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# Semi Implicit Hybrid Method with Vanishing Phase-lag and Amplification Error for Solving Second Order Oscillatory Problems

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**Abstract**

In this paper, we derived a semi-implicit hybrid method for solving special second order ordinary differential equations. The method is four-stage fifth-order with phase-lag or dispersion of order eight and dissipation or amplification of order seven. Modifications on the phase-lag and amplification error of order zero are infused to the method to improve the efficiency of the method. Hence resulting in a semi-implicit hybrid method with zero phase-lag and amplification error which is suitable for solving highly oscillatory problems over a very large interval.

**Keywords**: Semi-Implicit Hybrid Method, Two-step method, Phase-lag, Amplification error, Oscillatory problems.

**Subject Classification**: 65L05, 65L06

## INTRODUCTION

Second order ordinary differential equations (ODE) often arise in the field of mathematics, astronomy, population modeling, geometry, meteorology, satellite tracking and others. In this paper, we will focus our attention on method which can be used to solve special second order ODEs

$$y" = f(x, y),\ y(x_0) = x_0,\ y'(x_0) = y_0' \quad (1)$$

Where it does not contain the first derivative and the solutions to the problems are oscillatory in nature.

Runge-Kutta Nyström (RKN) method, hybrid method, and multistep method are often used to solve directly the equation without reducing it to a system of first order ODEs. Authors such as Van Der Houwen and Sommeijer [1], Attili et.al [2], Al-Khasawneh et.al [3], Ismail et.al [4] and Senu et.al [5] developed diagonally implicit Runge-KuttaNyström (DIRKN) methods for solving (1). Senu. et. al [5] derived diagonally implicit RKN method by minimizing the phase-lag or dispersion error and amplification error or dissipation error so that the method is suitable for solving oscillatory problems. Bursa and Nigro [6] were the first to introduce phase-lag in their work. Phase-lag is defined as the difference of the angle for the computed solution and the exact solution. Amplification is the distance of the computed solution from the cyclic solution. This type of truncation error often used in phase-fitting and optimizing RKN and hybrid methods for solving oscillatory problems, such work can be seen in Papadopoulos et.al [7], Kosti et.al [8], Simos [9] and Ahmad et.al [10].

Hence, in this paper we are going to construct a new semi-implicit hybrid method for directly solving second order ODEs. Phase-lag and amplification error of order zero are infuse into the method so that it will have zero phase-lag and zero dissipation. The performance of the original and modified methods as well as other existing methods are compared. The comparisons suggest the superiority of the modified hybrid method.

## THE ORDER CONDITIONS AND PHASE ANALYSIS.

The general formula of semi-implicit hybrid method for the numerical integration of the initial value problems (IVPs) is given as

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s} a_{ij} f\big(x_n + c_j h, Y_j\big), \quad (2)$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i). \quad (3)$$

where $i = 1, \ldots, s,$ and $i \geq j$. The equations of the form (2) and (3) are defined as

$$Y_1 = y_{n-1}, Y_2 = y_n, \quad (4)$$

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{i} a_{ij} f\left(x_n + c_j h, Y_j\right), \quad (5)$$

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \left(b_1 f_{n-1} + b_2 f_n + \sum_{i=3}^{s} b_i f(x_n + c_i h, Y_i)\right). \quad (6)$$

for $i = 3, \ldots, s$ and the first two nodes are $c_1 = -1$, $c_2 = 0$ and functions $f_{n-1} = f(x_{n-1}, y_{n-1})$ and $f_n = f(x_n, y_n)$.

The coefficients of $b_i$, $c_i$ and $a_{ij}$ can be tabulatedutcher tableau as in Table 1.

**Table 1: The $s$-stage semi-implicit hybrid method**

| | | | | | |
|---|---|---|---|---|---|
| $-1$ | $0$ | | | | |
| $0$ | $0$ | $0$ | | | |
| $c_3$ | $a_{3,1}$ | $a_{3,2}$ | $a_{3,3}$ | | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\ddots$ | |
| $c_s$ | $a_{s,1}$ | $a_{s,2}$ | $\dots$ | $a_{s,s-1}$ | $a_{s,s}$ |
| | $b_1$ | $b_2$ | $\dots$ | $b_{s-1}$ | $b_s$ |

The order conditions for hybrid method which were given in Coleman [11] for up to order six are listed as below:

