International Journal of Applied Engineering Research ISSN 0973-4562 Volume 12, Number 20 (2017) pp. 10298-10304

Stability Analysis and Fractional Order Controller Design for Control System

Parvendra Kumar[1] and Sunil Kumar Chaudhary[2]

[1,2] Electrical Engineering Department, SunRise University, Alwar, Rajasthan, India

[1]Orcid: 0000-0001-6763-7864

Abstract

In this paper, a new approach to stability for fractional order control system is proposed. Here a dynamic system whose behavior can be modeled by means of differential equations involving fractional derivatives. Applying Laplace transforms to such equations, and assuming zero initial conditions, causes transfer functions with no integer powers of the Laplace transform variable s to appear. In recent time, the application of fractional derivatives has become quite apparent in modeling mechanical and electrical properties of real materials. Fractional integrals and derivatives have found wide application in the control of dynamical systems when the controlled system and the controller are described by a set of fractional order differential equations. In the existing work, a fractional order system has been signified by a higher integer order system.

Fractional calculus provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. Fractional derivatives have better flexibility as the comparison to classical integer order models, in which system dynamics not taken into account. Modern times have a wide use of field fractional derivatives and integral as well in the field of dynamic control systems, where the controlled system and the controller are defined by a set of fractional differential equations. Here the stability of fractional order system is checked at the different level and it is found that the stability region is large in the complex plane. This large stability region provides the more flexibility for system implementation in the control engineering.

Keywords: Fractional Order Controller; Fractional Order System; Fractional Order Calculus; Stability; Performance Analysis; MATLAB; Function Under Class.

INTRODUCTION

Many research and study using the notion of fractional-order may be a reliable step because a real process in the industry is generally fractional. However for many real processes fractional is very low. A typical example of fractional (non-integer) order system is, relationship of voltage-current in semi-infinite resistor with losses and capacitor (RC) line or the diffusion of heat in a semi-infinite solid, where the heat flow $q(t)$ is naturally equal to the semi-derivative of temperature $T(t)$ [1], as describe by the fractional order differential equation(FODE) which is given as

$$\frac{d^{0.5}T(t)}{dt^{0.5}} = q(t) \tag{1}$$

Clearly, an ordinary differential equation (ODE) for integer order describes the above system and it may differ significantly from the concrete situation.

The analysis of dynamic systems modeled by FODEs can be found in [2–9]. The consequence of fractional order control system is that it is a simplification case of classical control theory where it adequate for more robust control performance. Despite this fact, the integer-order control is still more welcome due to the absence of accurate solution methods for fractional order differential equations (FODEs). A plant to be controlled can also be modeled as a dynamic system describes by a fractional order differential equations (FODE). In theory, control systems can include both the fractional-order dynamic system or plant to be controlled and the fractional-order controller. In the field of control engineering, it is a common practice to consider the fractional-order controller for analysis the system. This is due to the fact that the plant model may have already been obtained as an integer-order model in a classical sense. Generally, the objective for application of fractional-order control (FOC) is to enhance the system control performance. A PID controller which is known as an industrial controller has been modifying using the notion of fractional order integer and differentiator. It has been shown that the performance of traditional PID controller improves by adding two degrees of freedom by the use of a fractional-order integer and differentiator. The main objective of this paper is to investigate the stability and performance of the fractional order control system by illustrating some design examples. Some MATLAB function files are used in this paper to simulate the fraction order dynamic system [10]. The rest of the article is organized as follows: In section 2, a brief introduction of fractional calculus and fractional order system has been presented. Section 3 presents the stability analysis of fraction order systems. Section 4 presents the analysis of three illustrative examples to check stability and controller design. Section 5 concludes this paper.

