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Drawing Parameter Spaces of Optimal Cubic-order Multiple-zero Finder

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Abstract

The complex dynamical analysis of the cubic-order iterative family is made to draw the fractal images by means of Möbius conjugacy map applied to a prototype polynomial of the form $(z-A)^m\,(z-B)^m$. The related dynamics is best displayed through various stability surfaces and parameter spaces.

Keywords: Conjugacy ; Cubic-order; Critical orbit; Multiple-zero solvers

INTRODUCTION

Most equations used in industrial science, engineering, physics and earth sciences are nonlinear. In order to find the solutions for these nonlinear equations, numerical methods are sought. According to the form and property of these problems, the iteration method is one of the generally used schemes. With the help of initial values and recurrence equations, initial guesses are improved continuously until desired accuracy is obtained. In the past decade, many researchers[1-3] have set their focus on the development of the higher-order solver to locate the roots of nonlinear equations. Numerous methods were proposed based on various considerations[4-9].

The optimal cubic-order methods[10] is designed as follows:

$$\begin{cases} y_n = x_n - \mu\,\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} = x_n - \frac{\lambda f^2(x_n)}{f'(x_n)\{f(x_n)-f(y_n)\}} \end{cases} \qquad (1)$$

where $\lambda = m(1-t^m)$, $\mu = m(1-t)$, $m \in N$ is a multiplicity of the sought zero and $t \in C$ is a free parameter.

The aim of this paper is to analyze the parameter spaces associated with the free critical points relating to the cubic-order multiple-zero solvers. The rest of this paper is made up of two sections. In section 2, conjugacy maps along with the p roperty of dynamical analysis for the aforementioned numeric al methods are studied and the stability surfaces of the strange fixed points for the conjugacy map are displayed. Section 3 sh ows the relevant parameter spaces and conclusion

COMPLEX DYNAMICS FEATURES

A nonlinear equation (1) can be written in a generic form as a discrete dynamical system

$$x_{n+1} = R_f\,(x) \qquad (2)$$

where $R_f\,(x)$ is the iteration function. As a result, we have a complex discrete dynamical system:

$$z_{n+1} = R_f\,(z_n) = z_n - \frac{f(z_n)}{f'(xz_n)}\,H_f(z_n) \qquad (3)$$

where $H_f(z_n) = \frac{\lambda\,f(x_n)}{f(x_n)-f(y_n)}$ and $y_n = x_n - \mu\,\frac{f(x_n)}{f'(x_n)}$.

The following definition and remark are useful to construct the conjugacy map and to investigate the relevant dynamics.

Definition 1. Let *F: X→X* and *G: Y→Y* be two functions (dynamical systems). We say that F and G are conjugate if there is a function

h: X→Y such that $h \circ F = G \circ h$. Then the map *h* is called a conjugacy[11].

Remark 1. Note that a conjugacy indeed preserves the dynamical behavior between the two dynamical systems; for instance, if *F* is conjugate to *G* via *h*, and ξ is a fixed point of *F*, then *h(ξ)* is a fixed point of *G*. Furthermore, if *h* is a homeomorphism, i.e., if *F* is topologically conjugate to *G* via *h*, and ς is a fixed point of *G*, then $h^{-1}(\varsigma)$ is a fixed point of *F*. In addition, we find

$$G = h \circ F \circ h^{-1}$$

and

$$\begin{aligned} G^n &= (h \circ F \circ h^{-1})\circ(h \circ F \circ h^{-1}) \cdots \circ(h \circ F \circ h^{-1}) \\ &= h \circ F^n \circ h^{-1}. \end{aligned}$$

If *F* and *G* are invertible, then the topological conjugacy *h* maps an orbit

$$\cdots F^{-2}(x),\ F^{-1}(x), x, F(x),\ F^2(x), \cdots$$

of *F* onto an orbit

$$\cdots G^{-2}(y),\ G^{-1}(y), y, G(y),\ G^2(y), \cdots$$

of *G*, where $y = h(x)$ and the order of points is preserved.

Hence, the orbits of the two maps behave similarly under homeomorphism h or h^{-1}.