(i) Order 2

$$\sum_{i=1}^{s} b_i = 1\,. \tag{7}$$

(ii) Order 3

$$\sum_{i=1}^{s} b_i c_i = 0. \tag{8}$$

(iii) Order 4

$$\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{6}\ , \quad \sum_{i=1}^{s} b_i a_{ij} = \frac{1}{12} \tag{9}$$

(iv) Order 5

$$\sum_{i=1}^{s} b_i c_i^3 = 0\ , \quad \sum_{i=1}^{s} b_i c_i a_{ij} = \frac{1}{12}\ , \quad \sum_{i=1}^{s} b_i\, a_{ij} c_j = 0. \tag{10}$$

(v) Order 6

$$\sum_{i=1}^{s} b_i c_i^4 = \frac{1}{15}\ , \quad \sum_{i=1}^{s} b_i c_i^2 a_{ij} = \frac{1}{30}\ , \quad \sum_{i=1}^{s} b_i c_i a_{ij} c_j = -\frac{1}{60},$$

$$\sum_{i=1}^{s} b_i a_{ij} a_{ik} = \frac{7}{120}\ , \quad \sum_{i=1}^{s} b_i a_{ij} c_j^2 = \frac{1}{180}\ , \quad \sum_{i=1}^{s} b_i a_{ij} a_{jk} = \frac{1}{360}. \tag{11}$$

where value of $i \geq j \geq k$ . For $c_i$ , the methods need to satisfy the simplifying condition which is:

$$\sum_{i}^{s} a_{ij} = \frac{(c_i^2 + c_i)}{2}\quad , \quad i = 3, \dots, s. \tag{12}$$

To derive the new semi-implicit hybrid method, we use the order conditions, simplifying conditions and minimization of the error constant $C_{p+1}$ of the method. The error constant is defined by

$$C_{p+1} = \left\| \left(e_{p+1}(t_1)\right), \dots, e_{p+1}(t_k) \right\|_2 \tag{13}$$

where$k$ is the number of trees of order $p + 2(p(t_i) = p + 2)$, for the $p$th – order method and $e_{p+1}(t_i)$ is the local truncation error.

The phase analysis is investigated using the second order homogeneous linear test equation

$$y''(t) = -v^2 y(t), \text{for } \lambda > 0. \tag{14}$$

Applying (14) into the method (equations (2) and (3)) we have the following

$$\boldsymbol{Y} = (\boldsymbol{e} + \boldsymbol{c})y_n - \boldsymbol{c}y_{n-1} - z^2 \boldsymbol{AY}, \tag{15}$$

$$y_{n+1} = 2y_n - y_{n-1} - z^2 \boldsymbol{b^T Y}, \tag{16}$$

where $z = vh$, $\boldsymbol{Y} = (Y_1, \dots, Y_s)^T$ ,$\boldsymbol{c} = (c_1, \dots, c_s)^T$ , and $\boldsymbol{e} = (1, \dots, 1)^T$. From (15) we obtain

$$\boldsymbol{Y} = (\boldsymbol{I} + z^2\boldsymbol{A})^{-1}(\boldsymbol{e} + \boldsymbol{c})y_n - (\boldsymbol{I} + z^2\boldsymbol{A})^{-1}\boldsymbol{c}y_{n-1}. \tag{17}$$

Substituting equation (17) into equation (16), the following recursion relation is obtained

$$y_{n+1} - S(z^2)y_n + P(z^2)y_{n-1} = 0\ , \tag{18}$$

where

$$S(z^2) = 2 - z^2 \boldsymbol{b}^{\boldsymbol{T}}(\boldsymbol{I} + z^2\boldsymbol{A})^{-1}(\boldsymbol{e} + \boldsymbol{c}), \tag{19}$$

$$P(z^2) = 1 - z^2 \boldsymbol{b}^{\boldsymbol{T}}(\boldsymbol{I} + z^2\boldsymbol{A})^{-1}\boldsymbol{c}. \tag{20}$$

*Definition 1.* The phase-lag is define as $\varphi(z) = h - \phi$. If $\varphi(z) = O(z^{q+1})$, then the hybrid method is said to be dissipative of order *q*. The quantity $d(z) = 1 - \sqrt{P(z^2)}$ is called amplification error. If $(z) = O(z^{r+1})$ , the method is said to have dissipation of order *r*.