FRACTIONAL ORDER SYSTEM FUNDAMENTALS

A. The introduction to fractional calculus.

The term “fractional-order calculus” is by no means new. It is a generalization of ordinary differentiation by non-integer derivatives. The theory of fractional-order derivatives was developed mainly in the 19th century [5, 7, 11 and 12]. In the development of fractional order calculus, there appeared different definitions of fractional-order differentiation and integration. To reduce to a general form fractional calculus from integration and differentiation to the fractional order fundamental operator $\alpha D_t^{\beta} f(t)$, where α and t are the limit and $\beta \in R$ is the directive of operation. The continuous integration differential operator is [10]

$$\alpha D_t^{\beta} f(t) = \begin{cases} \frac{d^{\beta}}{dt} \cdots \beta > 0 \\ 1 \cdots \beta = 0 \\ \int_{\alpha}^{t} (d\tau)^{-\beta} \cdots \beta < 0 \end{cases} \quad (2)$$

There are various definitions for fractional integration and differentiation. Some of the definitions spread out directly as of integer-order calculus. The deep-rooted descriptions include the Cauchy integral formula, the Grunwald–Letnikov (GL) definition and Riemann–Liouville (RL) definitions are given [10] as

Definition 1: - Cauchy integral formula

$$D^{\gamma} f(t) = \frac{\Gamma(\gamma - 1)}{2\pi j} \int_{c} \frac{f(\tau)}{(\tau - t)^{\gamma+1}} d\tau \quad (3)$$

Where c is the smooth curve encircling the single value function f (t)

Definition 2: - Grunwald–Letnikov (GL) definition

$$\alpha D_t^{\beta} f(t) = \lim_{h \to 0} h^{-\beta} \sum_{j=0}^{\left[\frac{t-\alpha}{h}\right]} (-1)^j \binom{\beta}{j} f(t - jh) \quad (4)$$

Here [.] represent the integer part.

Definition 3: - Riemann-Liouville (RL) definition

$$\alpha D_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_{\alpha}^{t} \frac{f(\tau)}{(t-\tau)^{\beta-n+1}} d\tau \quad (5)$$

The following function given below is obtained by Laplace Transform of the GL and RL fractional differential-integral. The zero initial conditions and order β gives the following result

$$\ell[\alpha D_t^{\pm\beta} f(t); s] = s^{\pm\beta} F(s) \quad (6)$$

B. Fractional order system

The fractional-order system is the extension form of the traditional integer order systems. Fractional order system is gained from the fractional-order differential equations. A classic n-term linear fractional order differential equation (FODE) is assumed by

$$\alpha_n D_t^{\beta_n} y(t) + \ldots + \alpha_1 D_t^{\beta_1} y(t) + \alpha_0 D_t^{\beta_0} y(t) = 0 \quad (7)$$

Let considering the control function on which input signal $u(t)$ is applied to FODE system (7) as follows:

$$\alpha_n D_t^{\beta_n} y(t) + \ldots + \alpha_1 D_t^{\beta_1} y(t) + \alpha_0 D_t^{\beta_0} y(t) = u(t) \quad (8)$$

After Laplace transform of equation (8), we get

$$\alpha_n s_t^{\beta_n} Y(t) + \ldots + \alpha_1 s_t^{\beta_1} Y(t) + \alpha_0 s_t^{\beta_0} Y(t) = U(t) \quad (9)$$

From Eq. (9), we can obtain a fractional-order transfer function as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{\alpha_0 s^{\beta_0} + \alpha_1 s^{\beta_1} + .. + \alpha_n s^{\beta_n}} \quad (10)$$

In broad, for a dynamic system with single variable and fractional order transfer function of a system can be defined as

$$G(s) = \frac{b_0 s^{\gamma_0} + b_1 s^{\gamma_1} + \ldots + b_m s^{y_m}}{a_0 s^{\beta_0} + a_1 s^{\beta_1} + \ldots + a_n s^{\beta_n}} \quad (11)$$

Here $b_i (i = 0,1..m), a_i (i = 0,1..n)$ are constant and $\gamma_i (i = 0,1..m), \beta i (i = 0,1..n)$ are random real or rational number and without lacking generality, can be prescribed as $\gamma_m > \gamma_{m-1} > ..\gamma_0$ and $\beta_m > \beta_{m-1} > ..\beta_0$

The incommensurable fractional order system Eq. (11) can also be expressed incommensurable form by the multi-valued transfer function

