Active Vibration Control of Sandwich FGM Beam with Piezoelectric Sensor/Actuator

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Abstract
This work presents a dynamic study of sandwich beams containing layers of Functionally Graded Materials (FGM) and piezoelectric layers, using the Euler-Bernoulli theory. The equations of motion are obtained by applying the Hamilton principle. The vibration frequencies are found by solving the eigenvalues problem. The theory proposed for the dynamic behavior of the sandwich Functionally Graded piezo-beams is then exploited in the formulation of the active control laws using the optimal control LQG accompanied by the Kalman filter.

Keywords: Multi-layered beam, Functionally Graded Materials, Piezoelectricity, Vibration, Finite element, Euler-Bernoulli, LQG-Kalman control.

INTRODUCTION
Modern industry is looking for materials with functional advantages such as lightness, good mechanical and chemical resistance, long service life, reduced maintenance, reduced manufacturing time… A conventional material does not make it possible to combine all these characteristics. To achieve this goal, composite materials must be used.

Multilayer composite materials are widely used in various engineering structures because of their performance. For example, a homogeneous ceramic elastic layer can be bonded to the surface of a metal structure and acts as a thermal barrier in a high temperature environment. However, due to a distinct interface between the two materials (ceramic and metal), the material properties across the interface undergo a sudden change, which produces the force jump and can further cause the detachment or breakage of the interface. One possible solution for this problem is the use of functionally graded materials. These materials are a new generation of composites presented in 1990 by Japanese scientists [1], for which material properties, such as Young's modulus, density and Poisson's coefficient, vary continuously, conferring a considerable advantage over homogeneous and laminated materials in maintaining the integrity of the structure.

Among the harmful physical phenomena that act on structures during their lifetime, the phenomenon of vibrations and environmental conditions. Therefore the active control of the vibrations of composites in a thermal environment deserves to be studied. Piezoelectric materials are considered the most appropriate intelligent materials to use for the active and semi-active control of structures such as beams, plates and shells. Lam et al. (1997) [2] developed a finite element model based on the classical laminated plate theory for active vibration control of a composite plate containing distributed piezoelectric sensors and actuators. Peng et al. [3] developed a finite element model based on third-order laminate theory for the active location and vibration control of composite beams with distributed piezoelectric sensors and actuators. Bruant et al. [4] have worked on improving the active control of a FGM beam with piezoelectric elements. Panda et al. [5] treated active fiber composites for vibration control of functional beams (FGM).

The analysis of composite structures has developed with the emergence of numerical methods including the finite element method. The objective of this work is to analyze the vibrational behavior of a PZT / FGM / PZT multi-layer beam using the finite element method and then its active control via the PZT piezoelectric elements.

MATHEMATICAL MODEL
Consider a beam of length L in the x-direction, rectangular cross-section with the width b in the y-direction and the thickness h in the z-direction as shown in FIG. 1. The beam consists of a functionally graded materials composed of metal and ceramic, partially covered by two piezoelectric actuators and two piezoelectric sensors bonded to the upper and lower faces of the FGM core [6].

When the structure presents a symmetry both geometric and material with respect to its neutral plane, as in our case, the mechanical part of these equations is then similar to that of a
homogeneous beam per sections, where there is decoupling between traction and bending, a curvature of the beam produces only a bending moment [7], [8]. We thus place ourselves in a case of pure bending.

Figure 1: Geometry of an embedded-free FGM beam containing piezoelectric layers.

In this study, a simple power law is considered to describe the variation of the material properties from the pure metal in the underside up to pure ceramic on the upper surface of the beam.

\[ V_c = \left( \frac{z}{h} \right)^k = 1 - V_m \cdot (k \geq 0) \]  

Where \( V_c \) and \( V_m \) are the volume fractions of the ceramic and metal components respectively, and \( k \) is the exponent of the volume fraction which dictates the variation profile of the material through the thickness of the beam.

Figure 2: Variation of the volume fraction as a function of the thickness.