From definition 1, it follows the nomenclature given by [1]

$$\phi(z) = z - \cos^{-1}\left(\frac{S(z^2)}{2\sqrt{P(z^2)}}\right), \quad d(z) = 1 - \sqrt{P(z^2)}, \tag{21}$$

are called the dispersion error (phase-lag) and the dissipation error (amplification error), respectively. The stability polynomial of hybrid method (18) can be written as

$$\xi^2 - S(z^2)\xi + P(z^2) = 0. \tag{22}$$

The numerical solution defined by the difference (18) should be periodic. The necessary conditions are $P(z^2) \equiv 1$,and$|S(z^2)| < 2$, $\forall z \epsilon \left(0, z_p\right)$and interval $\left(0, z_p\right)$ is known as the periodicity interval of the method.

The method is called zero dissipative which is when $d(z) = 0$ if it satisfies the conditions (21). Otherwise, as the method possesses a finite order of dissipation, the integration process is stable if the coefficients of polynomial in (22) satisfy the conditions $P(z^2) < 1$,and $|S(z^2)| < 1 + P(z^2)$,$\forall z\epsilon(0, z_s)$and interval $(0, z_s)$ is known as interval of absolute stability of the method.

## DERIVATION OF SEMI-IMPLICIT HYBRID METHOD (SIHM).

In this section a new four-stage fifth-order SIHM is constructed using the algebraic conditions up to order five (using equations (7) up to equation (10), simplifying condition in (12) and dispersion error up to order eight. We obtained the the coefficients of the method in terms of $a_{44}$, $b_4$, and $c_4$ and are listed as below:

$$c_3 = -\frac{6b_4{c_4}^2 + 6b_4c_4 - 1}{6b_4{c_4}^2 + 6b_4c_4 - 1},$$

$$a_{31} = \frac{(56a_{44} - 1)(6b_4c_4^3 - 1 + 6b_4c_4^2)}{56(-1 + 30a_{44})(6c_4b_4 - 1 + 6b_4c_4^2)},$$

$$a_{32} = \frac{\begin{pmatrix} -9408b_4a_{44}c_4^2 + 324b_4c_4^2 + 249b_4c_4^3 + 18{b_4}^2c_4^2 + 75b_4c_4 \\ -450{b_4}^2{c_4}^3 - 7392b_4{c_4}^3a_{44} + 42336{b_4}^2{c_4}^4a_{44} + 12096{b_4}^2{c_4}^3a_{44} \\ -504{b_4}^2{c_4}^6 - 27 + 44352{b_4}^2{c_4}^5a_{44} + 784a_{44} - 1008{b_4}^2{c_4}^2a_{44} \\ -1458{b_4}^2{c_4}^4 + 15120{b_4}^2{c_4}^6a_{44} - 2016b_4c_4a_{44} - 1494{b_4}^2{c_4}^5 \end{pmatrix}}{28(-1 + 30a_{44})(6b_4c_4 - 1 + 6b_4{c_4}^2)^2},$$

$$a_{33} = \frac{56a_{44} - 1}{56(-1 + 30a_{44})},$$

$$a_{41} = \frac{\begin{pmatrix} 30240{b_4}^2{c_4}^4a_{44} - 540{b_4}^2{c_4}^4 - 1080{b_4}^2{c_4}^3 + 60480{b_4}^2{c_4}^3a_{44} \\ +30240{b_4}^2{c_4}^2a_{44} - 540{b_4}^2{c_4}^2 - 10080b_4{c_4}^2a_{44} + 12b_4{c_4}^2 \\ +151200b_4{c_4}^2{a_{44}}^2 - 151200b_4c_4{a_{44}}^2 + 12b_4c_4 + 13 \end{pmatrix}}{10080b_4(-1 + 30a_{44})(-1 + 3b_4{c_4}^3 + 6b_4{c_4}^2 + 3b_4c_4)},$$