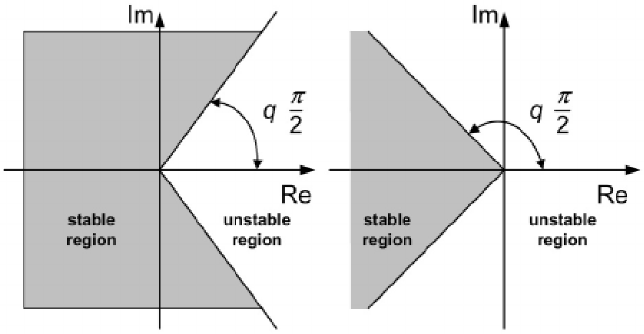
$$H(s) = \frac{b_0 s + b_1 s^{\frac{1}{v}} + \ldots + b_m s^{\frac{m}{v}}}{a_0 s + a_1 s^{\frac{1}{v}} + \ldots + a_n s^{\frac{n}{v}}}, (v > 1). \quad (12)$$

Note that every fractional order system may be represented in the form of (12) and domain of H(s) meaning is a Riemann sheets.

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 12, Number 20 (2017) pp. 10298-10304

STABILITY OF FRACTIONAL ORDER SYSTEM

Stability is one of the most frequent terms used in literature when we deal with the dynamical systems and their behaviors. In mathematical terminology, stability theory addresses the convergence solutions of differential or difference equations. A system (LTI) is said to be stable if the roots of characteristics polynomial are had negative real part. In the case of fractional order system (LTI), the stability is not same as of integer one. Important point is that, for a fractional order system, the roots may lie on the right half of complex plane (Fig.1).



(a) $0 < q < 1$ (b) $1 < q < 2$

Figure 1: Stable and unstable region of LTI fractional order system

Theorem: - According to Matignon's stability theorem the fractional order transfer function $G(s) = \frac{N(s)}{D(s)}$ is stable if and only if $\left|\arg(\sigma_i)\right| = q\frac{\pi}{2}$, where $\sigma = s^q$, (0 < q < 2) with $\forall \sigma_i \in C$, i^th^ root of $D(\sigma) = 0$.

If $s = 0$, is a single root of D(s), the system cannot be stable.

Above theorem stability region is shown in Fig. (1), Indicate the wholes s plane where $q = 0$. It shows the Routh-Hurwitz stability and $q = 1$ tends to the negative real axis for $q = 2$.

As we know that only the poles play an important role in the stability of a system. So the stability assessment is done by denominator only and numerator does not affect the stability of a FOTF. The stability of fractional order system can be analyzed in another way also. Let considering here, the characteristic equation of a general fractional order system as:

$$\alpha_0 s^{\beta_0} + \alpha_1 s^{\beta_1} + \ldots + \alpha_n s^{\beta n} = \sum_{i=0}^{n} \alpha_i s^{\beta_i} = 0 \qquad (13)$$

For $\beta_i = \frac{v_i}{v}$, we can transform the Eq.(13) into σ-plane.

$$\sum_{i=0}^{n} \alpha_i s^{\frac{v_i}{v}} = \sum_{i=o}^{n} \alpha_i \sigma^{v_i} = 0 \qquad (14)$$

Here $\sigma = s^{\frac{k}{m}}$ and m is the least common multiple of v.

For a given α_i, if the absolute phase of all roots of transform equation (14) is $\left|\phi_\sigma\right| = \left|\arg(\sigma)\right|$, we can close the following points for the stability of fractional order systems.

1. The stability condition is as $\frac{\pi}{2m} < \left|\arg(\sigma)\right| < \frac{\pi}{m}$.

2. The oscillation condition is as $\left|\arg(\sigma)\right| = \frac{\pi}{2m}$.

If any linear time invariant (LTI) fractional order system satisfy the above two points then the system is stable otherwise unstable.

