International Journal of Applied Engineering Research ISSN 0973-4562 Volume 12, Number 20 (2017) pp. 10077-10083

Some Homological Groups Related To Simplicial Complexes

Majid Mohammed Abed[1] and Faisal Ghazi AL-Sharqi[2]

[1]*Department of Mathematics, University of Anbar-Iraq.*

[1]*Orcid: 0000-0002-1825-0097*
[2]*Department of Mathematics, University of Anbar-Iraq.*

Abstract

The aim of the paper is to study some of the homological groups in general and related these groups with simplicial complexes. Characterization these groups revealed the successful method to study the simplicial complex which has the following two properties: (a) each q-simplex determines (q +1) faces of dimension q-1, (b) the faces of a simplex determine the simplex and a semi-simplicial complex K is a collection of elements {f} called simplexes together with two functions. The main examples of homological groups are r-chain group, r-cycle group and r-boundary group. When we calculating the Euler characteristic of surface, we need to building a multi-surface equivalent to the original surface, therefore in this paper we achieved that the homological groups are a type of improvement for the Euler characteristic. If there is no simplex of order two (2-simplexs) in K, then $B_1(K)$ and $H_1(K)$ are equal to $Z_1(K)$. Also if K is a simplex complex**,** then r-chain ($C_r(K)$ is a group. We obtained that if three points and three lines such that is triangulation of the rings and there is no simplex of order two (2-simplexs) in K, in this case the boundary homological group equal zero and $H_1(K)=Z_1(K)$.

Keywords: Simplex complex, Homology group**,** Chain group, Cyclic group, Bounded group.

Mathematics Subject Classification: MSC (2010): Primary: 20E15; Secondary: 20F19.

INTRODUCTION AND PRELIMINARIES

Some researchers considered a simplicial complex is a pair K=(V, S); where V is a finite set whose elements are called the vertices of K and S is a set of non-empty subsets of V if p in V, then {p} in S and σ in S, τ subset of σ then τ in S. Simplicial sets are, essentially, generalizations of the geometric simplicial complexes of elementary algebraic topology (in some cases quite extreme generalizations) [1]. A simplicial complex has the following two properties: (a) each q-simplex determines q+1 faces of dimension q-1, (b) the faces of a simplex determine the simplex and a semi-simplicial complex K is a collection of elements {f} called simplexes together with two functions. The chief example of such a "semi-simplicial" complex is the singular complex S(X) of a topological space X [2]. There is a strong relationship between finite spaces and finite simplicial complexes, which was discovered by McCord and explicitly, given a finite simplicial complex K, one can associate to K a finite T_0-space X(K) which corresponds to the poset of simplices of K ordered by inclusion [3]. Let X be a simplicial set, $x_{n+1} > X_{n+1}$ is called degenerate simplex if there is $n \geq i \geq 0$, and $x_n > X_n$ such that $x_{n+1}= s^n{}_i(x_n)$ [4]. Now we have aim to define the homology groups for more general spaces than the Δ-complexes. Let X be a complex space, a singular n-simplex in X is a continuous map $\sigma: \Delta_n \to X$. We form a CX such that $C_nX= \bigoplus Z\ |\sigma|$ (a n-chain) singular simplex. An element is a formal linear combination: $c=\sum u_\sigma \sigma$ where $u_\sigma \in Z$, finitely many of them are zero [5]. The idea of a Δ-complex is to generalize constructions like these to any number of dimensions. The *n* dimensional analog of the triangle is the *n* simplex, this is the smallest convex set in a Euclidean space R^m containing *n*+1 points $v_0,\ldots,v_n$ that do not lie in a hyperplane of dimension less than *n*, where by a hyperplane we mean the set of solutions of a system of linear equations. In [6], Sieradski studied algebraic topology for two dimensional complexes. Definition of a CW-complex, CW-complexes form a nice collection of topological spaces which include most spaces of interest in geometric and algebraic topology [7]. In the 1930`s one view of gown topology ought to develop was as combinatorial. The homeomorphism classification of finit simplicial complexes had been attacked by introducing elementary equivalent, if one could get from K to L a finite sequence of such moves [8]. A simplicial group G is a simplicial object in the category of groups. We will denote the category of simplicial group by Cim-gp and he said if G is a group, then it has a free sipmlicial resolution F [9]. In order to focus on relating the combinatorial aspect of the theory to their geometric-topology origins and some other very interesting and important examples of simplicial maps, which will be critical for our development of simplicial sets, are the simplicial maps that collapse simplicials [10]. In [11], Johnathan introduced three important definitions:

Definition 1.1. The faces of the affine q-simplex determined by $\{P_0,\ldots,P_q\}$ are the affine simplices determined by subsets of this set.

Definition 1.2. A polyhedron in RN is a subset P which is the union of finitely many affine simplices (of varying dimensions). (We may assume that any two simplices meet along a common face, possibly empty).

Definition 1.3. A triangulation of a space X is a homeomorphism h:K→X from some polyhedron K (in some RN). In other words, a triangulation is a representation of X as a finite union of closed subsets, each homeomorphic to a simplex.