$$a_{42} = -\frac{\begin{pmatrix} 15120{b_4}^2{c_4}^5 - 453600{b_4}^2{c_4}^5a_{44} - 876960{b_4}^2{c_4}^4a_{44} + 29700{b_4}^2{c_4}^4 \\ +14040{b_4}^2{c_4}^3 - 393120{b_4}^2{c_4}^3a_{44} + 30240{b_4}^2{c_4}^2a_{44} - 540{b_4}^2{c_4}^2 \\ +151200b_4{c_4}^2{a_{44}}^2 - 2508b_4{c_4}^2 + 65520b_4{c_4}^2a_{44} - 2508b_4c_4 \\ +70560b_4c_4a_{44} - 151200b_4{a_{44}}^2 + 5040b_4a_{44} + 13 \end{pmatrix}}{5040b_4(-1 + 30a_{44})(-1 + 6b_4{c_4}^3 + 6b_4{c_4}^2)},$$

$$a_{43} = -\frac{\begin{pmatrix} 13 - 10080b_4c_4a_{44} + 120960{b_4}^2{c_4}^2a_{44} - 10080b_4{c_4}^2a_{44} \\ +120960{b_4}^2{c_4}^4a_{44} + 141920{b_4}^2{c_4}^3a_{44} + 151200b_4{c_4}^2{a_{44}}^2 \\ 151200b_4c_4{a_{44}}^2 - 66b_4{c_4}^2 - 66b_4c_4 - 612{b_4}^2{c_4}^2 - 612{b_4}^2{c_4}^4 \\ -1224{b_4}^2{c_4}^3 + 3240{b_4}^3{c_4}^3 + 9720{b_4}^3{c_4}^5 + 3240{b_4}^3{c_4}^6 + 9720{b_4}^3{c_4}^4 \\ -362880{b_4}^3{c_4}^6a_{44} - 1814400{b_4}^2{c_4}^4{a_{44}}^2 - 3628800{b_4}^2{c_4}^3{a_{44}}^2 \\ -1814400{b_4}^2{c_4}^2{a_{44}}^2 + 5443200{b_4}^3{c_4}^6{a_{44}}^2 + 16329600{b_4}^3{c_4}^5{a_{44}}^2 \\ -362880{b_4}^3{c_4}^3a_{44} - 1088640{b_4}^3{c_4}^5a_{44} - 1088640{b_4}^3{c_4}^4a_{44} \\ +5443200{b_4}^3{c_4}^3{a_{44}}^2 + 16329600{b_4}^3{c_4}^4{a_{44}}^2 \end{pmatrix}}{10080b_4(-1 + 6b_4{c_4}^3 + 6b_4{c_4}^2)(-1 + 3b_4{c_4}^3 + 6b_4{c_4}^2 + 3b_4c_4)(-1 + 30a_{44})},$$

$$b_1 = \frac{-1 + 12b_4{c_4}^2}{12(-1 + 3b_4{c_4}^3 + 6b_4{c_4}^2 + 3b_4c_4)},$$

$$b_2 = \frac{36b_4{c_4}^3 + 24b_4{c_4}^2 - 6b_4c_4 + 6b_4 - 5}{6(-1 + 6b_4{c_4}^3 + 6b_4{c_4}^2)},$$

$$b_3 = -\frac{(6b_4{c_4}^2 + 6b_4c_4 - 1)^3}{12(-1 + 3b_4{c_4}^3 + 6b_4{c_4}^2 + 3b_4c_4)(-1 + 6b_4{c_4}^3 + 6b_4{c_4}^2)}.$$

By assuming the value of $b_4 = \frac{2}{11}$ and $c_4 = \frac{3}{4}$ , using minimization of error norm we get the value of $a_{44} = \frac{1}{300}$. The principal local truncation error coefficient for $y_n$ as $\left\|\tau^{(6)}\right\|_2 = 1.1472 \times 10^{-1}$ , where $\left\|\tau^{(6)}\right\|_2$ is the norm of all the error equation for the sixth order method. The method has dispersion of order eight and dissipation of order seven and denoted as SIHM4(5)(8,7) and given in Table 2:

**Table 2: SIHM4(5),(8.7) method**

| | | | | |
|---|---|---|---|---|
| $-1$ | | | | |
| $0$ | | | | |
| $\frac{13}{76}$ | $\frac{25250849}{829664640}$ | $\frac{44456123}{829664640}$ | $\frac{61}{3780}$ | |
| $\frac{3}{4}$ | $\frac{13725721}{161481600}$ | $\frac{9560591}{23587200}$ | $\frac{5333053}{32800950}$ | $\frac{1}{300}$ |
| | $\frac{20}{267}$ | $\frac{43}{39}$ | $-\frac{13718}{38181}$ | $\frac{2}{11}$ |

