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The Asymptotic Distributions of the Absolute Maximum of the Generalized Wiener Random Field

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Abstract

The technique is presented for obtaining the asymptotic approximations of the distribution functions of the absolute maximum of the generalized Wiener random fields. Based on the proposed approaches, the closed expressions are found for the distribution functions of the absolute maximum of Wiener random fields with the constants and piecewise constant drift and diffusion. By means of statistical simulation, it is established that the introduced analytical formulas successfully agree with the corresponding experimental data referring to a wide range of the random field parameters values.

Keywords: Wiener random field, Distribution function of the absolute maximum, Local Markov approximation, Statistical simulation.

INTRODUCTION

The task of finding the probabilistic distributions for the absolute (greatest) maxima of Gaussian random processes and fields arises in various applications of physics, including radio physics and radio engineering [1, 2]. Despite a great number of theoretical and experimental papers devoted to the study of the maxima of stochastic functions, the general expressions for the distributions for the absolute maximum of Gaussian random field remain undetermined yet.

In [3, 4], the asymptotically exact (with h increasing) expressions are presented for the distribution function $F(h)$ of the absolute maximum of the homogeneous Gaussian random field. In practice, the essentially heterogeneous Wiener processes and fields are commonly found. As Wiener random process $L_0(t)$, $t \geq 0$, we consider the purely diffusion and centered Gaussian Markov random process with the zero drift coefficient, the constant diffusion coefficient and the covariance function $R(t_1, t_2) = \min(t_1, t_2)$ [5]. Such process describes Brownian motion of particles in various environments, fluctuations of the generator oscillation phase in the presence of thermal and shot noise, as well as the behavior of decision statistics of the sequential detector of signals against Gaussian interferences, etc.

As the generalization of Wiener process in case of the two-dimensional definitional domain, Wiener field $L_0(\eta, \kappa)$, $\eta \geq 0$, $\kappa \geq 0$ is considered [3, 7, 8]. This field is Gaussian centered random field with the covariance function $R(\eta_1, \eta_2, \kappa_1, \kappa_2) = \min(\eta_1, \eta_2)\min(\kappa_1, \kappa_2)$. The sections of Wiener random field $L_0(\eta, \kappa)$ by the η and κ variables are Wiener random processes.

In [3] the asymptotically exact (with h increasing) expression is obtained for the probability $\alpha(h) = 1 - F(h)$ of the threshold h crossing by the absolute maximum of Wiener random field $L_0(\eta, \kappa)$ within the definitional domain set by the conditions $\eta \in [0,1]$, $\kappa \in [0,1]$. The analysis of the asymptotic expression [3] shows that its accuracy for the finite values of h may be unsatisfactory. Thus, the values of the probability $\alpha(h)$ calculated by the formula from [3] may be greater than one, while the corresponding values of the distribution function $F(h)$ may be less than zero. Besides, $\alpha(h) \to \infty$ and $F(h) \to -\infty$, if $h \to 0$, and this contradicts the understanding of the functions $\alpha(h)$ and $F(h)$ as the probability measures lying within the range of values from 0 to 1. It should also be noted that the researcher often need to know the distribution of the maximum of Wiener field within the intervals

$\eta \in [\eta_{\min}, \eta_{\max}]$, $\kappa \in [\kappa_{\min}, \kappa_{\max}]$, where $\eta_{\min} \geq 0$ and $\kappa_{\min} \geq 0$. The asymptotic formula [3] obtained for the case of $\eta_{\min} = 0$, $\kappa_{\min} = 0$ does not account for the influence of nonzero values of $\eta_{\min}$ and $\kappa_{\min}$ on the distribution of the absolute maximum of the field.

In a number of practical applications of statistical radio physics and radio engineering, it is necessary to calculate the probabilistic distributions for the absolute maxima of heterogeneous Gaussian random fields which are the additive superposition of both Wiener random field and a certain determined field. Such random fields (which further we will refer to as the generalized Wiener fields) are involved when we process of the quasi-deterministic and stochastic signals with unknown discontinuous power parameters [9-12] and the subimages with a priori unknown sizes, etc. Unlike Wiener random field, the generalized Wiener field has the mathematical expectation changing within the definitional domain. It does not allow using the technique [3, 4] for obtaining the asymptotic distributions of the absolute maximum of the generalized Wiener fields. In [9-11], by means of the methods of Markov processes theory [5, 6], the exact expression is found for the distribution function of the absolute maximum of the generalized Wiener random process with the constant and piecewise constant drift and diffusion coefficients. However, we do not have methods for the exact solution of a similar task in case of the generalized Wiener random field yet.

