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Determination of order quantity with piecewise-linear demand function with saturation

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Abstract: The paper considers the model of the retailer's behavior when the expected demand for goods is a piecewise linear function characterized by some predetermined saturation level. The demand function is constructed as a result of a special marketing research. Let's assume that the following sale of goods scheme is used. The ordered goods are divided into two parts. The first batch of goods comes at once and is sold to consumers with immediate delivery. At time T_1 the first batch is completely sold. The second batch of goods is sold during the following period, but now the delivery only occurs at the end of the period at time T. The need to consider this scheme comes from the fact that trading firms usually have limited capacity warehouses and cannot accommodate the entire ordered volume of goods. In addition, a manufacturer may not be able to offer the entire ordered volume of goods because all goods cannot be produced at the initial (zero) moment of time when the order is placed. A distinctive feature of this problem is the choice of parameters on which the optimization will be carried out. The paper considers the choice of the optimal strategies for ordering goods by the retailer.

Keywords: The level of product inventory; Piecewise linear demand; The goods deficit; Reduction.

INTRODUCTION

Classical economic order quantity (EOQ) model contains the main idea that a retailer must pay immediately on receiving items. However, in many recently published articles a manufacturer or a supplier may offer to retailers a delay period (called trade credit) to settle the payment. Goyal (1985) [9] first developed an EOQ model under trade credit financing. Dave (1985) [4] corrected Goyal's model (1985) by addressing the fact that the selling price is necessarily higher than the purchase price. Aggarwal and Jaggi (1995) [1] extended Goyal's (1985) model for deteriorating items. Jamal et al. (1997) [12] further generalized Aggarwal and Jaggi's model (1995) [1] to allow shortages. Huang (2003) [10] extended Goyal's (1985) model in which the supplier provides the retailer a permissible delay period. More related articles can be found in Huang and Hsu (2008) [11], Giri and Sharma (2015) [8] and their references. Khanra et al. (2011) [13] proposed an inventory model with quadratic time varying demand under permissible delay in payments. Recently, Khanra et al. (2013) [14] developed the EOQ model with time varying quadratic demand, shortages and permissible delay in payments. Chen et al. (2013) [3] developed an economic production quantity (EPQ) model for deteriorating items with up-stream full trade credit and down-stream partial trade credit. Some other questions connected with determining the order quantity were considered in Bure, Karelin and Polyakova (2016) [2]. Giri and Sharma (2016) [7] proposed the EOQ model under two levels of trade credit assuming that demand rate is a linear increasing function of time, the retailer offers a partial trade credit to its credit-risk customers to reduce the default risk and shortages in the retailer's inventory are allowed. In this paper, we propose an EOQ model assuming that demand rate is a piecewise linear function characterized by some predetermined saturation

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level. Using the piecewise linear demand function with saturation leads to a more complex mathematical model, but the model is more adequate from the point of view of economic content. The paper formulates and investigates the model of the retailer behavior who sells off a certain volume of goods ordered from the manufacturer when the expected demand for goods is piecewise linear characterized by some predetermined saturation level. It is assumed that as a result of a special marketing research the demand function is accurately determined for goods procured from a certain manufacturer according to the goods popularity among buyers. The retailer places an order to the manufacturer and sells merchandise through stores located on a certain territory.

