overline{z_j}}},$$

then

$$c_k = \frac{2B(z_0)(\overline{z_0} - z_k) y_k}{y_0 (z_k - z_0)^3 \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{z_k - z_j}{z_k - \overline{z_j}}} (z_k = x_k + iy_k; k = 1, \dots, n).$$

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