A polyhedron is a special case of a cell complex in which all the defining maps are one-to-one. Conversely, any such cell complex admits a triangulation as a polyhedron, but in general more simplices are needed than cells. In [12], Andre Joyal Myles denoted by Cat the category of small categories and functors. There is a functor Δ→Cat which sends [n] into [n] regarded as a category via its natural ordering. Again, by the universal property of Δ−Set this functor can be extended to a functor R: S →Cat so as to give a commutative triangle. In [13], John denoted L_q the 3-dimensional lens manifold of type (7, q), suitably triangulated and let σ^n denote an n-simplex and he said the finite simplcial complex X, is obtained from the product $L_n \times \sigma^n$ by adjoining a cone over the boundary $L_n \times \partial\sigma^n$. The dimension of X_n is n+3. Also John found, the following result:

Theorem 1.4. For $n+3 \geqq 6$ the complex X_1 is homeomorphisc to X_2.

The author found [14] very useful when trying to understand the idea of simplicial sets and [15] illustrating the derivation of some relations about the subject. In [16], Jean introduced a new data structure, called simplex tree, to represent abstract simplicial complexes of any dimension. All faces of the simplicial complex are explicitly stored which whose nodes are in bijection with the faces of the complex. This data structure allows to efficiently implement a large range of basic operations on simplicial complexes. We provide theoretical complexity analysis as well as detailed experimental results. We more specifically study Rips and witness complexes. A semi-simplicial group Γ is a semi-simplicial complex together with a simplicial map ΓxΓ→Γ which, restricted to each of the sets Γ_q is a group composition [17]. The fundamental groupoid of a simplicial set, as a functor, is seen to be adjoint to the classifying space construction on small categories. The Dold-Kan correspondence is established between simplicial Abelian groups and chain complexes [18]. For simplex, i.e. monomorphemic, nouns in Dutch, he studied the effect of the frequency of the monomorphemic noun itself as well as the effect of the frequencies of morphologically related forms on the processing of these monomorphemic nouns [19]. In [20], Jakob introduced the set-theoretic concepts and notation related to abstract simplicial complexes. Throughout the section, all sets and families are finite. Whenever appropriate, we extend our definitions to arbitrary families of sets rather than restricting to the special case of simplicial complexes. In [21], Edward said there are two basic properties of simplicial groups, both due to More. First, each simplicial group satisfies the extension condition. Second, its homotopy groups $\beta_*(G)$ can be obtained as the homology, i.e., Ker(∂/image ∂) of a certain chain complex (NG, ∂). and he found a simplicial group considered as a simplicial set satisfring the extension condition will be a minimal complex if and only if the chain complex (NG, ∂) is minimal, i.e., each ∂_n is the null homomorphism. In this paper we focus on homological groups and related them with simplicial complexes.

SIMPLICIAL COMPLEXES

In order to study the relations of simplicial complexes inside algebraic topology we need to give a definition of simplexes and simplicial complexes and we must do connect for this simplexes with another algebraic structure for instance, abelian groups instead of a number as Euler characteristic**.** As with much else in the theory of simplicial sets, the extension condition comes from a fairly straightforward idea that is often completely obfuscated in the formal definition. To explain the idea, we first need the following definitions.

Definition 2.1. As a simplicial complex, the kth horn $|\Lambda_k^n|$ the n-simplex $|\Delta_k^n|$ is the sub-complex of $|\Delta_k^n|$ obtained by removing the interior of $|\Delta_k^n|$ and the interior of the face $d_k\Delta^n$.

Definition 2.2. [3]. "A simplexes are polyhedron 0-simplex $\langle p_0 \rangle$ is vertices or point, 1-simplex $\langle p_0\, p_1 \rangle$ is a side or line, 2-simplex $\langle p_0\, p_1\, p_2 \rangle$ is a triangle and inside it is full and 3-simplex $\langle p_0\, p_1\, p_2\, p_3 \rangle$ is a solid pyramid and by the same way we can obtain r-simplex $< p_0\, p_1\, p_2\, p_3\, p_4 \ldots p_r \rangle$. We call a simplex has dimension r".

Definition 2.3. [3]. "The n-simplex, Δ^n is the simplest geometric figure determined by a collection of n+1 points in Euclidean space R^n. Geometrically, it can be thought of as the complete graph on (n+1) vertices, which is solid in n dimensions".

Remark 2.4. The vertices p_i is independent geometric, in other word there is no (1-r) overhead surface and contain all the vertices (r+1).

Definition 2.5. Let $p_0\, p_1\, p_2\, p_3\, p_4 \ldots p_r$ be an independent geometric vertices in R^m such that $m \geq r$. Then r-simplex $\sigma_r = < p_0\, p_1\, p_2\, p_3\, p_4 \ldots p_r \rangle$ is:

$\sigma_r = \{x \in R^m : x = \sum c_i\, p_i,\ i=0,\ldots,r \ni c_i \geq 0 \text{ and } \sum c_i = 1\}$.