It is assumed that for selling goods the following scheme is used. The whole batch of ordered goods is divided into two parts. The first part comes and is sold for a certain period of time. We assume that at the time T_1 the entire first batch is completely sold out to consumers and after some time interval of length $T - T_1$ the second batch of goods is received and during this period the goods are also sold but the delivery of goods to customers occurs at the time T. The necessity to consider such a scheme of sale is due to various reasons and often occurs in practice, because firstly, warehouses of the retailer have limited capacity and cannot accommodate the full ordered amount of goods from the manufacturer immediately, and, secondly, the manufacturer, as often happens, cannot offer immediately the whole batch of ordered goods, since not all the goods can be produced at the initial (zero) point of time, when the order is placed. Moreover, for production of ordered goods the funds may be required, which will go to the manufacturer from the retailer only after some of the goods have been sold. A distinctive feature of the problem is the choice of parameters which are subsequently optimized. For the retailer time moments T_1 and T are of importance. By time T_1 the retailer will have sold the first batch of goods and receives the funds, part of which it pays the manufacturer which will allow the manufacturer to continue the activities. This time is important for the retailer, as it will allow them to fulfil various financial obligations related to payment of loans and other expenses. Time T is also of great importance for the retailer as it will mean successful completion of full realization of all the purchased goods. Assume that as a result of marketing research we can accurately determine the demand function for goods, it means that the choice of time points T_1 and T allows in principle to determine the volume of the first shipment of the ordered goods q_0 and the total volume of ordered goods from manufacturers Q. Of course, all the calculated values represent some evaluations and the accuracy of obtained estimates depends on the quality of marketing research. But the choice of commercial decisions can only be made on the basis of the obtained estimates and certainly contains elements of commercial risk, namely for the retailer whose income depends entirely on the commercial decisions quality so the choice of the key moments T_1 and T is decisive.

Further in the paper the choice of the optimal ordering strategy for the retailer is considered. Problems similar in formulation were considered earlier in the works ([1]–[4], [7]–[15]).

Notations used in the paper

c - unit purchasing cost;
p, $(p > c)$, - unit selling price at the retailer stores;
s - unit price discount (applied when the goods are in deficit and the delivery cannot occur immediately);
T_1 - length of stock-end cycle (decision variable);
T - the moment of time when the ordered shipment is sold (decision variable);
Q - order quantity;
$I(t)$ - inventory level (deficit);
$D(t)$ - demand rate;
V - losses associated with shortage of goods in a trading firm;
$TP(T_1, T)$ - the average income of a trading company.

MAIN RESULTS

In the paper we assume that the demand rate $D(t)$ is linearly increasing to the point of saturation t_n. Let

$$D(t) = \begin{cases} a + bt, & t < t_n, \\ a + bt_n, & t \geq t_n, \end{cases}$$

where constants $a > 0$ and $b > 0$. There are three choices of time points T and T_1:

1. $t_n \geq T > T_1$,
2. $T > t_n \geq T_1$,
3. $T > T_1 \geq t_n$.

The first variant was considered in [7]. This variant assumes that the demand grows linearly without saturation, which usually does not occur in practice. In this paper we studied in detail a linear demand saturation for the second variant. Note that the third variant is less interesting since it assumes the saturation of the demand before even the first batch is sold, a scenario in which the retailer is unlikely to consider ordering beyond the warehouse capacity at all.

From the above model it follows that, if $T > t_n \geq T_1$, then the current inventory level $I(t)$ for $t \in [0, T_1]$ can be described by the differential equation

$$\frac{dI(t)}{dt} = -D(t) = -(a + bt), \quad 0 \leq t \leq T_1$$

with the boundary condition $I(T_1) = 0$.

Then it is easily seen that

$$\int_t^{T_1} dI(t) = -\int_t^{T_1} D(t)dt = -\int_t^{T_1} (a + bt)dt,$$

and therefore the current inventory level $I(t)$ if $t \in [0, T_1]$ is equal to

$$I(t) = a(T_1 - t) + \frac{b}{2}(T_1^2 - t^2), \quad 0 \leq t \leq T_1.$$

Following [7] the current level of shortages (backorders) if $T_1 \leq t \leq t_n$ is determined by the following equation:

$$I(t) = a(T_1 - t) + \frac{b}{2}(T_1^2 - t^2), \quad T_1 \leq t \leq t_n.$$

On the interval $[t_n, T]$, the demand has reached the saturation level, therefore, the shortage level (short supply) is defined by the expression:

$$I(t) = I(t_n) - (a + bt_n)(t - t_n), \quad t_n \leq t \leq T.$$

The behavior of the inventory system for $t \in [0, T]$ is shown in Fig. 1.