Via Möbius conjugacy map and its inverse

$$M(z) = \frac{z - A}{z - B}, \quad M^{-1}(z) = \frac{B\, z \;-\; A}{z \;-\; 1}$$

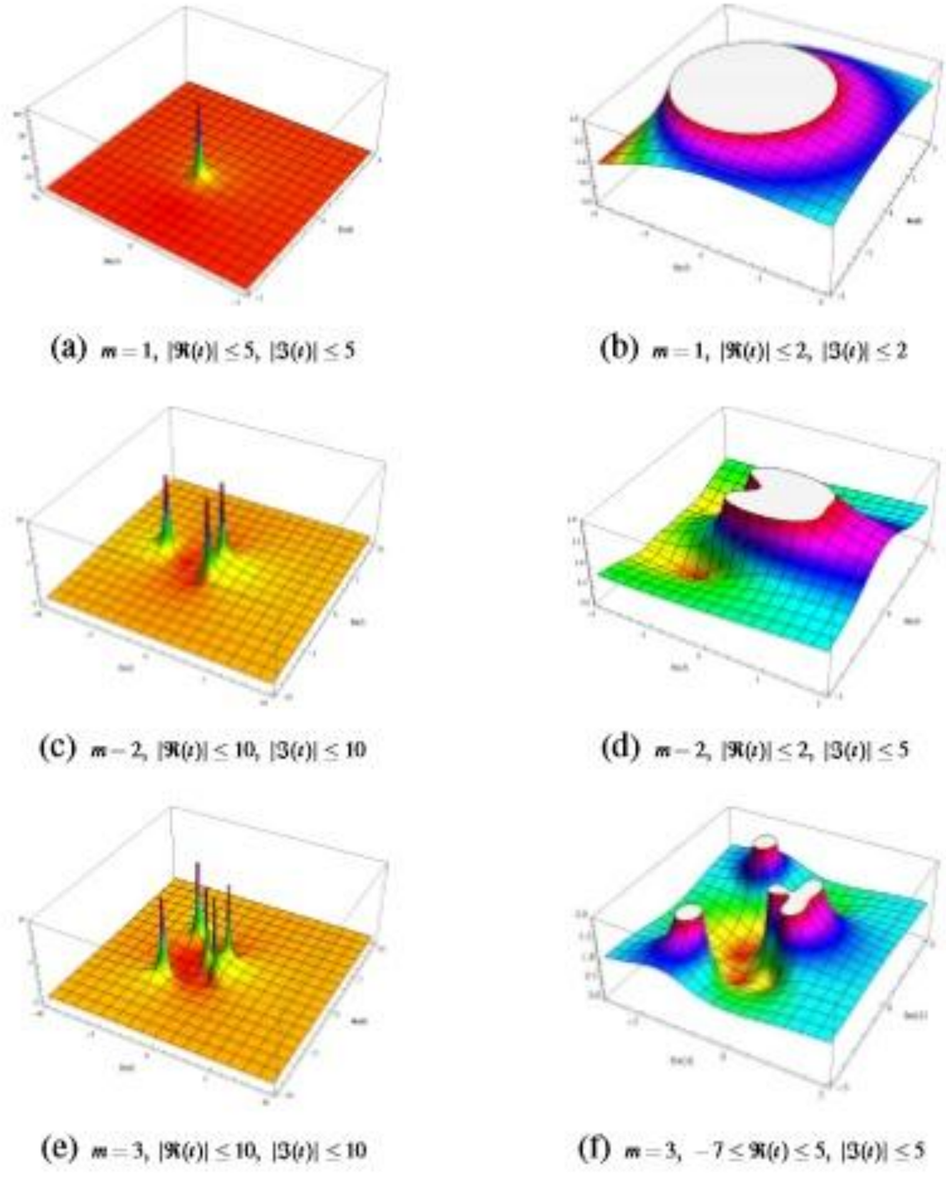
with $z,\ A \neq B,\ A,\ B \in C \cup \infty$ considered by Blanchard[12], $R_f\ (x)$ in (3) is conjugated to J satisfying

$$J(z; t) = z\ \frac{r_1 r_2 (1+z)\ - r_3 (t^m + z)}{r_1 r_2 (1+z)\ - r_3 (1\ + t^m\, z)} \tag{4}$$

with

$r_1\ = (t + z)^m,\ r_2\ = (1 + t\ z)^m\ and\ r_3\ = (1 + z)^{2m}$, when applied to $f(z) = (z - A)^m\ (z - B)^m$.

We find that $z = 0$ (corresponding to fixed point *A* of R_f or root *A* of $f(z) = (z - A)^m\ (z - B)^m$) and $z = \infty$ (corresponding to fixed point *B* of R_f or root *B* of *f(z)*) are two of their fixed points of the conjugate map *J(z; t)*, regardless of *t*-values from the study of *J(z; t)*. And $z = 1$ is a strange fixed point [13-15] of *J* (that is not a root of $f(z) = (z - A)^m\ (z - B)^m$) due to the fact of *J(z;t)=1*, irrespective of *t*-values. The stability of these fixed points for *m=1, 2* and *3* are drawn by illustrative conical surface shown in Figure 1.