Note that σ_r is bounded and closed. Hence is compact in R^m. Let q be integer number; $0 \leq q \leq r$. If q + r is vertices $p_{i0}, p_{i1}, \ldots, p_{iq}$ of $p_0, p_1, \ldots, p_r$, then the q+1 vertices defined by q-simplex; $\sigma_q = \langle p_{i0}, p_{i1}, \ldots, p_{iq} \rangle$ which called the face-q of the simplex or face of σ_r. If $\sigma_q \leq \sigma_r$.

Example 2.6. The pyramid figure has four faces**.**

Note that p_0 is 0-simplex is 0-face and 2-face is □ p_1 p_2 p_3 □ of 3-simplex □p_0 p_1 p_2 p_3□. Also there is 3-face one and six and 1-face and four 0-facein 3-simplex. We can verification by $\binom{r+1}{q+1}$ q-face of r-simplex. The 0-simplex has no proper face.

Definition 2.7. Let K be a finite set of simplexes in R^m. If these simplexes are well ordered, then K is called simplicial complex.

Note that we can rewrite above definition by another way:

A simplicial complex K is a finite set of simplices satisfying the following conditions:

1. For all simplices A ∈ K with α a face of A, we have α ∈K.
2. A; B ∈ K ⟹ A; B are properly situated.

Remarks 2.8. We mean well ordered:

a) Any random face for simplex of K belong to the set K, it means if σ ∈ K, there exists σ/ ≤ σ, then σ/ ∈ K.

b) If σ and σ/ are two simplexes of K, then σ∩σ/=Ø or common face inside σ and σ/. i.e. if σ and σ/ **∈ K** ⟹ σ∩σ/ =Ø or σ∩σ/ ≤σ and σ∩σ/ ≤ σ/ .

c) The dimension of a complex is the maximum dimension of the simplices contained in k.

Let q be integer number; 0≤ q ≤r. If q+r is vertece p_{i0}, p_{i1}, …p_{ir}, then q+1 vertices defined by q-simplex; $\sigma_q = <$ pi0, pi1, …$pi_q >$ which called the q-face of σ_r and we write $\sigma_q < \sigma_r$ if σ_q is a surface or face σ_r. If $\sigma_q \neq \sigma_r$, then we say that σ_r proper face of σ_q and denoted by $\sigma_q < \sigma_r$.

Remark 2.9. The dimension of simplex k is the large dimension for simplexes in k.

Example 2.2.10. Let σ_r be the set of r-simplexes. Let k = { σ/ │ σ/ ≤ σ_r } be the set of faces of σ_r and K is simplex complex has a dimension equal r. For example the following pyramid:

Σ_3 = □p_0 p_1 p_2 p_3□. Then K={p_0, p_1, p_2, p_3, □p_0 p_1□ □p_0 p_2□ □p_0 p_3□ □p_1 p_2□ □p_1 p_3□ □ p_2 p_3□ □p_0 p_1 p_2□ □p_0 p_1 p_3□ □p_0 p_2 p_3□ □p_1 p_2 p_3□ □p_0 p_1 p_2 p_3□}. The simplex complex K is a set which has elements are simplexes.

Remarke 2.11. **(**1) If every simplex is subset of $\mathbb{R}^m$ (m □ dimK), then the union of an simplexes is a subset of $\mathbb{R}^m$ and this union is called Polyhedron or |K| of simplex complex.

(2) The dimension |K| as a subset of $\mathbb{R}^m$ is the same dimension of K So dim |K| = dimK.

Definition 2.12. Let X be a topological space. If there exists simplex complex K and equivalent topology f: |K|→X , then X is portability triangulation and the pair (K, f) is called (triangulation) X. Let X be a topological space .Then the triangulation not unique. We can triangulation the space X by many times.

Now we add a new direction to r-simplex for r ≥1 and we use ( ) instead of □ □ for all not oriented simplex. The symbole σ_r represent a simplex in both cases □ □ or ( ).

Example 2.13. The oriented 1-simplex σ_0 (p_0 p_1) is a line has direction (oriented) such that transition from $p_0 \rightarrow p_1$ and the (p_0 p_1) difference (p_1 p_0) because (p_0 p_1) = - (p_1 p_0) such that is opposite direction (i.e. the inverse in group means if it moved from p_0 to p_1 and later return from p_1 to p_0 it means don’t go to any place :

(p_0 p_1) + (p_1 p_0) = (p_0 p_1) – (p_0 p_1) = 0.

Also the oriental 2-simplex σ_2 is (p_0 p_1 p_2).

Remark 2.14. Note that the oriented p_0 p_1 p_2 is equal p_1 p_2 p_0, p_2 p_0 p_1 and opposite p_0 p_2 p_1, p_2 p_1 p_0, p_1 p_0 p_2. Therefore

$$\begin{aligned}(p_0\, p_1\, p_2) &= (p_1\, p_2\, p_0)\\ &= (p_2\, p_0\, p_1)\\ &= (p_0\, p_2\, p_1)\\ &= -(p_2\, p_1\, p_0)\\ &= -(p_1\, p_0\, p_2).\end{aligned}$$

Definition 2.15. Let $\mathbb{p}$ be a permutation of 0,1,2 $\mathbb{p} = \begin{pmatrix} 0 & 1 & 2 \\ i & j & k \end{pmatrix}$. The relationship can be expressed as a symbol: (p_i p_j p_k) = sgn(p) (p_0 p_1 p_2) such that sgn(p) = (-1)(+1) by the permutation P may be negative or positive .