Thus, the amount of the first batch of goods delivered at the initial moment of time is equal to:

$$q_0 = I(0) = aT_1 + \frac{b}{2}T_1^2.$$

The maximum level of shortages is equal to

$$B = -I(T) = a(T - T_1) - \frac{b}{2}(t_n^2 + T_1^2) + bt_nT,$$

From here we have that the total order quantity is determined by the following expression

$$Q = q_0 + B = aT + \frac{b}{2}t_n(2T - t_n)$$

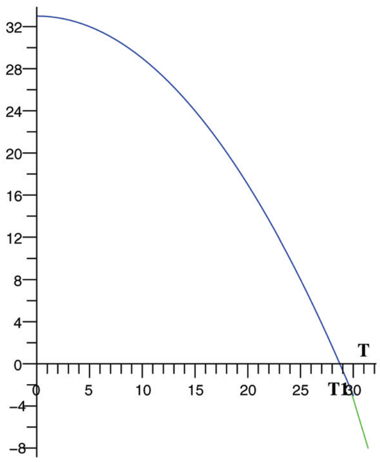

FIGURE 1. The graph of the function $I(t)$.

The losses of L associated with a shortage of production are determined by the expression [7]:

$$V = -s\int_{T_1}^{T} I(t)dt$$

In the variant under consideration we have

$$V = -s\left[\int_{T_1}^{t_n} I(t)dt + \int_{t_n}^{T} I(t)dt\right],$$

and

$$\int_{T_1}^{t_n} I(t)dt = -\frac{(T_1 - t_n)^2}{6}\left[3a + b(t_n + 2T_1)\right],$$

$$\int_{t_n}^{T} I(t)dt = I(t_n)(T - t_n) - (a + bt_n)\frac{(T_1 - t_n)^2}{2}.$$

Thus

$$V = -s\left[-\frac{(T_1 - t_n)^2}{6}\left[3a + b(t_n + 2T_1)\right] + I(t_n)(T - t_n) - (a + bt_n)\frac{(T_1 - t_n)^2}{2}\right].$$

Finally we get the expression for V in the following form:

$$V = -s\left[-\frac{(T_1 - t_n)^2}{3}(3a + 2b(t_n + T_1)) + \left[a(T_1 - t_n) + \frac{b}{2}(T_1^2 - t_n^2)\right](T - t_n)\right].$$

The total profit of the seller of $TP(T_1, T)$ is determined by the formula

$$TP(T_1,T) = \frac{1}{T}\Bigg\{(p-c)aT + \frac{b}{2}t_n\,(2T-t_n) + s\left[-\frac{(T_1-t_n)^2}{3}(3a+2b(t_n+T_1)) + \left[a(T_1-t_n)+\frac{b}{2}(T_1^2-t_n^2)\right](T-t_n)\right]\Bigg\}.$$

MATHEMATICAL OPTIMIZATION MODEL

Let's solve the problem of finding the maximum value of the function $f(T_1, T)$ arguments T_1, T on the set defined by inequalities, $T_1 \le t_n < T$.

Define functions

$$f(T_1,T) = -TP(T_1,T), \quad h_1(T_1,T) = T_1 - t_n, \quad h_2(T_1,T) = t_n - T, \quad h_0(T_1,T) = 0.$$

We have

$$\frac{\partial h_0(T_1,T)}{\partial T_1} = 0, \quad \frac{\partial h_0(T_1,T)}{\partial T} = 0, \quad \frac{\partial h_1(T_1,T)}{\partial T_1} = 1,$$

$$\frac{\partial h_1(T_1,T)}{\partial T} = 0, \quad \frac{\partial h_2(T_1,T)}{\partial T_1} = 0, \quad \frac{\partial h_2(T_1,T)}{\partial T} = -1.$$

Consider the optimization problem

$$f(T_1,T) \to \min, \quad (T_1,T) \in \mathcal{X}, \tag{1}$$

where

$$\mathcal{X} = \{(T_1,T) \in \mathbb{R}^2 \mid h_i(T_1,T) \le 0,\ i = 1,2\}.$$

For convenience of presenting the material we will introduce the notation $x = (x_1, x_2)$, $x_1 = T_1$, $x_2 = T$.

Then the set of $\mathcal{X}$ can be represented as follows

$$\mathcal{X} = \{x = (x_1,x_2) \in \mathbb{R}^2 \mid \varphi(x) = 0\},$$

where

$$\varphi(x) = \max_{i \in I} h_i(x), \ I = \{0,1,2\}.$$

It is easy to see that $\varphi(x) = 0$, if and only if when $x \in \mathcal{X}$. To solve the problem of conditional optimization, we will apply the methods used in the theory of nonsmooth optimization [17], [18].