FRACTIONAL ORDER CONTROLLER DESIGN

Maximum of the works in fractional order control systems are in hypothetical nature. Controller design and application in run-through is very small. In this paper, the core objective is to spread on the fractional order control (FOC) to examine the system control performance. The fractional-order $PI^{\lambda} D^{\mu}$ controller was proposed as a broad view of the PID controller with integrator of real order λ and differentiator of real order δ. The transfer function of such kind the controller in Laplace domain has form

$$C(s) = K_P + \frac{K_I}{s^{\lambda}} + K_D s^{\mu}, (\lambda, \mu > 0) \qquad (15)$$

Here K_P is the proportional gain constant K_I is the integral gain constant and K_D is the derivative gain constant. If λ=1and μ=1, we obtained a classical PID controller. If λ=0 and μ=0, we obtained a PD^{μ} and PI^{λ} controller respectively. These entire controllers are the case of $PI^{\lambda}D^{\mu}$ controller, which provides flexibility with an opportunity to adjust the dynamic property of fractional order control system. Two steps are used here to design such controllers.

Step 1: - Design of K_P

Overshoot in percentage [Pr], settling time in second [Ts] and static error in percentage [Et] belongs to Proportional gain K_P. In general, K_P can be obtained by

$$K_P \geq \left(\frac{100}{E_t}\right) \qquad (16)$$

Here Proportional gain K_P is selected for minimum static error.

Step 2: - Design of K_D, μ, K_I and λ

To determine these values for Fractional-Order controller design, the following synthesis scheme is used here.

Let the controller transfer function is C(s), Plant transfer function is G(s) and a unity feedback is applied to the system. Phase margin of controlled system [13 and 14] is

$$\Phi_m = \arg\left[C(j\omega_g)G(j\omega_g)\right] + \pi \tag{17}$$

Where $j\omega_g$ is the crossover frequency. Phase margin is an independent or constant phase. This can be accomplished by controller of the form

$$C(s) = k_1 \frac{k_2 s + 1}{s^v}, k_1 = \frac{1}{K_{plant}}, k_2 = \tau \tag{18}$$

Here K_{plant} is the gain of plant and τ is the time constant for the plant.

Now from the equation (17) and Eq. (18)

$$\begin{cases} \phi_m = \arg\left[C(j\omega_g)G(j\omega_g)\right] + \pi \\ \quad = \arg\left[\dfrac{k_1 k_{plant}}{j\omega^{(1+v)}}\right] + \pi \\ \quad = \arg\left[(j\omega)^{-(1+v)}\right] + \pi \\ \quad = \pi - (1+v)\dfrac{\pi}{2} \end{cases} \tag{19}$$

Here for a given plant, we fix the gain margin. Put the gain value in Eq. (19) one can find out the value of v. the other desired values k_1 and k_2 are obtained from Eq. (18).

Now using these constant in Eq. (18), we can obtain a fractional $I^{\lambda} D^{\mu}$ controller, which is a particular case of $PI^{\lambda}D^{\mu}$ controller has the form

$$C(s) = k_1 k_2 s^{(1-v)} + k_1 s^{-v}; K_D = k_1 k_2 and K_I = k_1 \tag{20}$$

If the value of K_P is given then the full transfer function of fractional order controller is

$$C(s) = K_P + K_D s^{(1-v)} + K_I s^{-v}$$
(21)

If do a comparison with Eq. (15), we can say

$\mu = (1-v) and \lambda = v$.

EXAMPLES

To check the effectiveness of stability concept for fractional order system, we have demonstrated three illustrative examples here.

Example1

The fractional-order transfer function [15] is given by

$$G_1(s) = \frac{0.96s^{1.59} + 1.2s^{1.2} + 2.4s^{0.39} + 3}{3.3s^{2.27} + 2.4s^{1.77} + 0.96s^{1.59} + 1.2s^{1.2} + 5.7s^{0.97} + 1.2s^{0.47} + 2.4s^{0.39} + 3} \tag{22}$$

The characteristics equation for the above Eq. (22) is given as

$$\begin{aligned} D_1(s) &= 3.3s^{2.27} + 2.4s^{1.77} + 0.96s^{1.59} + 1.2s^{1.2} \\ &+ 5.7s^{0.97} + 1.2s^{0.47} + 2.4s^{0.39} + 3 = 0 \end{aligned} \tag{23}$$