## MODIFICATION OF PHASE-LAG AND AMPLIFICATION

In this section, we will modified the method so that it will have vanishing phase-lag and amplification error. Using the general equation for hybrid method in (3), we modified the formula as

$$y_{n+1} = 2\gamma_1(z)y_n - \gamma_2(z)y_{n-1} + h^2\sum_{i=1}^{s} b_i\, f(x_n + c_ih, Y_i). \qquad (23)$$

where $z = vh$, $v$ the fitted frequency of the problem that we want to solve. Substituting equation (14) into (19)-(20) to obtain

$$S(z^2) = 2\gamma_1(z) - z^2\boldsymbol{b^T}(\boldsymbol{I} + z^2\boldsymbol{A})^{-1}(\boldsymbol{e} + \boldsymbol{c}), \qquad (24)$$

$$P(z^2) = \gamma_2(z) - z^2\boldsymbol{b^T}(\boldsymbol{I} + z^2\boldsymbol{A})^{-1}\boldsymbol{c}. \qquad (25)$$

In order to have phase-lag and amplification error of order zero, equation (21) have to be equal to zero, hence

$$S(z^2) = 2\cos(z), \qquad (26)$$

$$P(z^2) = 1. \qquad (27)$$

Using equations (24)-(27) and solving for $\gamma_1(z)$ and $\gamma_2(z)$ we have

$$\gamma_1(z) = \cos(z) + \frac{1}{2}z^2\boldsymbol{b^T}(\boldsymbol{I} + z^2\boldsymbol{A})^{-1}(\boldsymbol{e} + \boldsymbol{c}), \qquad (28)$$

$$\gamma_2(z) = 1 + z^2\boldsymbol{b^T}(\boldsymbol{I} + z^2\boldsymbol{A})^{-1}\boldsymbol{c}. \qquad (29)$$

By substituting the coefficients of SIHM4(5)(8,7) into (28) - (29), we find the values of $\gamma_1(z)$ and $\gamma_2(z)$ . The Taylor expansion of which can written as :

$$\gamma_1(z) = 1 - \frac{21}{60800}z^6 + \frac{23}{3420000}z^8 + \frac{440011}{21718368000000}z^{10} + O(z^{12}),$$

$$\gamma_2(z) = 1 - \frac{21}{30400}z^6 + \frac{23}{1710000}z^8 + \frac{116209}{517104000000}z^{10} + O(z^{12}).$$

Denote this new method as vanishing phase-lag and amplification semi-implicit hybrid method four-stage fifth-order (VPA-SIHM4(5)). When $z = 0$, the method is actually the original SIHM4(5)(8,7).

**PROBLEMS TESTED AND NUMERICAL RESULTS**

In this section, the new method VPA-SIHM4(5) is used to solve large interval test problems [0,10000]to signify that the new method is suitable for integrating oscillatory problems. The test problems are listed as below:

**Problem 1***(An almost Periodic Orbit problem studied by Stiefel and Bettis [12])*

$y_1''(x) + y_1(x) = 0.001\cos(x)$ ,$y_1(0) = 1$,$y_1'(0) = 0$ ,

$y_2''(x) + y_2(x) = 0.001\sin(x)$ ,$y_2(0) = 0$,$y_2'(0) = 0.9995$ ,

Exact solution is $y_1 = \cos(x) + 0.0005x\sin(x), y_2 = \sin(x) - 0.0005x\cos(x)$.

The fitted frequency is $v = 1$.

**Problem 2** *(Inhomogeneous system by Lambert and Watson [13])*

$\frac{d^2y_1(x)}{dt^2} = -v^2y_1(x) + v^2f(x) + f''(x), y_1(0) = a + f(0), y_1'(0) = f'(0)$,

$\frac{d^2y_2(x)}{dt^2} = -v^2y_2(x) + v^2f(x) + f''(x)y_2(0) = f(0), y_2'(0) = va + f'(0)$,

Exact solution is $y_1(x) = a\cos(vx) + f(x)$,$y_2(x) = a\sin(vx) + f(x)$, $f(x)$is chosen to be $e^{-0.05x}$ and parameters $v$ and $a$ are 20 and 0.1 respectively.