From the properties of the maximum function it is known that $\varphi(x)$ is directionally differentiable on any direction $g = (g_1, g_2) \in \mathbb{R}^2$, as that [19]

$$\varphi'(x,g) = \max_{v \in \partial\varphi(x)} \langle v, g\rangle,$$

$$\partial\varphi(x) = \text{co}\,\{h_i'(x) | i \in R(x)\},$$

$$R(x) = \{i \in I | h_i(x) = \varphi(x)\}.$$

Here the scalar product is denoted by $\langle *, *\rangle$, the subdifferential of the function $\varphi(x)$ at (x) is denoted by $\partial\varphi(x)$, the convex hull of A is denoted by co $\{A\}$. The set $R(x)$ is the set of indices of active constraints at (x). If $x \notin \mathcal{X}$ then $0 \notin R(x)$.

Let $\varepsilon > 0$. Introduce the set

$$\mathcal{X}_\epsilon = \{x \in \mathbb{R}^2 \mid \varphi(x) \le \epsilon\}.$$

Suppose that there exist such numbers $\epsilon > 0$, $a > 0$ and

$$\min_{v \in \partial\varphi(x)} \|v(x)\| \ge a \ \forall x \in \mathcal{X}_\epsilon \backslash \mathcal{X}.$$

Let

$$\varphi^{\downarrow}(x) = \liminf_{\alpha \downarrow 0, g' \to g} \frac{\varphi(x+\alpha g) - \varphi(T_1,T)}{\alpha}.$$

In this case it is possible to show that

$$\varphi^{\downarrow}(x) \le -a \qquad \forall x \in X_\epsilon \backslash \mathcal{X}.$$

Consider the function

$$F_\lambda(x) = f(x) + \lambda\varphi(x), \quad \lambda \ge 0,$$

by using results from [17], [18] we can prove the following theorem

Theorem 1. Let f be bounded from below, i.e.

$$\min_{x \in X} f(x) > -\infty,$$

and there exists $\lambda_0 > 0$ that for all $\lambda \ge \lambda_0$ there exists a point $x_\lambda \in \mathcal{X}$ for which

$$F_\lambda(x_\lambda) = \min_{x \in \mathcal{X}} F_\lambda(x),$$

then there exists $\lambda^* \ge \lambda_0$, such that for all $\lambda \ge \lambda^*$

$$\varphi(x_\lambda) = 0, \quad f(x_\lambda) = \min_{x \in \mathcal{X}} f(x).$$

From Theorem 1 it follows that if $x^* \in \mathcal{X}$ is a minimizer f on $\mathcal{X}$ then $x^* \in \mathcal{X}$ is a minimizer of $F_\lambda(x)$ on $\mathbb{R}^2$.

Remark 2. The function $F_\lambda(x)$ is subdifferential if $\lambda \geq 0$ and

$$\partial F_\lambda(x) = f'(x) + \lambda \partial \varphi(x),$$

where $f'(x)$ is the gradient of f at x

$$f'(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}\right) = \left(-\frac{\partial TP(T_1,T)}{\partial T_1}, -\frac{\partial TP(T_1,T)}{\partial T}\right).$$

We have

$$-\frac{\partial TP(T_1,T)}{\partial T_1} = \frac{s}{3T}\Big(-3bT_1^2 - 4aT_1bt_nT_1 + 3bT_1T + at_n + bt_n^2 + 3aT\Big),$$

$$-\frac{\partial TP(T_1,T)}{\partial T} = \frac{1}{6T^2}\Big(2sbT_1^3 + 4saT_1^2 + sbt_nT_1^2 - 2sat_nT_1 - 2sbt_n^2T_1$$
$$+6pa + 6pbt_n - 6ca - 6cbt_n - 3bpt_n^2 + 3bct_n^2 - 2sat_n^2 - sbt_n^3\Big).$$

By the necessary condition of minimum we have

$$0_2 = (0,0) \in \partial F_\lambda(x).$$

Method a hipodifferential descent

For solving minimization problem (1) there are enough effective solution methods, for example, a method a hipodifferential descent [16], [18]. Class of hypodifferentiable functions has been allocated by V.F. Demyanov among the nonsmooth functions [5], [6].