Example 2.16. The oriented 3-simplex σ_3 = (p_0 p_1 p_2 p_3) is ordered series of 4 vertices to triple oriented, take $\mathbb{p} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ i & j & k & l \end{pmatrix}$ be a permutation. We define p_i p_j p_k p_l = sgn(p) (p_0 p_1 p_2 p_3).

Now it is very easy build r-simplex oriented for r≥ 1. Take r +1 of independent geometric points p_1 p_0 p_2,..., p_r ∈ $\mathbb{R}^m$ and Let { p_{i0}, p_{i1},…p_{ir}} be a set of series points obtained it by permutation of points p_0 p_2,..., p_r. Then we say that { p_{i0}, p_{i1},…p_{ir}} and {p_0 p_2,..., p_r} are equivalent if $\mathbb{p} = \begin{pmatrix} 0 & 1 \ldots r \\ i0 & i1 \ldots ir \end{pmatrix}$ is even permutation. Clear that it is equivalent relation and the equivalent classes are oriented r-simplex and there are two cases of equivalent classes:

The first include even permutation of p_0 $p_{2,\ldots}$, p_r and the second odd permutation {p_0 $p_{2,\ldots}$, p_r} and denoted by $\sigma_r = (p_0\, p_1\, p_2\, p_3\, p_4 \ldots p_r)$ but the odd permutation is $-\sigma_r = (p_0\, p_1\, p_2\, p_3\, p_4 \ldots p_r)$. (i.e. $p_{i0}, p_{i1}, \ldots p_{ir} = \mathrm{sgn}(p)\, (p_0\, p_1\, p_2\, p_3\, p_4 \ldots p_r)$). If r = 0, then we define the oriented 0-simplex is only the point $\sigma_0 = p_0$. Let K= { σ_x } be simplex complex with n dimension . Suppose that $\sigma_x \in K$ as oriented simplex and we denoted by σ_x.

GROUPS RELATED TO SIMPLICIAL COMPLEXES

Theorem 3.1. (Composite Mapping Theorem). The composite mapping $(\rho_r \circ \rho_{r+1})$: $C_{r+1}(K) \rightarrow C_{r-1}(K)$ is zero mapping $(\rho_{r+1}\, c) = 0\ \forall\ c \in C_{r+1}(K)$.

Proof. We must prove that $\rho_r \circ \rho_{r+1}$: $C_{r+1}(K) \rightarrow C_{r-1}(K) \cong 0$ (i.e. $\rho_r(\rho_{r+1}\, c) = 0$). Since ρ_r is a linear operator on $C_r(k)$, then we will prove that the composites $\rho_r\ \rho_{r+1} = 0$ of the elements which are generates from $C_{r+1}(K)$. If r = 0, then Hence $\rho_2 \circ \rho_1 = 0$ because ρ_r is zero function or zero operator. If r□0, we need to take $\sigma = (p_0 p_1 \ldots p_r p_{r+1}) \in C_{r+1}(K)$. To find

$$\rho_r(\rho_{r+1}\sigma) = \rho_r \sum_{1=0}^{r+1} (-1)^i (p_0 \ldots \hat{p}_i \ldots p_{r+1})$$

$$= \sum_{i=0}^{r+1} (-1)^i \rho_r (p_0 \ldots \hat{p}_i \ldots p_{r+1})$$

$$= \sum_{j=0}^{i-1} (-1)^j (p_0 \ldots \hat{p}_j \ldots \hat{p}_i \ldots p_{r+1}) + \sum_{j=i+1}^{r+1} (-1)^{j-1} (p_0 \ldots \hat{p}_j \ldots \hat{p}_i \ldots p_{r+1}))$$

$$= \sum_{j<i} (-1)^j + 1 (p_0 \ldots \hat{p}_j \ldots \hat{p}_i \ldots p_{r+1}) \sum_{j<i} (-1)^{j+1} (p_0 \ldots \hat{p}_i \ldots \hat{p}_j \ldots p_{r+1}) = 0.$$

Thus completing the proof.

Theorem 3.2. If X is a Kan complex, then being in the same path component is an equivalence relation.

Proof.

Reexivity. This one is easy: for any vertex [a], s_0[a] is a path from a to a.

Transitivity. Consider Δ^2 = [0; 1; 2]. If p_1 is a path from a to b and p_2 is a path from b to c, then let f : $\Lambda^2{}_1 \rightarrow$ X take [0; 1] to p_1 and [1; 2] to p_2. The Kan condition lets us extend f to $\bar{f}$: $\Delta^2 \rightarrow$ X, and $\bar{f}$ [0; 2] gives us a path from a to c.

Symmetry. This is only slightly more tricky than the transitivity condition. Let p be a path in X from a to b. We need a path the other way. Think of p as the [0, 1] side of Δ^2. Let the [0, 2] side of Δ^2 represent s_0[a], which must exist since X is a simplicial set. Notice that $d_0 s_0[a] = d_1 s_0[a] = [a]$. At this point, we can label the three vertices [0, 1, 2] of Δ^2 as [a; b; a], and we have a simplicial map on $\Delta^2{}_0$ taking [0, 1] to p and [0, 2] to s_0[a]. The Kan condition tells us that this map can be extended to all of Δ^2 and [1, 2] gets taken to a path from b to a.