(a) $m = 1,\ |\Re(t)| \le 5,\ |\Im(t)| \le 5$

(b) $m = 1,\ |\Re(t)| \le 2,\ |\Im(t)| \le 2$

(c) $m - 2,\ |\Re(t)| \le 10,\ |\Im(t)| \le 10$

(d) $m - 2,\ |\Re(t)| \le 2,\ |\Im(t)| \le 5$

(e) $m = 3,\ |\Re(t)| \le 10,\ |\Im(t)| \le 10$

(f) $m = 3,\ -7 \le \Re(t) \le 5,\ |\Im(t)| \le 5$

Figure 1: Stability surfaces of the strange fixed points J(z;t)

We figure out that $J'(z;t)$ can be reduced to a fraction of a common denominator as follows:

$$J'(z;t) = \frac{N(z;t)}{D(z;t)} \qquad (5)$$

$$N = r_1^{-\frac{1}{m}} r_2^{-\frac{1}{m}} (r_1^{2+\frac{1}{m}} r_2^{2+\frac{1}{m}} (1+z)^2 + r_1^{\frac{1}{m}} r_2^{\frac{1}{m}} r_3^2 \big(2z + t^m(1+z^2)\big) - r_1 r_2 r_3((1+t^2)z(\beta - t^m\gamma) + t(\tau + t^m\omega), D = \big(r_1 r_2(1+z) - r_3(1+t^m z)\big)^2,\ \beta = 1 + m(-1+z)^2 + z(4+z), \gamma = -1 + m(-1+z)^2 - z^2 \ and \ \omega = 2m(-1+z)^2 z + (1+z^2)^2$$

The critical points of the iterative method are given by the roots of $J'(z;t)=0$. $z = 0$ and $z = \infty$ are critical points associated with the roots A and B of the polynomial $(z-A)(z-B)$. In case of $m=1$, the critical points are $z = 0$, $z = -1$, $z = 1$ and $z = \infty$. In case of $m=2$ with a given t, three roots can be found.

CONCLUSION

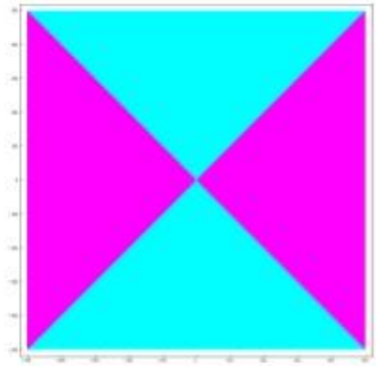
It is interesting to study the relevant complex dynamics from the viewpoint of parameter spaces. In Figures 2 and 3, parameter spaces P associated with critical points are shown. A point t in P is painted according to the coloring scheme listed in Table 1. Every point of the parameter space P whose color is none of cyan (root $z=A$), magenta (root $z=B$), yellow and red is not a good choice of t in terms of relevant numerical behavior. For convenience, we let P denote the parameter space associated with branch $cp(i)$ for $1 \le i \le 3$.

When $m = 1$, we have checked that there exist regions of finite period for stable q-cycles with $q \ge 1$. We find the fascinating fractal boundaries between the bulb associated with different cycles. For the remaining cases of $m = 2$, we have accomplished similar analysis to explore the relevant dynamics described in Figure 3.

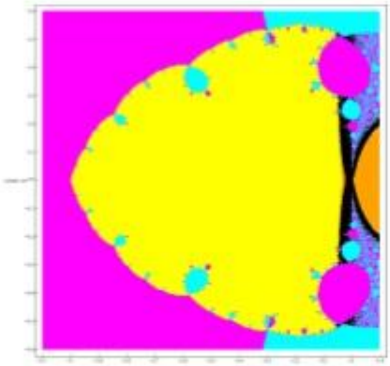
Table 1: Coloring scheme

q	C_q
$q=1$	$C_1 = \begin{cases} magenta, for\ fixed\ point\ \infty \\ cyan, for\ fixed\ point\ 0 \\ yellow, for\ fixed\ point\ 1 \\ red, for\ other\ strange\ fixed\ point \end{cases}$
$2 \le q \le 19$	$C_2 = orange, C_3 = light\ green, C_4 = dark\ red,$ $C_5 = dark\ blue, C_6 =$ dark green, $C_7 = dark\ yellow,$ $C_8 = floral\ white, C_9 = light\ pink, C_{10} = khaki,$ $C_{11} = dark\ orange, C_{12} = turquoise, C_{13} = lavender,$ $C_{14} = thistle, C_{15} = plum, C_{16} = orchid,$ $C_{17} = medium\ orchid, C_{18} = blue\ violet, C_{19} = dark\ orange$
$q=0^*$ or $q > 20$	$C_q = black$