Consequently, the material properties of the beam, the Young's modulus \( E \) and the mass density \( \rho \), vary continuously in the thickness direction (z-axis).

\[ E(z) = E_m + (E_c - E_m) \left( \frac{z}{h} \right)^k \]  
\[ \rho(z) = \rho_m + (\rho_c - \rho_m) \left( \frac{z}{h} \right)^k \]

Where the indices \( m \) and \( c \) denotes the constituents, metal and ceramic, respectively.

Considering the beam’s theory of the Euler-Bernoulli, the transverse and axial displacements \( w \) and \( u \) of any point of the beam are given by:

\[ u = z \frac{\partial w(x,t)}{\partial x} \cdot v = 0 \cdot w = w(x,t) \]  

Where, \( u \), \( v \) and \( w \) indicate the displacement components parallel to the directions, \( x \), \( y \) and \( z \) respectively. The components of the strain and stress corresponding to this displacement zone are given by:
being the total length of the beam.

\[
\epsilon_{xx} = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}, \quad \epsilon_{yy} = \epsilon_{zz} = \epsilon_{xy} = \epsilon_{yz} = \epsilon_{zx} = 0 \quad (5)
\]

\[
\sigma_{xx} = EZ \frac{\partial^2 w}{\partial x^2}, \quad \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \quad (6)
\]

Application of the Hamilton’s Principle

\[
\delta \int_{t_i}^{t_f} \left( E_{inc} - (E_{def} - W_{ext}) \right) dt = 0 \quad \forall \{\delta u\} \quad (7)
\]

With

\[
\{\delta u\}_{n=1}^{N} = \{\delta u\}_{n=N} = \{0\} \quad (8)
\]

leads to the Lagrange’s equations:

\[
\frac{d}{dt} \left( \frac{\partial E_{inc}}{\partial u} \right) + \frac{\partial E_{def}}{\partial u} - \frac{\partial W_{ext}}{\partial u} = 0 \quad i = 1, \ldots, 4 \quad (9)
\]

Either in matrix form:

\[
\{f_{nod}\} = [m]\{\ddot{u}\} + [k]\{\dot{u}\} - \{f\} \quad (10)
\]

The kinetic energy equals to:

\[
T = \frac{1}{2} \iiint_V \rho \left( \frac{\partial w}{\partial t} \right)^2 dV \quad (11)
\]

The strain energy is equal to:

\[
U = \frac{1}{2} \iiint_V \sigma_{xx} \epsilon_{xx} dV = \frac{1}{2} \iiint_V E_z \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dV \quad (12)
\]

The work of the external forces for displacement and rotation is equal to:

\[
W_{ext} = \int_0^L w(x) p_x(x) dx + \int_0^L \theta(x) m_z(x) dx + \{u\}^T \{f_{nod}\}
\]

\[
= \{u\}^T \{f\} + \{f_{nod}\} \quad (13)
\]

The dynamics due to the deformation behavior of the embedded-free element is governed by the following fourth-order differential equation [9]:

\[
\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w(x,t)}{\partial x^2} \right) + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} = f_{ext} \quad (14)
\]

With \(E, I, \rho, A\) and \(f_{ext}\) are, respectively, the Young’s modulus, the quadratic moment, the mass density, the surface of the element and the distribution of the forces applied to the end of the beam.

The solution of equation (14) is supposed to have a cubic form as a function of \(x\) and is written:

\[
w(x,t) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \quad (15)
\]

Hence the rotation of the straight sections:

\[
\theta(x,t) = \frac{\partial w}{\partial x} = a_2 + 2a_3 x + 3a_4 x^2 \quad (16)
\]

Constants \(a_i\) \((i = 1, \ldots, 4)\) are determined using boundary conditions at nodes 1 and 2 of the element. The Euler-Bernoulli equation is based on the assumption says that the plane normal to the axis of the beam before deformation remains normal after deformation [10].

The boundary conditions at the nodes, (embedded and end), are expressed as follows:

at \(x = 0\) (embedded section),

\[
w(x,t) = w_1 = a_1, \quad w(x,t) = \frac{\partial w}{\partial x} = a_2 = \theta_1 \quad (17)
\]

at \(x = L\) (end section),

\[
w(x,t) = w_2, \quad \dot{w}(x,t) = \frac{\partial w}{\partial x} = a_2 = \theta_2 \quad (18)
\]

Where, \(w_1, \theta_1\) and \(w_2, \theta_2\) are the DOF relating to the nodes 1 and 2 respectively, and \(L\) being the total length of the beam.