Definition 3.3. The r-chain group $C_r(K)$ of simplex complex K is abelean free group generated by r-simplexes oriented of K.

Remark 3.4. If dim(K) < r, then $(C_r(K)$ defined that equal zero. Every element of $C_r(K)$ is call r-chain.

Remark 3.5. Let I_r be a set of r-simplexes of K (σ_r, i) such that $1 \leq i \leq I_r$. Then any element $c \in (C_r(K)$ can expressed by the following: $C = \sum c_i\, \sigma_{r,i}$; $i = 1, \ldots I_r$ and $c_i \in Z$. The integer number c_i is called coefficients of C .

Theorem 3.6. Let K be a simplex complex**.** Then r-chain $(C_r(K)$ is a group.

Proof. We must satisfy all condition of group. Let $C = \sum c_i \sigma_{r,i}$; $i = 1, \ldots I_r$ and $c_i \in Z$ and $\bar{C} = \sum \bar{c}_i\, \sigma_{r,i}$, $i = 1, \ldots, I_r$ and $\bar{c}_i \in Z$. So C $+ \bar{C} = \sum_i (c_i + \bar{c}_i)\, \sigma_{r,i}$. Hence C + $\bar{C}$ in r-chain. Also it is very clear r-chain is assocuative.

The identity element of r-chain is 0 because $0 = \sum 0.\ \sigma_{ri}$ and The inverse element of C is $-$ C. (i.e. $-C = \sum_i (-c_i)\, \sigma_{r,i}$ and it's enough to say r-chain is group.

Remark 3.7. The opposite direction of r-simplex is $-\sigma_r$ and this is consistent with $(-1)\, \sigma_r \in (C_r(K)$. Hence $(C_r(K)$ is abelian free group of order I_r such that $(C_r(K)$ = Z + Z + Z + …+ Z.

Now we begin to define another homological groups namely, cyclic group and boundary group. But before that we need to define the boundary operators. Therefore, we denote the boundaries of any r-simplex σ_r by $\sigma_r\, \rho_r$ such that ρ_r is operator apply on σ_r to obtain his boundaries .

Examples 3.8. **(**1) 0-simplex (point or vertices) has no boundaries (i.e. $\rho_0 p_0 = 0$).

(2) 1-simplex $(p_0\, p_1)$ is $\rho_1(\sigma_1) = \rho_1(p_0\, p_1) = p_1 - p_0$.

Remark 3.9. (1) In figure (a) 1-simplex $(p_0\, p_1)$ has two parts $(p_0\, p_1)$ and $(p_1\, p_2)$ and the boundary of $(p_0\, p_2)$ are the points $\{p_0\} \cup \{p_2\}$, therefore $(p_0\, p_2) = (p_0\, p_1) + (p_1\, p_2)$.

(2) If $\rho_1(p_0\, p_2)$ is addition two points $(p_0 + p_1)$, then

$$\rho_1(p_0\, p_2) = \rho_1(p_0\, p_1) + \rho_1(p_1\, p_2)$$

$$= \rho_1(p_0\, p_1) + \rho_1(p_1\, p_2)$$

$$= p_0 + p_1 + p_1 p_2$$

$$= 2p_1 + p_2$$

and this is undesirable because the point p1 imaginary , but if we take

$$\rho_1(p_0\, p_2) = p_2 - p_0 \text{ . So}$$

$$\rho_1(p_0\, p_2) = \rho_1(p_0\, p_1) + \rho_1(p_1\, p_2)$$

$= p_1 - p_0 + p_2 - p_1$

$= p_2 - p_0$

3) In the figure (b) the triangle is second triangle and it is sum of three 1-simplexes oriented

$(p_0\, p_1) + (p_1\, p_2) + (p_2\, p_0)$. See the boundaries not founded.

4) If we say $p_1 + p_0 = \rho_1 (p_0\, p_1)$, then $\rho_1 (p_0\, p_1) + \rho_1 (p_1\, p_2) + \rho_1 (p_2\, p_0) = p_0 + p_1 + p_1 + p_2 + p_2 + p_0$, but this is undesirable.

5) If $\rho_1 (p_0\, p_1) = p_1 - p_0$, then $\rho_1 (p_0\, p_1) + \rho_1 (p_1\, p_2) + \rho_1 (p_2 p_0) = p_1 - p_0 + p_2 - p_1 + p_0 - p_2 = 0$

6) If $\sigma_r (p_0 \ldots p_r)$, r □ 0 r-simplex oriented, then the boundaries $\rho_r\, \sigma_r$ is (r–1)-chan and define by: $\rho_r\, \sigma_r = \sum (-1)^i (p_0, p_1, \ldots p'_r, \ldots p_r)$ such that the element p_i under □ is deleted, for example

$\rho_2 (p_0\, p_1 p_2) = (p_1\, p_2) - (p_0\, p_2) + (p_0\, p_1)$. Also $\rho_2 (p_0\, p_1 p_2\, p_3) = (p_1\, p_2\, p_3) - (p_0\, p_2\, p_3) + (p_0\, p_1\, p_3) - (p_0\, p_1 p_2)$ when r = 0 imply $\sigma_0\, \rho_0 = 0$. Note that his operator ρ_r is linear on the element $C = \sum_i c_i\, \sigma_{ri}$ of $C_r(K)$. So $\rho_r\, C = \sum_i c_i\, \rho_r, \sigma_{ri}$ **…** (*)

The right side of (*) is an element of $C_{r-1}(K)$, so $\rho_r : C_r (K) \rightarrow C_{r-1}(K)$. Here ρ_r is called boundary operator and it is homomorphism.