Below, the asymptotically exact expressions are introduced for the distribution function of the absolute maximum of the generalized Wiener random filed with both constant and piecewise constant drift and diffusion, without the previously specified flaws [3]. The applicability limits of the obtained asymptotic expressions are experimentally established by means of computer statistical simulation.

THE DISTRIBUTION OF THE ABSOLUTE MAXIMUM OF THE WIENER RANDOM FIELD

According to [3, 7, 8], under Wiener field we understand the centered heterogeneous Gaussian random field $L_0(\eta, \kappa)$, $\eta \geq 0$, $\kappa \geq 0$ with the covariance function

$$R(\eta_1, \eta_2, \kappa_1, \kappa_2) = \langle L_0(\eta_1, \kappa_1) L_0(\eta_2, \kappa_2) \rangle = \\ = \min(\eta_1, \eta_2) \min(\kappa_1, \kappa_2). \quad (1)$$

Let us find the asymptotically exact (with h increasing) expression for the distribution function $F(h)$ of the absolute maximum of Wiener random field within the definitional domain Λ set by the conditions $\eta \in [\eta_{\min}, 1]$, $\kappa \in [\kappa_{\min}, 1]$, $0 \leq \eta_{\min} \leq 1$, $0 \leq \kappa_{\min} \leq 1$. It should be noted that the distribution function of the absolute maximum of Wiener field within the arbitrary definitional domain $\eta \in [\eta^*_{\min}, \eta^*_{\max}]$, $\kappa \in [\kappa^*_{\min}, \kappa^*_{\max}]$ can be easy calculated by the formula $F(h) = F^*\left(h / \sqrt{\eta^*_{\max} \kappa^*_{\max}}\right)$, where $F^*(h)$ is the distribution function of the absolute maximum of the field under $\eta \in [\eta^*_{\min} / \eta^*_{\max}, 1]$, $\kappa \in [\kappa^*_{\min} / \kappa^*_{\max}, 1]$.

It is known from [3] that the distribution $F(h)$ of the absolute maximum of Wiener random field $L_0(\eta, \kappa)$ under the large values of h is determined by the behavior of the field in the neighborhood of the values $\eta = \eta_m$, $\kappa = \kappa_m$ belonging to the definitional domain Λ and setting the field dispersion as the maximum one. In the present case, it is obvious that $\eta_m = 1$, $\kappa_m = 1$. Therefore, in order to find the asymptotically exact (with h increasing) expression for the distribution function $F(h)$, it is sufficient to study the behavior of the covariance function of the field $L_0(\eta, \kappa)$ in the small neighborhood of point (1,1). Under $\delta_R = \max(|\eta_1 - 1|, |\eta_2 - 1|, |\kappa_1 - 1|, |\kappa_2 - 1|) \to 0$, the covariance function (1) can be asymptotically presented as follows

$$R(\eta_1, \eta_2, \kappa_1, \kappa_2) = R_0(\eta_1 - 1/2, \eta_2 - 1/2) + \\ + R_0(\kappa_1 - 1/2, \kappa_2 - 1/2) + o(\delta_R), \quad (2)$$

where

$$R_0(t_1, t_2) = \max[0, \min(t_1, t_2)] \quad (3)$$

and $o(\delta_R)$ denotes the higher-order infinitesimal terms compared with δ_R.

We designate $L_i(t)$, $i = 1,2$ as the statistically independent and jointly Gaussian centered random processes with the covariance functions $R_0(t_1, t_2)$ (3). Such processes are identically equal to zero under $t < 0$, and they are Wiener random processes under $t \geq 0$ [5]. From Eqs. (1), (2), it follows that the covariance functions of Gaussian random fields $L_0(\eta, \kappa)$ and $L^*(\eta, \kappa) = L_1(\eta - 1/2) + L_2(\kappa - 1/2)$ coincide asymptotically, while $\delta_R \to 0$. Therefore, Wiener random field $L_0(\eta, \kappa)$ converges in distribution to the sum of the statistically independent Gaussian random processes $L_1(\eta - 1/2)$ and $L_2(\kappa - 1/2)$ under $\delta = \max(|\eta - 1|, |\kappa - 1|) \to 0$. Thus, the distribution function $F(h)$ of the absolute maximum of the random field $L_0(\eta, \kappa)$ can be presented in the form of