By applying the boundary conditions of equations (17) and (18) in equation (15), the final form \(w(x,t)\) becomes:

\[
[w(x,t)] = [f_1(x) \quad f_2(x) \quad f_3(x) \quad f_4(x)] \quad \left[ w_1 \quad \theta_1 \quad w_2 \quad \theta_2 \right]^T \quad (19)
\]

where \([a]\) contains the interpolation functions \(f_i(x)\) to \(f_4(x)\) of the element.

Therefore, the first and second spatial derivatives and the time derivative are written:

\[
[w(x,t)] = [a_2^T] \quad [q], \quad [\dot{w}(x,t)] = [a_1^T] \quad [q] \quad (20)
\]

The strain energy \(U\) and the kinetic energy \(T\) of the piezoelectric element and the core element FGM at the cross-section of the beam are written [10]:

\[
A. \quad FOR \quad THE \quad PIEZOELECTRIC \quad ELEMENT:
\]

\[
U = \frac{E_p}{2} \int_{L_p} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx = \frac{E_p l}{2} \int_{L_p} \left[ \frac{\partial w}{\partial x} \right]^2 \left[ \frac{\partial w}{\partial x} \right] dx \quad (21)
\]
Where $\rho_p$ is the mass density, $A_p$ is the area of the crosssection, $I_p$ is the quadratic moment, $l_p$ is the length and $E_p$ is the Young’s modulus of the piezoelectric material.

### B. FOR THE FGM ELEMENT:

\[ U = \frac{1}{2} \int_V E(z) z^2 \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dV \]

\[ T = \frac{1}{2} \int_V \rho(z) \left( \frac{\partial w}{\partial t} \right)^2 dV \]

\[ \frac{d}{dt} \left[ \frac{\partial T}{\partial q_1} \right] + \frac{\partial U}{\partial q_1} = [Z_i] \]

The local elementary matrices of mass and stiffness of the piezo-core element are written:

\[ [M] = [M_{FGM}] + 2[M_p] \]

\[ [K] = [K_{FGM}] + 2[K_p] \]

with

\[ [M_{FGM}], [K_{FGM}] \text{ and } [M_p], [K_p] \] are the elementary matrices of mass and stiffness of the core and piezoelectric elements, respectively.

### SENSOR / ACTUATOR EQUATIONS

The linear piezoelectric coupling between the elastic field and the electric field can be expressed by the direct and inverse piezoelectric constitutive equations [7], [9] as:

\[ D_z = d_{31} \sigma + e \varepsilon \varepsilon_f , \quad D_z = d_{31} \varepsilon F + S^F \sigma \]

where $D_z$ is the electric displacement, $d_{31}$ is the piezoelectric constant, $\sigma$ is the stress, $e$ is the permittivity of the medium (dielectric constant), $\varepsilon_f$ is the electric field, $\varepsilon$ is the strain and $S^F$ is the compliance of the piezoelectric medium.

The equation of the sensor is derived from the direct piezoelectric equation which is used to calculate the total load by the strain in the structure. Since no external field is applied to the sensor layer, the electrical displacement developed on the sensor surface is directly proportional to the strain acting on it.

If the polarization is carried out along the direction of the thickness of the sensors with the electrodes on the upper and lower surfaces, the electric displacement $D_z$ is given as:

\[ D_z \propto \varepsilon_x = e_3 \varepsilon_x \]

The total load $Q$ developed on the surface of the sensor is the spatial summation of all the point charges developed on the sensor layer and which is given by:

\[ Q(t) = \int_A D_z dA \]

\[ i(t) = \frac{dQ(t)}{dt} = \frac{d}{dt} \int_A D_z dA = \frac{d}{dt} \int_A e_3 \varepsilon_x dA \]