**Problem 3***(Inhomogeneous system studied by Franco [14])*

$$y''(x) = \begin{pmatrix} \frac{101}{2} & -\frac{99}{2} \\ -\frac{99}{2} & \frac{101}{2} \end{pmatrix} y(x) = \delta \begin{pmatrix} \frac{93}{2}\cos(2x) & -\frac{99}{2}\sin(2x) \\ \frac{93}{2}\sin(2x) & -\frac{99}{2}\cos(2x) \end{pmatrix}$$

$y(0) = \begin{pmatrix} -1+\delta \\ 1 \end{pmatrix}, y'(0) = \begin{pmatrix} -10 \\ 10+2\delta \end{pmatrix}$for $\delta = 10^{-3}$.

Exact solution is given by $y(t) = \begin{pmatrix} -\cos(10x) - \sin(10x) + \delta\cos(2x) \\ \cos(10x) + \sin(10x) + \delta\sin(2x) \end{pmatrix}$.

The Eigen-value of the problem are $v = 10$and $v = 1$.

The fitted frequency is choose to be $v = 10$ because it is dominant than $v = 1$.

**Problem 4***(Homogeneous by Chakravarti and Worland [15])*

$y''(x) = -y(x), y(0) = 0, y'(0) = 1$.

Exact solution is $y(x) = \sin(x)$.

The fitted frequency and $v = 1$.

**Problem 5***(Homogenous given in Attili et.al [2])*

$y''(x) = -64y(x)$ , $y(0) = {}^1/_4$ , $y'(0) = -{}^1/_2$.

Exact solution is $y = {}^{\sqrt{17}}/_{16}\sin(8x + \theta), \theta = \pi - \tan^{-1}(4)$.

The fitted frequency, $v = 8$.

**Problem 6***(Inhomogeneous equation studied by Papadopoulos et.al [7])*

$y''(x) = -v^2y(x) + (v^2 - 1)\sin(x), y(0) = 1, y'(0) = v + 1$.

Exact solution is $y(x) = \cos(vx) + \sin(vx) + \sin(x)$.The fitted frequency,$v = 10$.

The methods are compared using a measure of the accuracy,$absolute\ error$ which is defined by

$$Absolute\ error = max\{\|y(t_n) - y_n\|\}$$

where$y(t_n)$ is the exact solution and $y_n$ is the computed solution.

The following notations are used in Figures 1-6:

1. **VPA-SIHM4(5)** : A method of vanishing phase-lag and amplification error of four-stage fifth-order semi-implicit hybrid method developed in this paper.
2. **SIHM4(5)(8,7)** : A four-stage fifth-order semi-implicit hybrid method with dispersion of order eight and dissipation of order seven developed in this paper.
3. **RKN4(5)** : Explicit Runge-KuttaNyström method four-stage fifth-order by Hairer et al. [16].
4. **PFRKN4(4)** : Phase-fitted explicit Runge-KuttaNyström method four-stage fourth-order by Papadopoulos et al. [7]
5. **DIRKN4(4)** : Diagonally Implicit Runge-KuttaNyström method four-stage fourth-order derived in Senu et al. [5].

