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On Co-Positive Approximation of Unbounded Functions in Weighted Spaces

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Abstract

The aim of present this article is to introduce a new method for establishing polynomials and spline approximation to unbounded functions in weighted space $\mathcal{L}_{p,\beta}(X)$ in terms of averaged modulus, modulus of smoothness and Ditzian-Totik modulus.

Keywords: Weighted space, unbounded functions, polynomials and spline.

Mathematics Subject Classifications: 41A05 and 41A15.

INTRODUCTION

Let $X=[-1,1]$, $\mathcal{L}_p(X)$ be a set of measurable functions on X, $1\le p<\infty$, such that

$$\|\lambda\|_p=\left(\int_X |\lambda(t)|^p dt\right)^{\frac{1}{p}}<\infty \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$W(\beta,t)$ be a set of each weighted functions on X (s. t) $\left|\frac{\lambda(t)}{\beta(t)}\right|\le M$ where $M\in\mathbb{R}^+$ and

$\beta: X\to\mathbb{R}^+$ be a weight function.

$\mathcal{L}_{p,\beta}(X)$, $1\le p<\infty$ be the space of each unbounded functions which the following rule

$$\|\lambda\|_{p,\beta}=\left(\int_X \left|\frac{\lambda(t)}{\beta(t)}\right|^p dx\right)^{\frac{1}{p}}<\infty \quad \ldots\ldots\ldots\ \ldots\ldots (2)$$

Now remember several definitions of modulus which are needs through our article. The k^{th} difference of λ is known by

$$\Delta_h^k(\lambda,t)=\begin{cases}\sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}\lambda(t+ih) & if\ \ t,t+ih\in X\\ 0 & otherwise\end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Then the k^{th} modulus smoothness of $\lambda\in\mathcal{L}_{p,\alpha}(X)$ is defined by $\omega_k(\lambda,\delta)_{p,\beta}=\underbrace{spu}_{0\le h\le\delta}\left\|\Delta_h^k(\lambda,.)\right\|_{p,\beta}$ ....(4)

Also an averaged modulus of k^{th} degree defined for every functions in $\mathcal{L}_{p,\beta}(X)$ by

$$\tau_k(\lambda,\delta)_{p,\beta}=\|\omega_k(\lambda,.)\|_{p,\beta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

We denoted k^{th} degree Ditzian-Totik modulus in the $\mathcal{L}_{p,\beta}(X)$ via

$$\omega_k^{\varphi}(\lambda,\delta)_{p,\beta}=\underbrace{spu}_{0\le h\le\delta}\left\|\Delta_{h\varphi}^k(\lambda,.)\right\|_{p,\beta} \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Where $\varphi(x)=\sqrt{1-x^2}$; $x\in X$.

Let $A_m=\{a_1,a_2,\ldots,a_m\,;\,a_0=-1<a_1<a_2<\cdots<a_m<a_{m+1}=1\}$

$m\ge 0$ And $\Delta^{\theta}(A_m)$ be the set of all functions λ such that $(-1)^{m-j}\lambda(t)\ge 0$ for $t\in[a_j,a_{j+1}]$, $0\le j\le m$ (i.e.) Every $\lambda\in\Delta^{\theta}(A_m)$ has $0\le m<\infty$ the sign will be change at the points in A_m and is non-negative proximate of 1.

In specific, if $m=0$, then $\Delta^{\theta}=\Delta^{\theta}(A_m)$ be the set of all non-negative functions on X. A function μ is said to be co-positive with function λ if $\lambda(t)\mu(t)\ge 0,\forall\, t\in X$.

We are concerned about approximation the functions from $\Delta^{\theta}(A_m)$ by polynomials belong to $\mathbb{P}_n$ of degree less than or equal n and spline with less than n knots that are copositive with λ, if $m=0$ this is also called positive approximation.

For $\lambda\in\mathcal{L}_{p,\beta}(X)$, let

$$E_n(\lambda)_{p,\beta}=inf\{\|\lambda-\rho_n\|_{p,\beta}\,;\,\rho_n\in\mathbb{P}_n\}\ldots\ldots (7)$$

Is the degree of best unconstraint approximation of unbounded function of λ to polynomial $\rho_n \in \mathbb{P}_n$, and let:

$$E_n^{(\theta)}(\lambda, A_m)_{p,\beta} = inf\{\|\lambda - \rho_n\|_{p,\beta} ; \rho_n \in \mathbb{P}_n \cap \Delta^\theta(A_m)\} \quad \ldots\ldots\ldots\ldots\ldots (8)$$

be the degree of co-positive polynomial approximation of $\lambda \in \mathcal{L}_{p,\beta}(X)$. In specific

$$E_n^{(\theta)}(\lambda)_{p,\beta} = E_n^{(\theta)}(\lambda, A_m)_{p,\beta} = inf\{\|\lambda - \rho_n\|_{p,\beta} ; \rho_n \in \mathbb{P}_n \cap \Delta^\theta\} \quad \ldots\ldots\ldots (9)$$

is the degree of best co-positive approximation.