Definition 3.10. Let K be a symbolical complex with n dimension. Then there exists exact sequence of abelian free group and homomorphism such that i: $0 \hookrightarrow C_r (K)$ is the containment function and 0 is identity element of $C_n (K)$. This sequence is called chain complex paired with K and denoted by C(K).

Now we begin to define another homological group namely group of r-cyclic.

Definition 3.11. If the element $c \in C_r(K)$ and satisfy $\rho_r\, c = 0$, then c is called r-cycle. The set of r-cycle elemented by $Z_r(K)$ is subset $C_r(K)$ and denoted the group of r-cycle. Note that $Z_r (K) = \ker(\rho_r)$ and if r = 0, then $\rho_r\, c = 0$. So $Z_0(K) = C_0(K)$.

Definition 3.12. Let K be a simplex complex with n-dimension and let c an element in $C_r(K)$. If there exists another element d in $C_{r+1}(K)$ such that $c = \rho_{r+1} d$, then c is a boundary element and denoted r-boundary.

Remark 3.13. (**1)** Note that c is boundary of the element d and the set r-boundary $B_r (K)$ subset of subgroup of the group $C_r(K)$ and denoted r-boundary group. Hence $B_r (K) = \mathrm{Img}(\rho_{r+1})$.

(2) $B_r(K)$ is defined to be zero such that K has one dimension.

Now we prove several the most important relationships between $B_r(k)$ and $Z_r(k)$, which is the nerve of the definition of the homology. Also, we will explain the relationship between cyclic group and boundary group of $C_r(k)$. Moreover, we introduce the relationship between Kan complex and the path component. Here we studied three groups are $C_r(K)$, $Z_r(K)$ and $B_r(K)$ which are connected with simplex complex K. Also, we try to answer of our question: How we can construct these groups a topology properties of simplex complex K or any topological space which has triangulation equivalent to K. On the other hand, we attempt to answer of the following question: Is C_r (K) reserves its topological properties under any homeomorphism ?.

Theorem 3.14. Let $Z_r(k)$ be a cyclic group and let $B_r(k)$ be a boundary group of $C_r(k)$. Then $B_r(k) \subseteq Z_r(k)$.

Proof. It is clear that any element $c \in B_r(k)$ can be written by : $C = \rho_{r+1}$ of $d \in C_{r+1}(k)$. Therefore $\rho_r\, c = \rho_r\, (\rho_{r+1}) = 0$. So $c \in Z_r(k)$. Thus $B_r(k) \subseteq Z_r(k)$ [3].

Note that, since the triangle and square closed curve, then they are equivalent by topology (homemorphic) $\triangle \cong \square$. But chain group of theorem not equivalent. See the following example:

Example 3.15. Find chain group of triangle $C_r(K)$. If r = 0, then $C_1(\triangle) = \{i(p_o p_1) + j(p_1 p_2) + k(p_2\, p_o) \mid i, j, k \in \mathbb{Z}\} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Example 3.16. Find chain group of the square at r =1. If r=0 , then $C_1(\triangle) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Thus $C_1(\triangle) \ncong C_1(\square)$

Definition 3.17. Let K be a simplex complex with n-dimension. The homology group of order r ; $H_r (k)$, $0 \leq r \leq n$ connected with K such that $H_r (k) = \frac{Z_r(k)}{B_r(k)}$.

Remarks 3.18. (1) $H_r (k)$ is a quotient group.

(2) We can define $H_r (k) = 0\ \forall\, n < r\ or\ r > 0$.

(3) If the build of this group depends on the integers, then we can written $H_r (k, \mathbb{Z})$ and if depends on the real numbers, then the homology group is $H_r(k, \mathbb{R})$ and if the coefficients of the group $\mathbb{Z}_2$, then the homology group is $H_r(k, \mathbb{Z}_2)$.

(4) Since $B_r (k)$ subgroup of $\mathbb{Z}_r(k)$, then $H_r (k)$ is well-define and $H_r(k)$ is the set of equivalent classes of cyclic element of order r (r-cycles) $H_r(k) = \{[z] \setminus z \in \mathbb{Z}_r(k)\}$ such that any equivalent class [z] is called homology class.

(5) Any two cycle elemets z_1 and z_2 are said equivalent if and only if $z_1 - z_2 \in B_r(k)$. (i.e. z_1, z_2 are equivalent homology) $(z_1 \sim z_2)$ or $[z_1] = [z_2]$.