$$F(h) = P\left[\sup_{\eta \in \Lambda_1} L_1(\eta) + \sup_{\kappa \in \Lambda_2} L_2(\kappa) < h\right] = \\ = \int_{-\infty}^{\infty} F_2(h - x) w_1(x) dx, \quad (4)$$

where $F_1(x)=P\left[\sup_{\eta\in\Lambda_1} L_1(\eta)<x\right]$, $F_2(x)=P\left[\sup_{\kappa\in\Lambda_2} L_2(\kappa)<x\right]$ are the distribution functions of the absolute maxima of the random processes $L_1(\eta)$, $L_2(\kappa)$, correspondingly, $w_1(x)=dF_1(x)/dx$ is the probability density of the absolute maximum of the random process $L_1(\eta)$ under $\eta\in\Lambda_1$, and Λ_1, Λ_2 are the intervals of the possible values of the parameters η and κ set by the conditions $\eta\in[\eta_{\min}-1/2,1/2]$, $\kappa\in[\kappa_{\min}-1/2,1/2]$, correspondingly. By applying the results presented in [9, 10], we obtain

$$F_i(x)=\frac{1}{\sqrt{2\pi\rho_i}}\int_0^\infty \exp\left[-\frac{\left(u-\sqrt{2}x\right)^2}{2\rho_i}\right]\times \left[2\Phi\left(\frac{u}{\sqrt{1-\rho_i}}\right)-1\right]du,\qquad i=1,2. \tag{5}$$

Here $\rho_1=\max(0,2\eta_{\min}-1)$, $\rho_2=\max(0,2\kappa_{\min}-1)$, and $\Phi(x)=\int_{-\infty}^{x}\exp\left(-t^2/2\right)dt/\sqrt{2\pi}$ is the probability integral [1]. Then, from Eq. (4), taking into account Eq. (5), we get

$$F(h)=\frac{1}{\pi}\sqrt{\frac{2}{\rho_2}}\int_{-\infty}^{\infty}\exp\left(-x^2\right)\Phi\left[x\sqrt{\frac{2(1-\rho_1)}{\rho_1}}\right]\times \left\{\int_0^\infty \exp\left[-\frac{\left(u-\sqrt{2}(h-x)\right)^2}{2\rho_2}\right]\left[2\Phi\left(\frac{u}{\sqrt{1-\rho_2}}\right)-1\right]du\right\}dx. \tag{6}$$

The accuracy of Eq. (6) increases with *h*.

In [3], under increasing *h*, the asymptotically exact expression is obtained for the probability $\alpha(h)=1-F(h)$ that the threshold *h* is exceeded by the realization of Wiener random field $L_0(\eta,\kappa)$ under $\eta_{\min}=0$, $\kappa_{\min}=0$, whence we find that

$$F(h)=1-2\sqrt{2}\exp\left(h^2/2\right)/\sqrt{\pi h} \tag{7}$$

Let us compare the asymptotically exact expressions (6) and (7). Under $h\to\infty$, from the formula (6) we get $F(h)=1-2\sqrt{2/\pi}\exp\left(h^2/2\right)\left[1/\sqrt{h}+o(1/h)\right]$. Therefore, the expressions (6) and (7) coincide asymptotically in the case of $h\to\infty$. For the finite values of *h*, the formula (7) can provide the essentially conservative values for the probability $\alpha(h)$, which may be greater than 1, while the appropriate values of the function $F(h)$ may appear less than zero. Then, by applying the formula (7), we obtain $\alpha(h)\approx 5.07$, $F(h)\approx -4.07$, if $h=0.3$, and $\alpha(h)\approx 15.88$, $F(h)\approx -14.88$, if $h=0.1$, while $\alpha(h)\to\infty$ and $F(h)\to-\infty$ under $h\to 0$. At the same time, the values of α calculated by the formula (6) do not exceed 1 and the corresponding values of $F(h)$ are nonnegative that fully complies with the meaning of the functions $\alpha(h)$ and $F(h)$ as the probability measures lying within the range of values from 0 to 1.