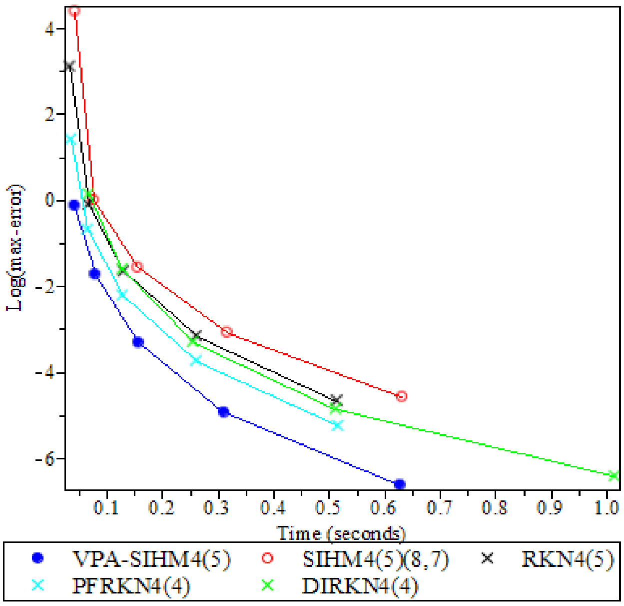


**Figure 1:** The efficiency curves forproblem in 1with $h = \frac{0.125}{2^i}$, for $i = 0, \ldots, 4$.

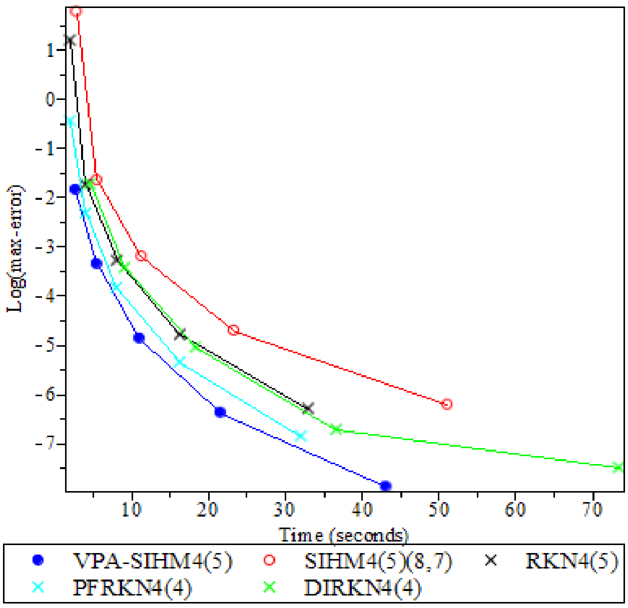


**Figure 2:** The efficiency curves for problem in 2 with $h = \frac{1.25}{2^i}$, for $i = 2, \ldots, 6$.

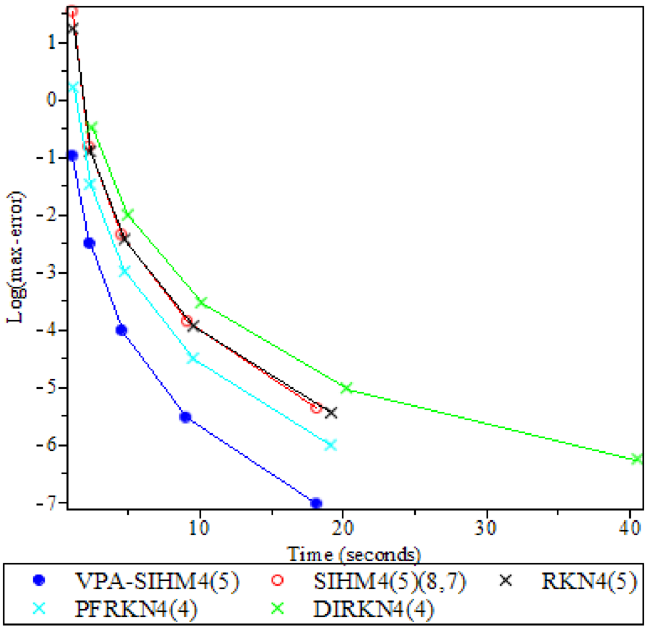


**Figure 3:** The efficiency curves for problem in 3 with $h = \frac{1.25}{2^i}$, for $i = 1, \ldots, 5$.

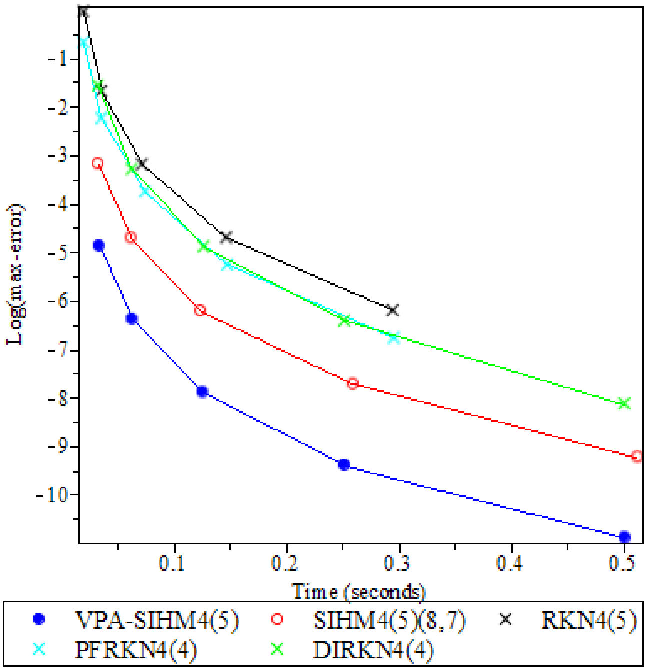


**Figure 4:** The efficiency curves for problem in 4 with $h = \frac{1.5}{2^i}$, for $i = 1, \ldots, 5$.

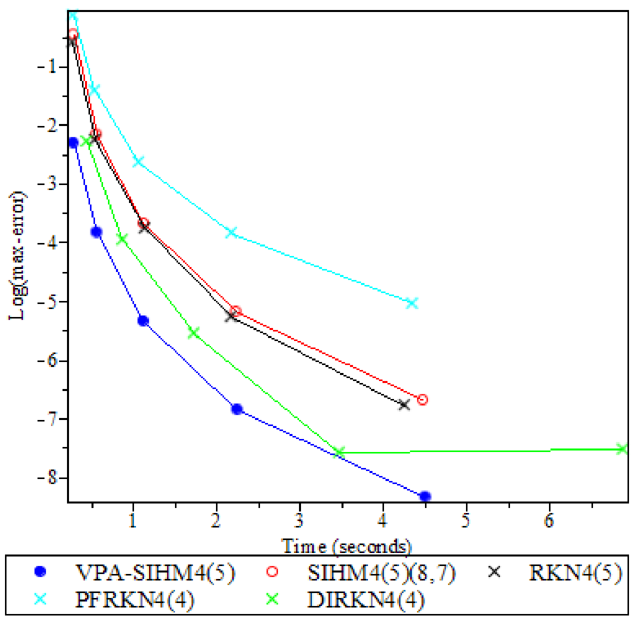


**Figure 5:** The efficiency curves problem in 5 with $h = \frac{1.0}{2^i}$, for $i = 3, \ldots, 7$.

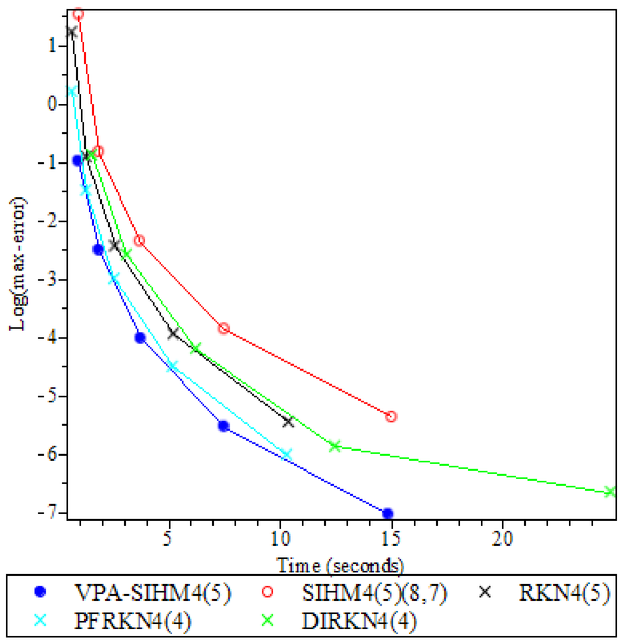


**Figure 6:** The efficiency curves for problem in 6 with $h = \frac{0.125}{2^i}$, for $i = 1, \ldots, 5$.

In analyzing the numerical results, methods of the same order or stage are compared. The results are given in Figures 1-6. We analyze the efficiency curves where the logarithm of the maximum global error are plotted against the CPU time taken in second.

From Figures 1-6, the modified method VPA-SIHM4(5) noticeably perform better compared to the original method SIHM4(5)(8,7) as well as other existing methods.

## CONCLUSION

In this paper we derived a new method semi implicit hybrid method of four-stage and fifth order with order of dispersion eight and order of dissipation seven denoted as SIHM45(8,7). The method is then phase-fitted so that it has vanishing phase-lag and amplification error and the new method is denoted as VPA-SIHM4(5). Numerical experiment over a very large interval for solving highly oscillatory problems proved that the new method is very efficient in solving oscillatory initial value problems compared to the original semi-implicit hybrid method and other robust methods in the scientific literature. Hence modifying the method so that it has zero phase-lag and zero dissipation really improved the efficiency of the method.

The authors declare there is no conflict of interest regarding the publication of this paper.

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