(6) By the definition, any boundary element $b \in B_r (k)$ is homology equivalent to zero such that $b - 0 \in B_r(k)$.

Theorem 3.19. The homology group is topological property or constant topology if X homemorphic to Y and if (K, f) and (L, g) triangulation of X and Y respectively, then $H_r(K) \cong H_r (L)$, r= 0,1,2,…

Corollary 3.20. If (K, f) and (L, g) in Theorem 3.19, such that they are triangulation to X only, then $H_r (K) \cong H_r (L)$, r=0,1,2,**…**

Theorem 3.21. Let K={P_0}. Then the chain element of order 0 (0-chain) is $C_0(K)=\{ip_o:I \in Z\}\cong \mathbb{Z}$.

Proof. It is clear that $C_o(k)=Z_o(K)$ and $B_o(K)=\{0\}$ because P_o not boundary of anything ($\rho_0P_0=0$).

Thus

$$H_0(K) \cong \frac{Z_0(K)}{B_0(K)} = \frac{C_0(K)}{\{0\}} = C_0(K) \cong \mathbb{Z}.$$

Corollary 3.22. If k={p_0, p_1} is a simplex and contain two simplexes has order 0 (0-simplex), then:

$$H_r(k) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z}, & r=0 \\ \{0\} & r\neq 0 \end{cases}$$

Corollary 3.23. Let K ={p_0, p_1, {p_0, p_1}} two points and one line. Then $C_0(k)=\{ip_0 + jp_1 \mid i, j\in Z$ and $C_1(k)=\{k(p_0p_1) : k\in Z\}$.

To understand above corollaries, see the following:

Since the line not boundary for any simplex inside K (because it is larger peace), therefore $B_1(k)=\{0\}$. So

$$H_1(k) \cong \frac{Z_1(k)}{B_1(k)} = \frac{Z_1(k)}{\{0\}} = Z_1(k).$$

On the other hand, if $Z = m(p_0p_1) \in Z_1(K)$, then $\rho_1 z = m\partial_1(p_op_1) = m[p_1 - p_o] = mp_1 - mp_o = 0$. Thus m=0 and hence. $Z_1(k) = 0$. So $H_1(k) = 0$. But to compute $H_o(k)$, we have $Z_o(k) = C_o(k) = \{ip_0+jp_1\}$ and $B_0(k) = im\rho =\{\rho_1: (p_0\ p_1) \mid I \in Z\}=\{i(p_0 - p_1): I \in Z\}$.

Note that; we define homomorphism onto f: $Z_0(K) \rightarrow Z$: $f(ip_0 + jp_1) = I + j$. Hence $\ker(f) = f^{-1}(0) = B_0(K)$. So by the fundamental theorem of algebra we obtain $\frac{\mathbb{Z}_o(K)}{Ker(f)} \cong im(f) = \mathbb{Z}, but \ker(f) = B_o(K)$. Then $H_o(k) = \frac{\mathbb{Z}_o(K)}{B_o(K)} \cong im(f) = \mathbb{Z}$. Thus $H_0(k)=Z$.

Theorem 3.24. Let $K=\{p_0, p_1, p_2, (p_0p_1), (p_1p_2), (p_2p_3)\}$ three points and three lines such that is triangulation of the rings. If there is no simplex of order two (2-simplexs) in K, then $B_1(K).=\{0\}$ and $H_1(K) = Z_1(K)$.

Proof. Let $Z=i(p_0p_1) + j(p_1p_2) + k(p_2p_3) \in \mathbb{Z}_1(k)$.Then $\partial_1 z=(p_1-p_o) + j(p_2 - p_1) + k(p_o - p_2) = (k - i)p_o + (i - j)p_1 + (j - k)p_2 = 0$ and this satisfy I=j = k. So $Z_1(K)=\{i(p_0p_1) + (p_1p_2) + (p_2\ p_0) \in Z\}$ and $Z_1(K)\cong Z$. Thus $H_1(K)\cong Z_1(K)=Z)$. Now to compute $H_0(K)$, we have $Z_0(K)=C_0(K)$. We define homomorphism onto $f:Z_o(K) \rightarrow \mathbb{Z}$ by: $f(ip_o + jp_1 + kp_2) = i + j + k$ and $Kerf=f^{-1}(0) = B_o(K)$. So by the fundamental theorem of algebra we obtain $\frac{Z_o(k)}{Kerf} \cong imf\cong \mathbb{Z}$. So $\frac{Z_o(k)}{B_o(K)} \cong Z$. Hence $H_o(K) \cong \mathbb{Z}$. Note that K her is triangulation of the rings, therefore $H_1(s)= Z$ and $H_0(s) =Z$.

Corollary 3.25. Let K={p_0, p_1, p_2, p_3, ($p_0\,p_1$), ($p_1\,p_2$), ($p_2\,p_3$), ($p_3\,p_0$)} be a simplex as polygonal (square). Then $H_0(K) \cong Z$ and $H_1(K) = Z$.