In order to establish the borders of applicability of the asymptotically exact formulas (6), (7) for the finite values of *h*, the statistical computer simulation is carried out of the value of the absolute maximum of Wiener random field $L_0(\eta,\kappa)$ with the covariance function (1). During the simulation, the samples $X_{ij}=L_0(i\Delta,j\Delta)=\Delta\sum_{l=1}^{i}\sum_{k=1}^{j}n_{lk}$, $i,j=1,2,\ldots$ of Wiener random field are formed with the discretization step $\Delta=0.005$ by the η and κ variables. Here n_{lk} are independent Gaussian random numbers with the zero mathematical expectation and the unity dispersion. The value X_m of the absolute maximum of the field $L_0(\eta,\kappa)$ within the definitional domain Λ is found as the greatest value X_{ij} for all $i\in[\{\eta_{\min}/\Delta\},\{1/\Delta\}]$, $j\in[\{\kappa_{\min}/\Delta\},\{1/\Delta\}]$, where $\{\cdot\}$ is the integer part. The experimental values for the probabilities $\alpha(h)$ and $F(h)$ for the various thresholds *h* are determined as the relative frequencies of the variable X_m exceeding, or not exceeding the thresholds, respectively.

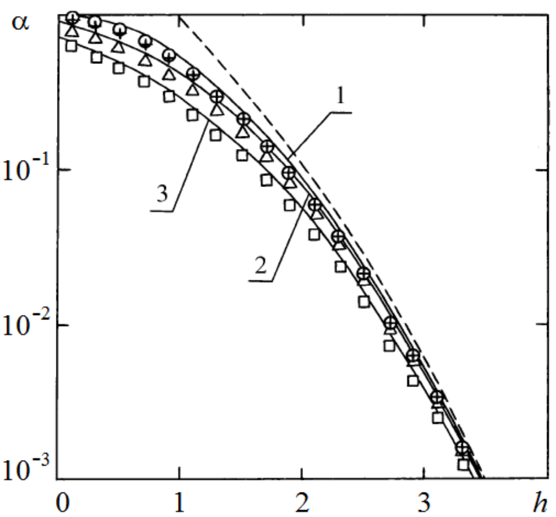

Figure 1: The probability of the absolute maximum of Wiener random field exceeding the threshold

In Fig. 1 the experimental values of the probability $\alpha(h)=1-F(h)$ are presented obtained during the processing of no less than 10^4 realizations of Wiener random field. Here, by continuous lines the corresponding theoretical dependences $\alpha(h)$ are also drawn calculated by the formula (6). The confidence limits deviate from the experimental values for $\alpha(h)$ with the probability of 0.95 and by no more than 20 % under $\alpha\geq 0.01$ and by no more than 40 % under $\alpha\geq 0.003$.

Curve 1 and crosses correspond to $\eta_{\min} = \kappa_{\min} = 0.4$, curve 2 and triangles – to $\eta_{\min} = \kappa_{\min} = 0.7$, curve 3 and squares – to $\eta_{\min} = \kappa_{\min} = 0.9$. By circles, the experimental values for α are shown under $\eta_{\min} = \kappa_{\min} = 0$, and the corresponding theoretical curve calculated by the formula (6) coincides with the curve 1. For the comparison, by dashed line the theoretical dependence $\alpha(h)$ is plotted calculated by the formula (7) for the case of $\eta_{\min} = \kappa_{\min} = 0$. From Fig. 1 and other simulation results, it follows that the theoretical formula (6) introduced above successfully approximates the experimental data for all the values of $h \geq 0$, $\eta_{\min}$ and $\kappa_{\min}$. The formula (7) obtained in [3] has a low accuracy under small values of *h*, and that is why it can be used for calculating $F(h)$ under $h \geq 3$, when $\alpha \leq 0.01$.

THE DISTRIBUTION OF THE ABSOLUTE MAXIMUM OF THE GENERALIZED WIENER RANDOM FIELD WITH CONSTANT DRIFT AND DIFFUSION

Let us consider the generalized Wiener random field $L(\eta, \kappa)$, $\eta \geq 0$, $\kappa \geq 0$ with the mathematical expectation

$$S(\eta, \kappa) = \langle L(\eta, \kappa) \rangle = -z\eta\kappa, \qquad z > 0 \qquad (8)$$

and the covariance function

$$\begin{aligned} R(\eta_1, \eta_2, \kappa_1, \kappa_2) &= \langle [L(\eta_1, \kappa_1) - \langle L(\eta_1, \kappa_1) \rangle] \times \\ &\times [L(\eta_2, \kappa_2) - \langle L(\eta_2, \kappa_2) \rangle] \rangle = \min(\eta_1, \eta_2) \min(\kappa_1, \kappa_2). \end{aligned} \qquad (9)$$