Equation (23) can be written as

$$\begin{aligned} D_1(s) &= 3.3s^{\frac{227}{100}} + 2.4s^{\frac{177}{100}} + 0.96s^{\frac{159}{100}} + 1.2s^{\frac{120}{100}} \\ &+ 5.7s^{\frac{97}{100}} + 1.2s^{\frac{47}{100}} + 2.4s^{\frac{39}{100}} + 3 = 0 \end{aligned} \tag{24}$$

The transformed equation for Eq. (24) in σ-plane is as

$$\begin{aligned} D_1(\sigma) &= 3.3\sigma^{227} + 2.4\sigma^{177} + 0.96\sigma^{159} + 1.2\sigma^{12} \\ &+ 5.7\sigma^{97} + 1.2\sigma^{47} + 2.4\sigma^{39} + 3 = 0 \end{aligned} \tag{25}$$

For solving the Eq. (25), MATLAB function *solve ( )* is used here the obtained roots are 227. We are not showing here all roots. We are discussing the root which satisfy the stability condition, the argument of it is $\sigma = |\arg(\sigma)| = 0.0243$, satisfying the stability condition $-\frac{\pi}{m} < \arg(\sigma) < \frac{\pi}{m} \Rightarrow$ -0.03 < 0.0243 < 0.03 and $|\arg(\sigma)| > \frac{\pi}{2m} \Rightarrow$ 0.0243 > 0.0157. According to the theorem, the value of m is 100 here. Hence it is shown that the fractional order system $G_1(s)$ is stable. The function isstable checked the denominator of $G_1(s)$, $3.3s^{2.27}+2.4s^{1.77}+0.96s^{1.59}+1.2s^{1.2}+5.7s^{0.97}+1.2s^{0.47}+2.4s^{0.39}+3$ and it is found that K=1, indicate the system is stable Fig. (2), with q=0.01. Here we can see that $G_1(s)$ has more stability region. Now it is easy to summarize here that the system is stable even if its pole lies on the right side of the imaginary axis. Fig. (4), Shows that the step response of the system is converging which indicate, that the system is stable.

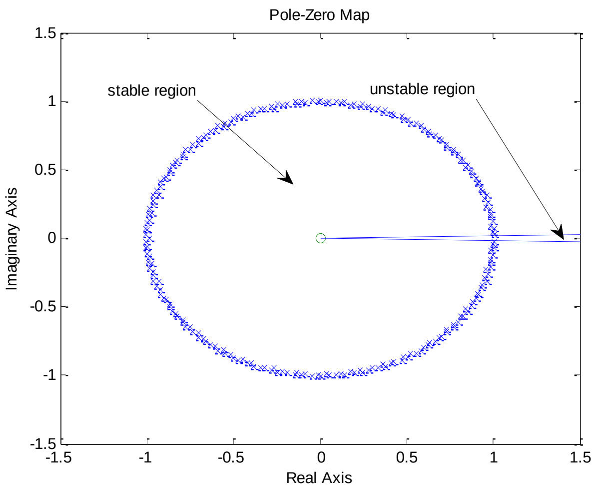


Figure 2: Poles position in complex plane for $G_1(s)$

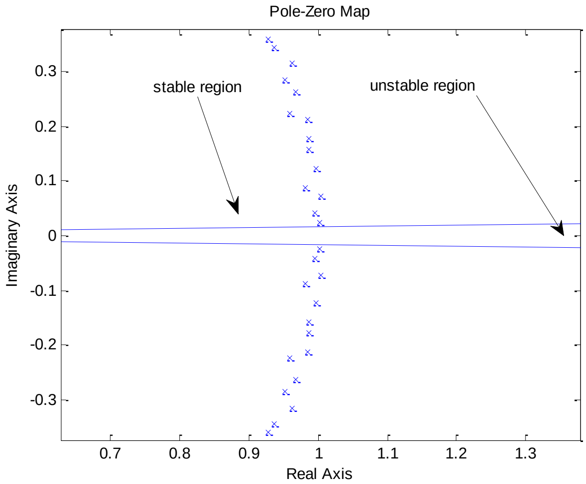


Figure 3: Poles position in complex plane $G_1(s)$ (zoomed graph)