Since the strain $\varepsilon_x$ of the structure can be expressed in terms of the second spatial derivative of displacement $w(x,t)$ in...
\( e_x = \frac{d^2 w}{dx^2} \), which \( z \) is the coordinate of the point on the beam (neutral axis), the equation can be written as:

\[
i(t) = \frac{d}{dt} \int_A e_3 z \ n_i^T q dA + \int_A e_3 z \ n_i^T \frac{d}{dt} (q) dA
\]

\[
i(t) = e_3 z \ b \int_0^t n_i^T q d\tau
\]  

The voltage will be applied as input to the actuator with an accurate gain depending on the degree of damping desired,

\[
V^c(t) = G_e e_3 z b \int_0^t n_i^T q d\tau
\]

where \( z = \left[ \frac{l_b}{2} + e_3 \right] \), \( e_3 \) is the piezoelectric constant, \( n_i^T \) is the second derivative with respect to the \( x \) of the interpolation function and \( q \) is the derivative with respect to the time of the nodal vector \( \dot{q} \).

\[
V_1^c(t) = S_e [0 \ -1 \ 0 \ 1] q
\]

\[
V_2^c(t) = -S_e [0 \ -1 \ 0 \ 1] q
\]

which \( S_e = G_e e_3 z b \) is called the sensor constant.

The last equations of the two sensors can be written in the form:

\[
V_1^c(t) = -V_2^c = p^T \dot{q}
\]

The control input \( u \) is given by:

\[
V^e(t) = u = \text{Gain control} \times V^c(t)
\]

The resulting moment acting on the element due to the voltage \( V^e \) is determined by the integration of the stress over the entire beam thickness, which becomes after simplification:

\[
M_u = E_p d_{31} \bar{z} V^e(t)
\]  

With \( \bar{z} = \frac{l_a + l_b}{2} \) is the distance between the axes of symmetry of the core and the piezoelectric actuator.

The control force \( f_{ctr} \) produced by the actuator can be expressed as:

\[
f_{ctr1} = h V_1^e(t) = h u(t)
\]

\[
f_{ctr2} = h V_2^e(t) = -h u(t)
\]

Where \( h \) is a constant vector which depends on actuator characteristics:

\[
k^T = E_p d_{31} \bar{z} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = a_e \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}
\]

where \( a_e = E_p d_{31} \bar{z} \) being the actuator constant.

**DYNAMIC EQUATION AND STATE SPACE MODEL**

The equation of motion of the whole structure and the control equation are written, respectively:

\[
M \ddot{q} + K q = f_{ctr} + f_{ctr1} + f_{ctr2} = f^i
\]

\[
y_i(t) = V_i^c(t) = p_i^T q \cdot i = 1,2
\]

In order be interested in the first modes of vibration, generalized coordinates are introduced into equations (47) and (48), using the transformation \( q = Tg \).

Here, \( T \) being the modal matrix which contains the vectors representing the desired number of vibration modes of the embedded beam, \( q \) is the vector of the generalized coordinates and \( g \) is the principal coordinates.

Equations (47) and (48) become:

\[
M T \ddot{g} + K \dot{g} = f_{ctr} + f_{ctr1} + f_{ctr2} = f^i
\]

\[
y_i(t) = V_i^c(t) = p_i^T q \cdot i = 1,2
\]

Multiplying the equation (49) by \( T^T \), it can be rewritten as follows:

\[
M^* \ddot{g} + K^* \dot{g} = f_{ctr} + f_{ctr1} + f_{ctr2}
\]

where the matrices \( M^* \) and \( K^* \) are called the generalized mass and stiffness matrices.

Using Rayleigh’s proportional damping [11], [12]:

\[
C^* = \alpha M^* + \beta K^*
\]

where \( \alpha \) and \( \beta \) are respectively, the friction damping constant and the structural damping constant.