By the side even if λ has only signal change, following its signal is not so easy and order of approximation fails. It was publicized by Zhou [1] there exists $\lambda \in C^1(X) \cap \Delta^\theta(A_m)$, $m \geq 1$ such that $\lim_{n\to\infty} \sup \frac{E_n^{(\theta)}(\lambda, A_m)_\infty}{\omega_4(\lambda, n^{-1})_\infty} = +\infty$

Hu and Yu [2] with [3], [4], [5], [6],[7] and [8] showed that

$$E_n^{(\theta)}(\lambda, A_m)_{p,\beta} \leq C_p(A_m)\omega_3(\lambda, \frac{1}{n})_\infty$$

and Kopotun [9] shows that,

$$E_n^{(\theta)}(\lambda, A_m)_{p,\beta} \leq C_p(A_m)\omega_3^\varphi(\lambda, \frac{1}{n})_\infty$$

NOTATION AND DEFINITION

Let $k > 0$ be an integer and

$$-1 = a_0 < a_1 < a_2 < \cdots < a_{k-1} < a_k = 1$$

be the divider of X. Define the so support knots see [10] by

$$a_i = -1 + i\Delta a_0, \quad i = -q+1, \ldots, -1$$

And $a_i = 1 - (i-k)\Delta a_{k-1}$,

$i = k+1, k+2, \ldots, k+q-1$,

Let $X_i = [a_i, a_{i+1}]$, $I = [a_{i-q+1}, a_{i+q}]$.

Denote $S_k = \{a_i\}_{i=-q+1}^{k+q-1}$, $\Delta S_k = \max\{\Delta a_i\}$

then for $i = -q+1, \ldots, k-1$.

$$b_i = \frac{a_{i+q} - a_i}{q}, \qquad a_i^* = \frac{(a_{i+1}, \ldots, a_{i+q-1})}{q-1},$$

$$b_i^* = 2\min \frac{(a_i^* - a, a_{i+q} - a_i^*)}{q}$$

And $\quad X_i^* = [a_i^* - \frac{b_i^*}{2}, a_i^* + \frac{b_i^*}{2}]$.

For every $\lambda \in \mathcal{L}_{p,\beta}(X)([a_{-q+1}, a_{k+q-1}])$,

We shall define the operator S by:

$$S(\lambda) = \sum_{i=-q+1}^{k-1} \Gamma_i \Upsilon_i ; \quad \Gamma_i = \Gamma_i(\lambda) = c_i^{*-1} \int_{X_i^*} \lambda$$

Such that $\Upsilon_i(x) = \Upsilon_{q,i}(x) = \Upsilon(x, a_i, \ldots, a_{i+q})$

on $a_i, \ldots, a_{i+1}$ standardized so that $\sum_i^{i+q} \Upsilon_i(x) \cong 1$ Hence S preserver linearity that is $S(p_n) = p_n$ for any $p_n \in \mathbb{P}_n$.

THE MAIN THEOREMS

We summarize all the result in this paper by the following theorems. The first theorem shows that co-positive spline approximation of unbounded functions in terms the modulus of smoothness of order 2 for $1 \leq p < \infty$, the second theorem shows that Ditzian-Totik modulus, is indeed easy to get to, thus belong the order of copositive polynomials approximation of unbounded functions, also the third theorem construct the inequality in terms averaged modulus smoothness of co-positive spline approximation to $\lambda \in \mathcal{L}_{p,\beta}(X)$.

Theorem 3.1: Let $\lambda \in \mathcal{L}_{p,\beta}(X), 1 \leq p < \infty$.Then spline $Q = S(\lambda)$ of the order r (natural number) on the knots sequence S_k satisfies

$$E_n^{(\theta)}(\lambda, A_m)_{p,\beta} \leq \Gamma_q(\delta)\omega_2(\lambda, \delta)_{p,\beta} \quad \ldots\ldots\ldots\ldots(10)$$

Where Γ is a constant depends on q and $\delta = \underbrace{\min}_{0\leq i\leq m} |a_{i+1} - a_i|$.