We see that the sections $v_1(\eta) = L(\eta, \kappa^*)$, $\kappa^* = \text{const}$ and $v_2(\kappa) = L(\eta^*, \kappa)$, $\eta^* = \text{const}$ of the field $L(\eta, \kappa)$ by the η and κ variables are Gaussian Markov diffusion random processes with the constant drift $K_{1\eta} = -z\kappa^*$, $K_{1\kappa} = -z\eta^*$ and diffusion $K_{2\eta} = \kappa^*$, $K_{2\kappa} = \eta^*$ coefficients, correspondingly [5].

Let us find the asymptotically exact (with *z* increasing) expression for the distribution function $F(h)$ of the absolute maximum of the random field $L(\eta, \kappa)$ within the definitional domain Λ set by the conditions $\eta \in [1, \eta_{\max}]$, $\kappa \in [1, \kappa_{\max}]$. Using this expression makes it easy to write down the distribution function $F(h)$ for arbitrary definitional domain $\eta \in [\eta_{\min}, \eta_{\max}]$, $\kappa \in [\kappa_{\min}, \kappa_{\max}]$.

We take into account that within the domain Λ the mathematical expectation (8) of the random field $L(\eta, \kappa)$ reaches the absolute maximum at the point (1,1), while the field dispersion at this point is minimum. The field mathematical expectation decreases with η and κ increasing proportionally to *z*ηκ, while the field dispersion increases proportionally to the ηκ. Therefore, under $z \gg 1$, the position of the absolute maximum of the field $L(\eta, \kappa)$ is located in the small neighborhood of the point (1,1). Besides, if $z \to \infty$, then the coordinates (η_m, κ_m) of the field absolute maximum position converge to the values $\eta = 1$, $\kappa = 1$ in mean square [13]. Then, in order to find the asymptotically exact (with *z* increasing) expression for the distribution function $F(h)$ of the absolute maximum of the field $L(\eta, \kappa)$, it is sufficient to study the behavior of its mathematical expectation and covariance function in the small neighborhood of the point (1,1). Under $\delta = \max(|\eta - 1|, |\kappa - 1|) \to 0$, the mathematical expectation (8) is represented in the form of

$$S(\eta, \kappa) = S_1(\eta - 1) + S_1(\kappa) + o(\delta), \quad S_1(t) = -zt. \quad (10)$$

In turn, if $\delta_R = \max(|\eta_1 - 1|, |\eta_2 - 1|, |\kappa_1 - 1|, |\kappa_2 - 1|) \to 0$, then the correlation function (9) takes the form

$$R(\eta_1, \eta_2, \kappa_1, \kappa_2) = R_1(\eta_1 - 1, \eta_2 - 1) + R_1(\kappa_1, \kappa_2) + o(\delta_R), \quad (11)$$

$$R_1(t_1, t_2) = \min(t_1, t_2). \quad (12)$$

We designate $L_i(t)$, $i = 1,2$ as statistically independent and jointly Gaussian random processes with the mathematical expectations $S_1(t)$ (10) and the covariance functions $R_1(t_1, t_2)$ (12). From Eqs. (8)-(12), it follows that the mathematical expectations and the correlation functions of Gaussian random fields $L(\eta, \kappa)$ and $L^*(\eta, \kappa) = L_1(\eta - 1) + L_2(\kappa)$ coincide asymptotically, if $\delta \to 0$, $\delta_R \to 0$. Therefore, under $\delta \to 0$, Gaussian random field $L(\eta, \kappa)$ converges in distribution to the sum of the statistically independent Gaussian random processes $L_1(\eta - 1)$ and $L_2(\kappa)$. Thus, similarly Eq. (4), the distribution function $F(h)$ of the absolute maximum of the random field $L(\eta, \kappa)$ can be presented as follows

$$\begin{aligned} F(h) &= P\left[\sup_{\eta \in \Lambda_1} L_1(\eta) + \sup_{\kappa \in \Lambda_2} L_2(\kappa) < h\right] = \\ &= \int_{-\infty}^{\infty} F_2(h - x) w_1(x) dx, \end{aligned} \qquad (13)$$