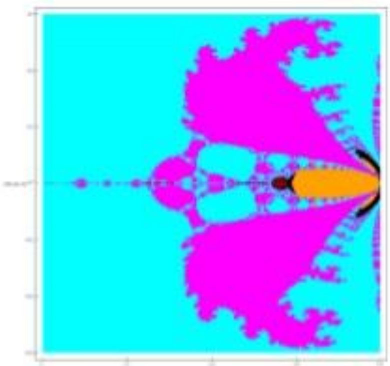
*: q=0 implies that the orbit is non-periodic but bounded.



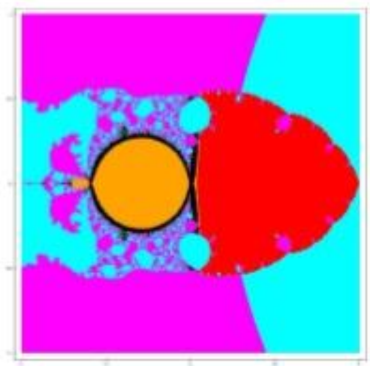
(a) $|\Re(t)| \leq 500,\ |\Im(t)| \leq 500$



(b) $-3.1 \leq \Re(t) \leq -1.9, |\Im(t)| \leq 0.6$

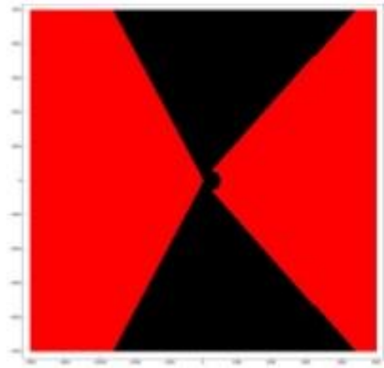


(c) $1 \leq \Re(t) \leq 1.4,\ |\Im(t)| \leq 0.3$

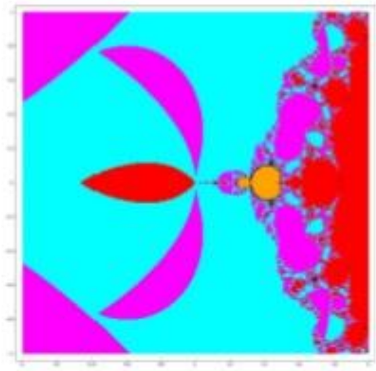


(d) $1 \leq \Re(t) \leq 3,\ |\Im(t)| \leq 1$

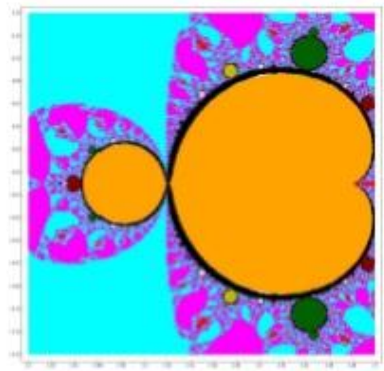
Figure 2. Parameter spaces associated with free critical points for m=1



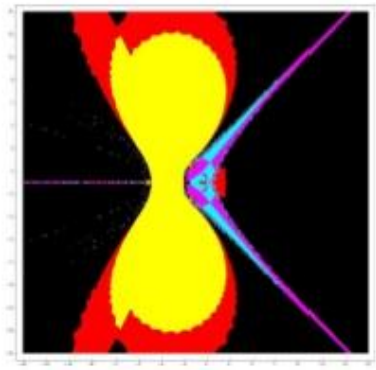
(a) $|\Re(t)| \leq 50000,\ |\Im(t)| \leq 50000$



(b) $0 \leq \Re(t) \leq 2,\ 0.3 \leq \Im(t) \leq 1$



(c) $1.2 \leq \Re(t) \leq 1.5,\ |\Im(t)| \leq 0.15$



(d) $|\Re(t)| \leq 15,\ |\Im(t)| \leq 15$

Figure 3. Parameter spaces associated with free critical points for m=2

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