Proof: By using Holder inequality, we have

$$|\Gamma_j| \leq \left|\sum_{j\in X} \Gamma_j \Upsilon_j\right| \leq \left(\sum_{j\in X} |\Gamma_j|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{j\in X} \Upsilon_j\right)^{\frac{1}{q}}$$

Since, $\sum_{j\in X} \Upsilon_j \equiv 1$ & $\int_{-\infty}^{+\infty} \Upsilon_j dt \cong b_j$

Thus $\quad |\Gamma_j| \leq \Gamma \|\lambda\|_{p,\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$

Also from de Boor [11] and Devore [10],

$$\|S(\lambda)\|_{p,\beta} = \left(\int_X \left|\frac{S(\lambda,t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}} = \left(\int_X \left|\frac{\sum_j^i \Gamma_j \Upsilon_j(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$\leq \Gamma\left(\int_X \left|\frac{\sum_j^i \Gamma_j b_j^{\frac{1}{p}}}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}} \leq \Gamma_1 \sum_j^i \|f\|_{p,\beta} \leq \Gamma_2\|\lambda\|_{p,\beta}$$

…..……….….. (12)

Let ρ_i^* be the best approximation to f on X. Then ρ_i^* is the best approximation on each $J \subset X$ and (see [12]), we have $\|\lambda - \rho_i^*\|_{p,\beta} \leq \Gamma\omega_2(\lambda,\delta)_{p,\beta}$ ……………………. (13)

From (11), (12) and (13), we obtain

$$E_n^{(\theta)}(\lambda, A_m)_{p,\beta} \leq \|\lambda - Q\|_{p,\beta} \leq \Gamma\|\lambda - S(\lambda)\|_{p,\beta}$$

$$\leq \Gamma_1\left\{\left(\int_X \left|\frac{\lambda(t)-\rho_i^*(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}} + \left(\int_X \left|\frac{\rho_i^*(t)-S(\lambda,t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}\right\}$$

$$\leq \Gamma_1\left\{\Gamma_2\omega_2(\lambda,\delta)_{p,\beta} + \left(\int_X \left|\frac{S(\rho_i^*,t)-S(\lambda,t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}\right\}$$

$$\leq \Gamma_3\omega_2(\lambda,\delta)_{p,\beta} + \Gamma_3\left(\int_X \left|\frac{\lambda(t)-\rho_i^*(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$\leq \Gamma_4\omega_2(\lambda,\delta)_{p,\beta} + \Gamma_5\omega_2(\lambda,\delta)_{p,\beta} \leq \Gamma_6\omega_2(\lambda,\delta)_{p,\beta}.$$

Thus (10) is proved.

Theorem 3.2:_ Let $A_m = \{a_1, a_2, \dots, a_m\}$ be given and $\delta = \min\{|a_{i+1} - a_i|, 0 \leq i \leq m\}$.

If $\lambda \in \mathcal{L}_{p,\beta}(X) \cap \Delta^\theta(A_m), 1 \leq p < \infty$,

$E_n^{(\theta)}(\lambda, A_m)_{p,\beta} \leq \Gamma\omega_k^\varphi(\lambda,\delta)_{p,\beta}$ ……. ….(14) Where

Γ is constant.

Proof: Suppose that (14) is hold for each function $\lambda \in \mathcal{L}_{p,\beta}(X) \cap \Delta^\theta(A_m)$ with

$A_{m-1} = \{a_1, a_2, \dots, a_{m-1}\}$, by proposition in [13], $\exists$ spline $\alpha \in \Delta^\theta(A_m)$ with knots $\{a_j\}_{j=0}^n$ satisfying (13),

Let $G(x) = \alpha(x)\sigma(x - a_m)$. Then

$G(x) \in \mathcal{L}_{p,\beta}(X) \cap \Delta^\theta(A_{m-1})$, And by supposition, $\exists$ $Q_n \in \mathbb{P}_n \cap \Delta^\theta(A_{m-1})$ such that

$\|G - Q_n\|_{p,\beta} \leq \Gamma\omega_k^\varphi(\lambda, \frac{1}{n})_{p,\beta}$ …… ... (15)

We define

$H_n(t) = Q_n(t)q_n(a_m, t)$, where $q_n(a_m, t)$ a polynomial which is co-positive with $\sigma(x - a_m)$ It is clear that, $H_n(t) \in \mathbb{P}_n \cap \Delta^\theta(A_m)$.