where $F_1(x) = P\left[\sup_{\eta \in \Lambda_1} L_1(\eta) < x\right]$, $F_2(x) = P\left[\sup_{\kappa \in \Lambda_2} L_2(\kappa) < x\right]$ are the distribution functions of the absolute maxima of the random processes $L_1(\eta)$, $L_2(\kappa)$; $w_1(x) = dF_1(x)/dx$ is the corresponding probability density of the absolute maximum of the random process $L_1(\eta)$; and Λ_1, Λ_2 are the intervals for the possible values of the η and κ parameters set by the conditions $\eta \in [0, \eta_{\max} - 1]$, $\kappa \in [1, \kappa_{\max}]$, respectively. The random processes $L_1(\eta)$ and $L_2(\kappa)$ are the continuous Gaussian Markov diffusion ones [5] with the $(-z)$ drift coefficients and the unity diffusion coefficients. The distributions of the absolute maxima of such processes are already known from [9, 10] and determined as

$$F_1(x)=\begin{cases}\Phi(zd_1+x/d_1)-\exp(-2zx)\Phi(zd_1+x/d_1), & x\ge 0,\\ 0, & x<0,\end{cases} \qquad (14)$$

$$F_2(x)=\frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left[-\frac{(y-x-z)^2}{2}\right]\times$$
$$\times\left[\Phi\left(zd_2+\frac{y}{d_2}\right)-\exp(-2zy)\Phi\left(zd_2-\frac{y}{d_2}\right)\right]dy,$$

where $d_1=\sqrt{\eta_{\max}-1}$, $d_2=\sqrt{\kappa_{\max}-1}$. Then, from Eq. (13) taking into account Eq. (14), we have

$$F(h)=\frac{1}{\sqrt{2\pi}}\int_0^\infty\left\{\sqrt{\frac{2}{\pi}}\frac{1}{d_1}\exp\left[-\frac{\left(x+zd_1^2\right)^2}{2d_1^2}\right]+2z\times\right.$$
$$\left.\times\exp(-2zx)\Phi\left(zd_1-\frac{x}{d_1}\right)\right\}\left\{\int_0^\infty\exp\left[-\frac{(y-h+x-z)^2}{2}\right]\times\right. \qquad (15)$$
$$\left.\times\left[\Phi\left(zd_2+\frac{y}{d_2}\right)-\exp(-2zy)\Phi\left(zd_2-\frac{y}{d_2}\right)\right]dy\right\}dx.$$

The accuracy of the expression (15) increases with z.

In order to establish the borders of applicability of the asymptotically exact formula (15) for the finite values of z, we carry out the statistical computer simulation determining the value of the absolute maximum of the generalized Wiener random field $L(\eta,\kappa)$ with the mathematical expectation (8) and the covariance function (9). During the simulation, the samples $X_{ij}=L(i\Delta, j\Delta)=\Delta\sum_{l=1}^{i}\sum_{k=1}^{j}n_{lk}-zij\Delta^2$, $i,j=1,2,\ldots$ of the random field are formed with the discretization step $\Delta=0.005$ by the η and κ variables. Here, as above, n_{lk} are independent Gaussian random numbers with the zero mathematical expectation and the unity dispersion. The value X_m of the absolute maximum of the field $L(\eta,\kappa)$ is found as the greatest value X_{ij} for all $i\in[\{1/\Delta\},\{\eta_{\max}/\Delta\}]$, $j\in[\{1/\Delta\},\{\kappa_{\max}/\Delta\}]$. The experimental values for the probabilities $\alpha(h)$ and $F(h)$ for the various thresholds h are determined as the relative frequencies of the variable X_m exceeding and not exceeding the thresholds, respectively.

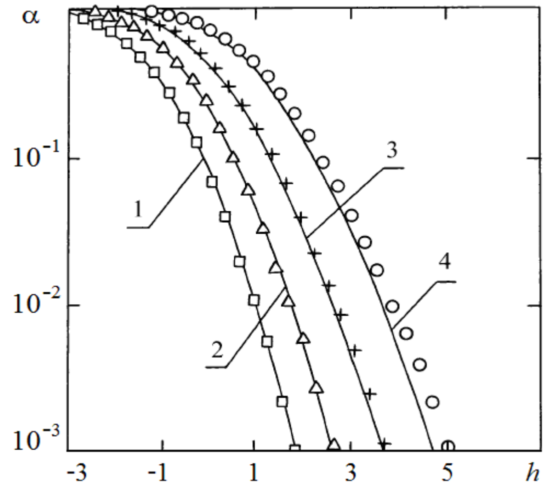

Figure 2: The probability of the absolute maximum of the generalized Wiener random field exceeding the threshold under $\eta_{\max}=\kappa_{\max}=2$.