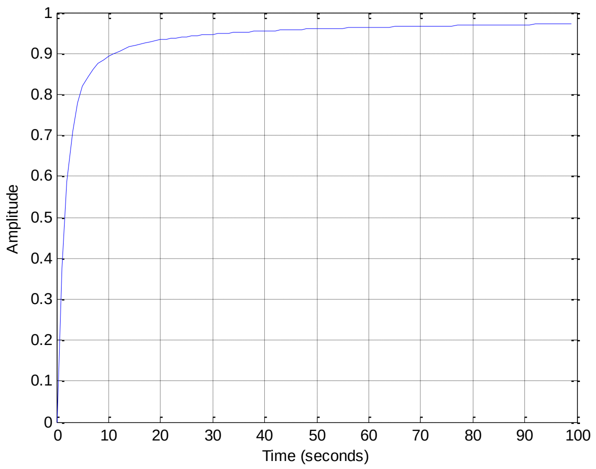


Figure 4: Step response of fractional order system, $G_1(s)$

Example2

Consider the fractional order system $G_2(s)=\dfrac{1}{-1.5s^{0.5}+1}$ the characteristic equation for the above transfer function is written as

$$D_2(s)=-1.5s^{0.5}+1=0 \tag{26}$$

The transformed equation for Eq. (26) in σ-plane for $\sigma=s^{\frac{1}{10}}$ and m=10 is given by

$$D_2(\sigma)=-1.5\sigma^5+1=0 \tag{27}$$

The roots of equation are

$$\sigma_1=0.9221$$

$$\sigma_{2,3}=-0.7460\pm0.5420i; \left|\arg(\sigma_{2,3})\right|=0.628$$

$$\sigma_{4,5}=0.2849\pm0.8769i; \left|\arg(\sigma_{3,4})\right|=1.25 \tag{28}$$

Thus no roots of Eq. (27) satisfy the stability condition $-\dfrac{\pi}{m}<\arg(\sigma)<\dfrac{\pi}{m}$ and $\left|\arg(\sigma)\right|>\dfrac{\pi}{2m}$. It indicates that the above system $G_2(s)$ is unstable. The function *isstable* checked the denominator of $G_2(s)$, $-1.5s^{0.5}+1$ and it is found that K=0, indicate the system is unstable. Step response of $G_2(s)$ is diverging Fig. (5). It shows the instability in the system.

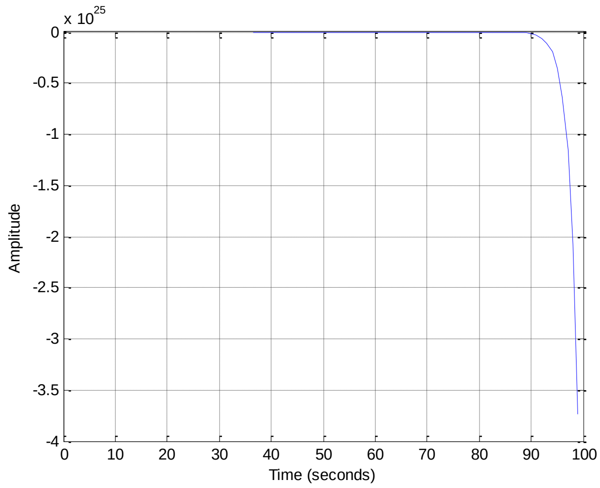


Figure 5: Step response of fractional order system, $G_2(s)$

Example 3

Consider the transfer function model of a DC motor [16], given by

$$G_{DCM}(s)=\frac{0.08}{s(0.05s+1)} \tag{29}$$

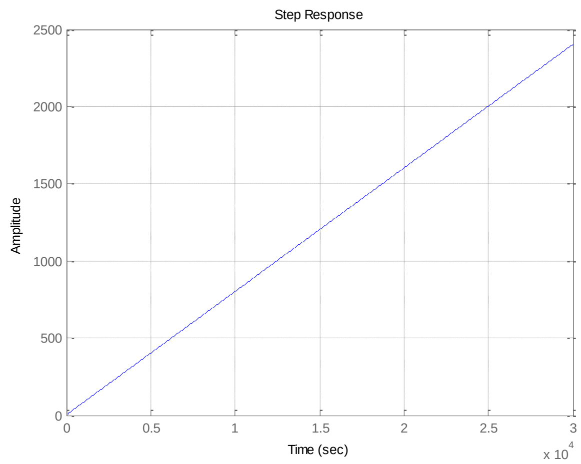


Figure 6: Step response of a DC motor without controller

The step response of DC motor Fig.(6), without a controller, shows that the system is unstable so to make system stable a fractional order model controller design is proposed here. The model is given in Fig.7, have controller transfer function C(s) and motor transfer function $G_{DCM}(s)$. The applied voltage Va controls the angular velocity ω of the system.