We need to estimate $\|G - H_n\|_{p,\beta}$ and, $\|\lambda - H_n\|_{p,\beta}$. By (15), we have

$$\|G - H_n\|_{p,\beta} = \left(\int_X \left|\frac{G(t) - H_n(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$= \left(\int_X \left|\frac{\alpha(t)\sigma(t-a_m) - Q_n(t)q_n(a_m,t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$\leq \left(\int_X \left|\frac{\alpha(t)\sigma(t-a_m) - \alpha(t)q_n(a_m,t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$+ \left(\int_X \left|\frac{\alpha(t)q_n(a_m,t) - Q_n(t)q_n(a_m,t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$\leq \chi + \left(\int_X \left|\frac{\alpha(t)q_n(a_m,t) - Q_n(x)q_n(a_m,t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

Such that

$$\chi = \left(\int_X \left|\frac{\alpha(t)\sigma(t-a_m) - \alpha(t)q_n(a_m,t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

From Kopotun [9], the lemma (3.5) in [13] and the properties of Ditzian-Totik modulus, We obtain

$$\chi^p \leq \Gamma^p(\omega_k^\varphi(G, \frac{1}{n})_{p,\beta})^p$$

So $\|G - H_n\|_{p,\beta} \le C_1\omega_k^\varphi(G,\frac{1}{n})_{p,\beta} + C_2\omega_k^\varphi(G,\frac{1}{n})_{p,\beta}$

We have $\Gamma_2\omega_k^\varphi(\alpha,\frac{1}{n})_{p,\beta} \le \Gamma\omega_k^\varphi(G,\frac{1}{n})_{p,\beta}$

Thus $\|G - H_n\|_{p,\beta} \le \Gamma_3\omega_k^\varphi(G,\frac{1}{n})_{p,\beta}$,

And

$$\|\lambda - H_n\|_{p,\beta} \le \left(\int_X \left|\frac{\lambda(t)-G(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}} + \left(\int_X \left|\frac{G(t)-H_n(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$\le \Gamma_4\omega_k^\varphi(\lambda,\frac{1}{n})_{p,\beta} + \Gamma_5\omega_k^\varphi(G,\frac{1}{n})_{p,\beta} \le \Gamma_6\omega_k^\varphi(\lambda,\frac{1}{n})_{p,\beta}$$

Because of $\omega_k^\varphi(G,\frac{1}{n})_{p,\beta} \le \omega_k^\varphi(\lambda,\frac{1}{n})_{p,\beta}$,

So $E_n^{(\theta)}(\lambda,A_m)_{p,\beta} \le \|\lambda - Q_n\|_{p,\beta} \le$

$$\left(\int_X \left|\frac{\lambda(t)-H_n(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}} + \left(\int_X \left|\frac{H_n(t)-G(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$+ \left(\int_X \left|\frac{G(t)-Q_n(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$\le \Gamma_7\omega_k^\varphi(\lambda,\frac{1}{n})_{p,\beta} + \Gamma_8\omega_k^\varphi(G,\frac{1}{n})_{p,\beta} + \Gamma_9\omega_k^\varphi(G,\frac{1}{n})_{p,\beta}$$

$\le \Gamma_{10}\omega_k^\varphi(\lambda,\frac{1}{n})_{p,\beta}$.Thus, (14) followed.

Theorem 3.3: If $\lambda \in \mathcal{L}_{p,\beta}(X) \cap \Delta^\theta(A_m), 1 \le p < \infty$, change its sign $k < \infty$ term at,

$-1 = a_0 < a_1 < \cdots < a_k < a_{k+1} = 1$, denote

$\delta = \min\{(a_{i+1} - a_i), 0 \le i \le k\}$, where $a_0 = -1$

and $a_{k+1} = 1$, then $\delta > \frac{1}{n}$, there exists quadratic spline Q_n with kth knots that copositive with λ

And satisfies $E_n^{(\theta)}(\lambda,A_m)_{p,\beta} \le \Gamma\tau_3(\lambda,\delta)_{p,\beta}$ …….(16)

Where Γ is constant.

Proof: Let $d_i = \frac{i}{2n}$, $I_i = [d_i, d_{i+1}]$ contaminated, if $d_i < a_i \le d_{i+1}$ $\exists$ a_i of sign change of λ, $1 \le i \le k$. We have $\delta > \frac{1}{n}$ $\exists$ one a_i in every of suitability, we correspondingly denote $l_0 = -1$ and $l_{k+1} = 2n$,

$l_i < l_i + 2 < l_{i+1}$, $0 \le i \le k$. That is amid I_{l_i} and $I_{l_{i+1}}$ for any $0 \le i \le k$ $\exists$ at least one interval I_i that is not contaminated. Note that λ dose not change sign amid I_{l_i} & $I_{l_{i+1}}$. If $l_{i+1} > l_i + 2$, $\exists$ at least two non-contaminated intervals amid I_{l_i} & $I_{l_{i+1}}$.