Proof. We have $\triangle \cong \uparrow$ by homology. Since $H_r(k)$ is a homology group and satisfy topology properties (topology constant), then $H_r(\triangle)\cong H_r(\uparrow)$. From Corollary **3.23** and Theorems **3.24**, we obtain $H_o(\triangle) \cong H_0(\square) = Z$ and $H_1(\triangle) \cong H_1(\uparrow).\cong \mathbb{Z}$.

CONCLUSION

This study has been demonstrated that the homological groups are a type of improvement for the Euler characteristic. The results showed that the developed simplex complex because there is a strong relation between simplex complex and other homology groups. We proved that if X is a Kan complex, then being in the same path component is an equivalence relation. The other result showed that if we have a simplex as polygonal (squar), then $H_0(K) \cong Z$ and $H_1(K) = Z$. Also suppose that there are three points and three lines such that is triangulation of the rings and there is no simplex of order two (2-simplexs) in K, then $B_1(K) = \{0\}$ and $H_1(K) = \mathbb{Z}_1(K)$. We studied three groups are $C_r(K)$, $Z_r(K)$ and $B_r(K)$ which are connected with simplex complex K. We constructed these groups by some topological properties on simplex complex K or any topological space which has triangulation equivalent to K. We proved that $C_r(K)$ reserves its topological properties under any homeomorphism. Finally we obtained that if three points and three lines such that is triangulation of the rings and there is no simplex of order two (2-simplexs) in K, in this case the boundary homological group equal zero and $H_1(K) = Z_1(K)$.

ACKNOWLEDGMENTS

The author would like to thank the referee, whose careful reading and thoughtful comments have helped improve the paper. Thank you very much to the Department of mathematics for the support given to me. Thanks to Dr. Alaa Mahmood for his support in to me.

REFERENCES

[1] Friedman, G., 2012, An elementary illustrated introduction to simplicial sets. Rocky Mountain Journal of Mathematics. 42, 353-424.

[2] Eilenberg, S., Zilber, J. A., 1950, Semi-Simplicial Complexes and Singular Homology. Annals of Mathematics. Second Series, 51(3), 499-513

[3] Barmak, J. A., Minian, E. G., 2006, Simple Homotopy Types and Finite Spaces. arXiv:math/0611158v2 [math.AT] 8 Nov.

[4] Alameddin, A., 2016, Homotopy Theory of Schemes. University of Liverpool, UK.

[5] Fenati, E., Westwood, T., 2014, M3/4/5P21- Algebraic Topology. Imperial College London.Lecturer: Professor Alessio Corti Spring January.

[6] Sieradski, A. J., 1993, Algebraic topology for two dimensional complexes. in Two-Dimensional Homotopy and Combinatorial Group Theory, ed. Hog-Angeloni, C. Metzler, W. and Sieradski, A. J. Cambridge Univ. Press. 51–96.

[7] Davis, J .F., Kirk, P., 1991, Lecture Notes in Algebraic Topology. Department of Mathematics. Indiana University. Bloomington. IN 47405.

[8] Cohen, M .M., 1972, A Course in Simple-Homotopy Theory. Springer-Verlag New York. Heidelberg.

[9] Mutlu, A., 1999, Free Crossed resolution from simplicial resolutions with given CW-basis. Cahiers de topologie et geometrie differentielle categoriques. tom 40 (4), 261-283.

[10] Friedman, G., 2012, Survey Article: An Elementary Illustrated Introduction To Simplicial Sets. Rocky Mountain Journal of Mathematics. 42(2), 353-423.

[11] Hillman,J.A., 2006, Algebraic Topology Notes. Part I: Homology. School of Mathematics and Statistics. University of Sydney. NSW. Australia.

[12] Joyal, A., 1999, Myles Tierney. An Introduction to Simplicial Homotopy Theory. Universite du Quebec a Montreal Rutgers University Preliminary Version. August 5.

[13] Milnor, J., 1961, Two Complexes Which Are Homeomorphic But Combinatorially Distinct. Annals of Mathematics. 74 (3), 575-590.

[14] Friedman, G., 2011, An Elementary Illustrated Introduction To Simplicial Sets. eprint arXiv:0809.4221v2.

[15] MacLane, S., 1971, Categories For The Working Mathematician. Second Edition. Graduate Texts In Mathematics 5. Springer-Verlag.

[16] Boissonnat, J.D., Maria, C., 2012, The Simplex Tree: An Efficient Data Structure for General Simplicial Complexes. HAL Id: hal-00707901. [Research Report] RR-7993. pp.20. https://hal.inria.fr/hal-00707901v1,

[17] Heller, A., 1955, Homotopy Resolution Of Semisimplicial Complexes. Harvard University. Cambridge. Mass.

[18] Goerss, P.G., Jardine, J.F., 2000, Bulletin (New Series) of The American Mathematical Society. S 0273-0979(00)00895-8. 38(2), 217-220.

[19] Schreuder, R., Baayen, H., 1997, How Complex Simplex Can. Journal of Memory and Language. DOI: 10.1006/jmla. 2510.

[20] Jonsson, J., 2008, Simplicial Complexes of Graphs. Springer-Verlag Berlin Heidelberg.

[21] Curtis, E.B., 1971, Simplicial Homotopy Theory. Advances In Mathematics. 6, 107-209.