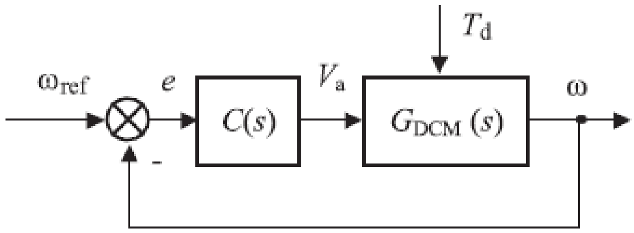


Figure 7: Feedback control loop

We are using here the technique proposed in section 4 for fraction order controller design. According to this

Step 1: - To design the K_P

For minimum static error the value of proportional gain $K_P=10$, from Eq. (15)

Step 2: - Design of KD, μ, KI and λ

The value of time constant τ = 0.05 and gain of DC motor $K_{DCM} = 0.08$ respectively Eq. (15).

If we fix to gain margin $\phi_m >= 60^0$ for the given control system. Then we find out the value of v = 0.3 by Eq. (19). The other desired value $k_1 = 12.5$ and $k_2 = 0.05$ obtained from Eq. (18). Now putting these values in Eq. (20), we got

$$C(s) = 0.625s^{0.7} + \frac{12.5}{s^{0.3}} \tag{30}$$

Now adding the value of $K_P = 10$ from step 1into Eq. (30), we got final transfer function of fractional order controller as

$$C(s) = 10 + 0.625s^{0.7} + \frac{12.5}{s^{0.3}} \tag{31}$$

The open loop control system for controller and plant (DC motor) is

$$G_o(s) = \frac{0.05s + 0.8s^{0.3} + 1}{0.05s^{2.3} + s^{1.3}} \tag{32}$$

The close-loop transfer function of given control system with unity feedback is obtained as

$$G_{cl}(s) = \frac{C(s)G_{DCM}(s)}{1 + C(s)G_{DCM}(s)} \quad \text{Or}$$

$$G_{cl}(s) = \frac{0.05s + 0.8s^{0.3} + 1}{0.05s^{2.3} + s^{1.3} + 0.05s + 0.8s^{0.3} + 1} \tag{33}$$

Here Fig. (8) Shows that system controlled by fractional order controller has more stability region and Fig. (9) Indicate that the complete designed system is stable even plant is unstable.

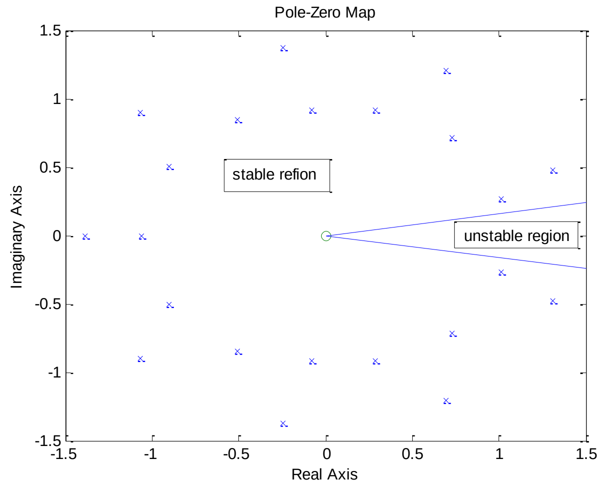


Figure 8: Poles position in complex plane for Gcl(s)