Consider the function $F_\lambda(x)$. It is not difficult to see that this function can be written as the maximum function:

$$F_\lambda(x) = f(x) + \lambda\varphi(x) = \max\{\psi_0(x), \psi_1(x), \psi_2(x)\},$$

where

$$\psi_0(x) = f(x) + \lambda h_0(x) = f(x),\ \ \psi_1(x) = f(x) + \lambda h_1(x),$$
$$\psi_2(x) = f(x) + \lambda h_2(x).$$

The function $F_\lambda(x)$ is continuously hypodifferentiable. A set

$$dF_\lambda(x) = \text{co}\ \left\{\bigcup_{i\in I} \big(\psi_i'(x), \psi_i(x) - F_\lambda(x)\big)^T\right\} \subset \mathbb{R}^2 \times \mathbb{R}$$

can be chosen as its continuous hypodifferential at x. In our case the set $dF_\lambda(x) \subset \mathbb{R}^3$ is a triangle.

Formulate the necessary condition for local minimizer of hypodifferentiable function dF_λ.

Theorem 3. For the point $x^* \in \mathbb{R}^2$ to be a minimum point of dF_λ on $\mathbb{R}^2$ it is necessary that

$$0_3 \in dF_\lambda(x^*). \tag{2}$$

If condition (2) does not hold at x then we project the point 0_3 onto $dF_\lambda(x)$, i.e. solve the optimization problem

$$\min_{z\in dF_\lambda(x)} \|z\| = \|z(x)\|,\ z(x) = (w(x), t(x))^T \in \mathbb{R}^2 \times \mathbb{R}.$$

Note that if $0_3 \notin dF_\lambda(x)$, then $w(x) \neq 0_2$. A direction $-w(x)$ is called a direction of hypodifferentiable descent of the function F_λ on $\mathbb{R}^2$ at the point x. It is continuous and unique. Consider the steepest descent method for minimizing hypodifferentiable functions. Let a point $x_k \in R^2$ have already been found and it is not a stationary point of F_λ on $\mathbb{R}^2$. Otherwise, put

$$x_{k+1} = x_k - \alpha_k \frac{w(x_k)}{||w(x_k)||} = x_k - \alpha_k \frac{w_k}{||w_k||},$$

where $-w(x_k) = -w_k$ is a hypodifferentiable descent direction at x_k and step size is chosen by using the one dimensional minimization. Consider the case when the sequence $\{x_k\}$ is infinite. Then the sequence $\{f(x_k)\}$ is monotonically decreasing, therefore, this method is a continuously relaxation method.

Theorem 4. Every limit point of the sequence $\{x_k\}$ is a stationary point of the hypodifferentiable function F_λ on $\mathbb{R}^2$.

Let us consider example of the application of the hypodifferential descent method.

Example 5. Let the following system be given

$$\begin{cases} \dfrac{dI(t)}{dt} = & -200 - 150t, \quad 0 \leq t \leq t_n \\ \dfrac{dI(t)}{dt} = & -200 - 150t_n, \quad t_n \leq t \leq T, \end{cases}$$

where $T > t_n \geq T_1$.
It is required to minimize the functional (1), provided that

$$c = 2\$/unit,\ \ p = 4\$/unit,\ \ s = 1.5\$/unit/year,$$
$$t_n = 2.428\ years.$$

Applying the proposed algorithm, we obtain the optimal solution as

$$T_1^* = 1.9950\ years,\ \ T^* = 2.8251\ years,$$

The optimal profit is $TP^*(T_1,T) = TP(T_1,T) = 607.697$ and the optimal order quantity is $Q^* = 1151.78$ units.

CONCLUSIONS

In this paper the model for determining the order quantity by the retailer in the presence of a piecewise-linear demand with saturation is formulated and studied. We show how the problem can be efficiently reduced to an optimization problem for which numerous existing techniques can be applied.

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