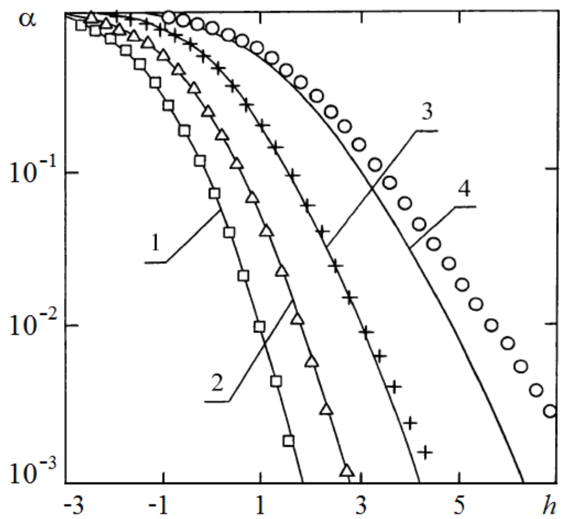

Figure 3: The probability of the absolute maximum of the generalized Wiener random field exceeding the threshold under $\eta_{\max}=\kappa_{\max}=4$.

In Fig. 2 the experimental values for the probability $\alpha(h)=1-F(h)$ are presented obtained during the processing of no less than 10^4 realizations of the random field under $\eta_{\max}=\kappa_{\max}=2$ and in Fig. 3 – under $\eta_{\max}=\kappa_{\max}=4$. In that Figures, by continuous lines the corresponding theoretical dependences of $\alpha(h)$ are drawn calculated by the formula (15). The confidence limits coincide with the corresponding ones in Fig. 1. Curves 1 and squares correspond to $z=2$, curves 2 and triangles – to $z=1.5$, curves 3 and crosses – to $z=1$, curves 4 and circles – to $z=0.5$. From Figs. 2, 3 and other simulation results, it follows that the theoretical formula (15) is already a successful approximation of the experimental data under $z\ge 0.5$.

The Distribution Of The Absolute Maximum Of The Generalized Wiener Random Field With Piecewise Constant Drift And Diffusion

Let us consider the generalized Wiener random field $L(\eta,\kappa)$, $\eta\geq 0$, $\kappa\geq 0$ with the mathematical expectation

$$S(\eta,\kappa)=(z_1+z_2)\min(1,\eta)\min(1,\kappa)-z_2\eta\kappa, \quad z_1>0, \quad z_2>0 \tag{16}$$

and the covariance function

$$R(\eta_1,\eta_2,\kappa_1,\kappa_2)=(1-g)\min(\eta_1,\eta_2)\min(\kappa_1,\kappa_2)+ + g\min(1,\eta_1,\eta_2)\min(1,\kappa_1,\kappa_2), \quad g<1. \tag{17}$$

We see that the sections $v_1(\eta)=L(\eta,\kappa^*)$, $\kappa^*=\text{const}$ and $v_2(\kappa)=L(\eta^*,\kappa)$, $\eta^*=\text{const}$ of the field $L(\eta,\kappa)$ by the η and κ variables are Gaussian Markov diffusion random processes with the piecewise constant drift and diffusion coefficients [5].

Let us find the asymptotically exact (with z_1 and z_2 increasing) expression for the distribution function $F(h)$ of the absolute maximum of the random field $L(\eta,\kappa)$ within the definitional domain Λ set by the conditions $\eta\in[\eta_{\min},\eta_{\max}]$, $\kappa\in[\kappa_{\min},\kappa_{\max}]$. We take into account that within the domain Λ the mathematical expectation (16) of the random field $L(\eta,\kappa)$ reaches the absolute maximum at the point (1,1). Indeed, $S(\eta,\kappa)=z_1\eta\kappa$, if $\eta<1$, $\kappa<1$, while $S(\eta,\kappa)=z_1-z_2(\eta\kappa-1)$, if $\eta\geq 1$, $\kappa\geq 1$. The field dispersion increases proportionally to the $b\eta\kappa$ with η and κ, where $b\leq 1$. Therefore, similarly to [9, 10, 12], we can show that under $z_1>>1$, $z_2>>1$ the position of the absolute maximum of the field $L(\eta,\kappa)$ is located in the small neighborhood of the point (1,1). Besides, if $z_1\to\infty$, $z_2\to\infty$, then the coordinates (η_m,κ_m) of the field absolute maximum position converge to the values $\eta=1$, $\kappa=1$ in mean square [13]. Then, in order to find the asymptotically exact (with increasing z_i, $i=1,2$) expression for the distribution function $F(h)$ of the absolute maximum of the field $L(\eta,\kappa)$, it is sufficient to study the behavior of its mathematical expectation and covariance function in the small neighborhood of the point (1,1). Under $\delta=\max(|\eta-1|,|\kappa-1|)\to 0$, the mathematical expectation (16) is written down in the form of