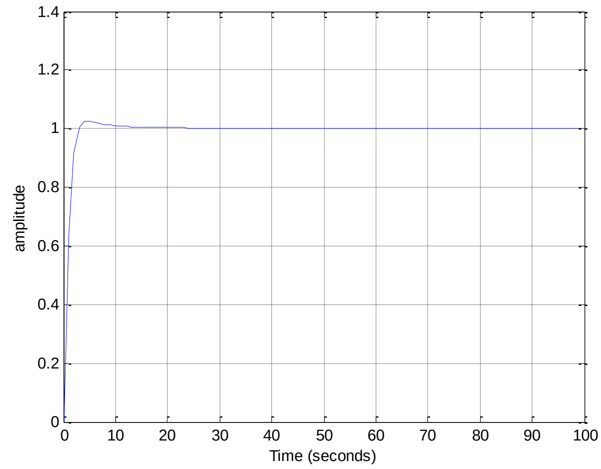


Figure 9: Step response of fractional order system, Gcl(s)

CONCLUSIONS

This paper investigated, three different fractional order control system for the stability and performance analysis. All basic ideas of fractional calculus, the stability of fractional order system and MATLAB function are presented here. The main purpose of the paper is to draw attention to fractional order system stability and analysis in a non-conventional way. Here an integer order unstable plant is controlled by fractional order controller. It concludes here that the fractional order system has a large region for stability which improves the performance of the system. We believe that the technique used in this paper is useful for stability analysis in the industry where the obtained model is fractional (stable or unstable) in nature.

REFERENCES

[1] Podlubny I., Dorcak L. Misanek J., “Application of fractional-order derivatives to calculation of heat load intensity change in blast furnace walls,” Transactions of the Technical University of Kosice, 1995, 5(5), pp. 137–144

[2] Axtell M., Bise E. M., “Fractional calculus applications in control systems,” In Proceeding of the IEEE 1990 National Aerospace and Electronics Conference, New York, 1990, pp. 563–566.

[3] Dorcak L., “Numerical models for simulation the fractional-order control systems,” UEF SAV, The Academy of Sciences Institute of Experimental Physics, Kosice, Slovak Republic, 1994, pp. 62–68

[4] Matignon D., “Stability result on fractional differential equations with applications to control processing,” In IMACS-SMC Proceedings, Lille, France, 1996, pp. 963–968

[5] Oldham K. B. and Spanier J., “The Fractional Calculus,” New York, Academic Press, 1974

[6] Podlubny I., “The Laplace transform method for linear differential equations of the fractional order,” In Proceedings of the 9th International BERG Conference, Kosice, Slovak Republic, 1997, pp. 119–119

[7] Podlubny I., “Fractional Differential Equations,” San Diego, Academic Press, 1999

[8] Woon S.C., “Analytic continuation of operators — operators acting complex s-times. Applications: from number theory and group theory to quantum field and string theories,” Reviews in Mathematical Physics, 1999, 11(4): pp. 463–501

[9] Zavada P., “Operator of fractional derivative in the complex plane,” Communications in Mathematical Physics, 1998, 192(2): pp. 261–285

[10] Y. Q. Chen, Ivo Petras and Dingyu Xue, “Fractional Order Control,” A Tutorial, American Control conference, June 10-12, 2009

[11] Miller K. S., Ross B., “An Introduction to the Fractional Calculus and Fractional Differential Equations,” New York, Wiley, 1993

[12] Samko S. G., Kilbas A. A. and Marichev O. I., “Fractional Integrals and Derivatives and Some of Their Applications,” Minsk: Nauka i Technika, 1987

[13] Bidan P., “commande diffusive dune machine electric introduction,” ESAIM proceeding, vol. 5, France, 1998, pp. 55-68

[14] VINAGRE, B. M. Podlubny, I. Dorcak, L. Feliu, V., “On Fractional PID Controllers: A Frequency Domain Approach,” Proceeding of the IFAC Workshop on Digital Control PID’00, Terrassa, Spain, 2000, pp. 53-55.

[15] D. Xue, Y. Q. Chen and D. P. Atherton, “Linear feedback control: Analysis and design with MATLAB,” Siam, 2007

[16] Dorf R.C., Bishop R.H., “Modern Control System,” Addison-Wesley, New York, 1990