The dynamic equation of the structure and the control equation are finally given by:

\[
M^* \ddot{g} + C^* \dot{g} + K^* g = f_{ctr} + f_{ctr1} + f_{ctr2}
\]

The state space model in MIMO mode is given by:

\[
\begin{align*}
\dot{x} &= Ax(t) + Bu(t) + Er(t) \\
y(t) &= C^T x(t) + Du(t)
\end{align*}
\]

\[
A = \begin{bmatrix} 0 & 1 \\ -M^{-1} K^* & -M^{-1} C^* \end{bmatrix}
\]
\[
B = \begin{bmatrix}
0 \\
-M^{-1}T^T h_1 \\
-M^{-1}T^T h_2
\end{bmatrix},
\]
\[
C^T = \begin{bmatrix}
0 \\
0 \\
-p_1^r T \\
0 \\
0 \\
p_2^x T
\end{bmatrix},
\]
\[
D = 0, \quad E = \begin{bmatrix}
0 \\
-M^{-1}T^T f
\end{bmatrix},
\]

Where \( r(t), u(t), A, B, C, D, E, x(t) \) and \( y(t) \) represent the input force, control input, system matrix, input matrix, output matrix, transmission matrix, external load matrix, state vector, and system output.

RESULTS AND DISCUSSION

In order to validate our control procedure, we consider a flexible beam embedded at its left end and composed of an FGM core, partially covered by four thin PZT piezoelectric layers. The geometrical and physical characteristics of the materials are shown in Table 1.

**Table 1:** Geometrical and physical characteristics of the structure.

<table>
<thead>
<tr>
<th>Physical properties</th>
<th>MGF core</th>
<th>Materials (PZT) Sensor/Actuator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (m)</td>
<td>( L_b = 0.25 )</td>
<td>( L_a = L_o = 0.25/4 )</td>
</tr>
<tr>
<td>Width (m)</td>
<td>( b = 0.03 )</td>
<td>( b_a = b_o = 0.03 )</td>
</tr>
<tr>
<td>Thickness (m)</td>
<td>( t_b = 0.002 )</td>
<td>( t_a = t_o = 0.00001 )</td>
</tr>
<tr>
<td>Density (Kg/m³)</td>
<td>( \rho_m = 2780 )</td>
<td>( \rho_p = 7700 )</td>
</tr>
<tr>
<td>( \rho_c = 3800 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Young’s Modulus (G Pa)</td>
<td>( E_m = 70 )</td>
<td>( E_p = 68.1 )</td>
</tr>
<tr>
<td>( E_c = 380 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PZT Strain constants (m/V)</td>
<td>( d_{31} = 1.25 \times 10^{-12} )</td>
<td></td>
</tr>
<tr>
<td>PZT Stress constant (V/m/N)</td>
<td>( g_{31} = 10.5 \times 10^{-3} )</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 6:** Displacement of ceramic beam, metal beam, FGM beam.

**Figure 7:** The first three modes of the beam vibration.

**Figure 8:** Impulse response of PZT/FGM/PZT beam (k=0.2, \( Q=10^7 \), \( R=1 \), \( R_c=100 \)).
Figure 9: Spectrum of the first three modes of PZT/FGM/PZT beam (k=0.2, Q=10^7, R=1, Rc=100).

Figure 10: Step response of PZT/FGM/PZT beam (k=0.2, Q=10^7, R=1, Rc=100).

Figure 11: Impulse response of PZT/FGM/PZT beam (k=1, Q=10^7, R=1, Rc=100).

Figure 12: Spectrum of the first three modes of PZT/FGM/PZT beam (k=1, Q=10^7, R=1, Rc=100).

Figure 13: Step response of PZT/FGM/PZT Beam (k=1, Q=10^7, R=1, Rc=100).

Figure 14: Impulse response of PZT/FGM/PZT beam (k=10, Q=10^6, R=1, Rc=100).

Figure 15: Spectrum of the first three modes of PZT/FGM/PZT beam (k=10, Q=10^6, R=1, Rc=100).

Figure 16: Step response of PZT/FGM/PZT beam (k=10, Q=10^6, R=1, Rc=100).
CONCLUSION

From modeling and numerical simulation results of a composite beam including mass and stiffness of geometrically and mechanically symmetric sensors / actuators, we find that the variation of the exponent of the volume fraction causes a change of the structural and vibratory characteristics of the beam on the one hand, and the active control of its vibrations on the other.

REFERENCES