Set two polynomials ξ_i and η_i such that $\eta_i \le \lambda(d) \le \xi_i$, $d \in [d_i, d_{i+2}] = X_i^*$

Where $i = l_i + 1, l_i + 2, \ldots, l_{i+1} - 2$,

$$\|\xi_i - \eta_i\|_{p,\beta} = \left(\int_{X_i^*} \left|\frac{\xi_i(t)-\eta_i(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}} \le \Gamma\tau_3(\lambda,\frac{1}{n})_{p,\beta}, t \in X_i^*.$$

And ξ_i is a co-positive with $\lambda \ge 0$ and η_i is co-positive with $\lambda \le 0$ and satisfies

$\|\xi_i - \lambda\|_{p,\beta} \le \|\xi_i - \eta_i\|_{p,\beta}$ and $\|\lambda - \eta_i\|_{p,\beta} \le \|\xi_i - \eta_i\|_{p,\beta}$

We make a local polynomial by interpolation on $[d_{l_{i-1}}, d_{l_{i+2}}]$,

$1 \le i \le k$, take the interpolation of λ at $d_{l_{i-1}}$ & $d_{l_{i+2}}$ by polynomial $\gamma_{l_{i-1}}$, its co-positive with λ on $[d_{l_{i-1}}, d_{l_{i+2}}]$. Used for its rate of approximation, we take two polynomials $\xi_{l_{i-1}}$ & $\eta_{l_{i-1}}$ are exists such that

$\eta_{l_{i-1}}(t) \le \lambda(t) \le \xi_{l_{i-1}}(t)$, $t \in [d_{l_{i-1}}, d_{l_{i+2}}]$, its holds true at $d_{l_{i-1}}$ & $d_{l_{i+2}}$.

Since $\left(\int_{X_i^*} \left|\frac{\xi_{l_{i-1}}(t)-\gamma_{l_{i-1}}(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$

$$\le \left(\int_{X_i^*} \left|\frac{\xi_{l_{i-1}}(t)-\eta_{l_{i-1}}(t)}{\beta(t)}\right|^p dt\right)$$

$$\le \left(\int_{X_i^*} \left|\frac{\xi_{l_{i-1}}(t)-\eta_{l_{i-1}}(t)}{\beta(t)}\right|^p dt\right) \le \Gamma\tau_3(\lambda,\frac{1}{n})_{p,\beta}, \forall t \in X_i^*$$

So $\left(\int_{X_i^*} \left|\frac{\lambda(t)-\gamma_{l_{i-1}}(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$

$$\le \left(\int_{X_i^*} \left|\frac{\lambda(t)-\xi_{l_{i-1}}(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}} + \left(\int_{X_i^*} \left|\frac{\xi_{l_{i-1}}(t)-\gamma_{l_{i-1}}(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$\le \left(\int_{X_i^*} \left|\frac{\lambda(t)-\xi_{l_{i-1}}(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}} + \left(\int_{X_i^*} \left|\frac{\xi_{l_{i-1}}(t)-\eta_{l_{i-1}}(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$\le \Gamma_1\tau_3(\lambda,\frac{1}{n})_{p,\beta} + \Gamma_2\tau_3(\lambda,\frac{1}{n})_{p,\beta} \le \Gamma_3\tau_3(\lambda,\frac{1}{n})_{p,\beta}$$

Now, we have construct local polynomials which are co-positive with λ and have estimate of order three, we now composite them for a kth spline approximation Q with the similar estimate order. If X_i is a non- contaminated interval, γ_{i-1} & γ_i similarity on X, then X must be non-contaminated also, or there would be no γ_i at very.

Construct spline on X these knots that connected with γ_{i-1} & γ_i at d_i & d_{i+1} respectively,

Furthermore, the display of Q_i mendacities amid these γ_{i-1}&γ_i , hence Q_i is too co-positive with λ and satisfies

$$\|\lambda - Q_i\|_{p,\beta} \leq$$

$$\left(\int_{X_i^*} \left|\frac{\lambda(t)-\gamma_i(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}} + \left(\int_{X_i^*} \left|\frac{\gamma_i(t)-Q_i(t)}{\beta(t)}\right|^p dt\right)^{\frac{1}{p}}$$

$$\leq \Gamma_1 \tau_3(\lambda, \frac{1}{n})_{p,\beta} + \Gamma_2 \tau_3(\lambda, \frac{1}{n})_{p,\beta} \leq \Gamma_3 \tau_3(\lambda, \frac{1}{n})_{p,\beta}$$

Thus, $E_n^{(\theta)}(\lambda, A_m)_{p,\beta} \leq \Gamma \tau_3(\lambda, \frac{1}{n})_{p,\beta}$

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