$$S(\eta,\kappa)=S_2(\eta-1/2)+S_2(\kappa-1/2)+o(\delta), \tag{18}$$

$$S_2(t)=\begin{cases}\min(1/2,t)-z_2t, & t\geq 0,\\ 0, & t<0.\end{cases} \tag{19}$$

In turn, if $\delta_R=\max(|\eta_1-1|,|\eta_2-1|,|\kappa_1-1|,|\kappa_2-1|)\to 0$, then the correlation function (17) permits the presentation

$$R(\eta_1,\eta_2,\kappa_1,\kappa_2)=R_2(\eta_1-1/2,\eta_2-1/2)+ + R_2(\kappa_1-1/2,\kappa_2-1/2)+o(\delta_R), \tag{20}$$

$$R_1(t_1,t_2)=\max[0,(1-g)\min(t_1,t_2)+g\min(1/2,t_1,t_2)]. \tag{21}$$

We designate $L_i(t)$, $i=1,2$ as statistically independent and jointly Gaussian random processes with the mathematical expectations $S_2(t)$ (19) and the covariance functions $R_2(t_1,t_2)$ (21).

Such processes are identically equal to zero under $t<0$, while under $t\geq 0$ they are continuous Gaussian Markov diffusion random processes with the piecewise constant drift and diffusion coefficients [5]. From Eqs. (16)-(18), (20), it follows that the mathematical expectations and the correlation functions of Gaussian random fields $L(\eta,\kappa)$ and $L^*(\eta,\kappa)=L_1(\eta-1/2)+L_2(\kappa-1/2)$ coincide asymptotically when $\delta\to 0$, $\delta_R\to 0$. Therefore, under $\delta\to 0$ Gaussian random field $L(\eta,\kappa)$ converges in distribution to the sum of the statistically independent Gaussian random processes $L_1(\eta-1/2)$ and $L_2(\kappa-1/2)$. Thus, similarly Eq. (4), the distribution function $F(h)$ of the absolute maximum of the random field $L(\eta,\kappa)$ can be presented as follows

$$F(h)=P\left[\sup_{\eta\in\Lambda_1}L_1(\eta)+\sup_{\kappa\in\Lambda_2}L_2(\kappa)<h\right]= =\int_{-\infty}^{\infty}F_2(h-x)w_1(x)dx, \tag{22}$$

where $F_1(x)=P\left[\sup_{\eta\in\Lambda_1}L_1(\eta)<x\right]$, $F_2(x)=P\left[\sup_{\kappa\in\Lambda_2}L_2(\kappa)<x\right]$ are the distribution functions of the absolute maxima of the random processes $L_1(\eta)$, $L_2(\kappa)$; $w_1(x)=dF_1(x)/dx$ is the corresponding probability density of the absolute maximum of the random process $L_1(\eta)$; and Λ_1, Λ_2 are the intervals for the possible values of the η and κ parameters set by the conditions $\eta\in[\eta_{\min}-1/2,\eta_{\max}-1/2]$, $\kappa\in[\kappa_{\min}-1/2,\kappa_{\max}+1/2]$, respectively. By applying the results obtained in [9, 10], we come to

$$F_i(x)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\left[-\left(\frac{y}{\sqrt{2}}-x+\frac{z_1}{2}\right)^2\right]\Psi_i(y)V_i(x,y)dy, \tag{23}$$

$$\Psi_i(y)=\Phi\left(\frac{z_2d_i}{\sqrt{1-g}}+\frac{y}{d_i\sqrt{2(1-g)}}\right)- -\exp\left(-\sqrt{2}\frac{yz_2}{1-g}\right)\Phi\left(\frac{z_2d_i}{\sqrt{1-g}}-\frac{y}{d_i\sqrt{2(1-g)}}\right),$$

$$V_i(x,y)=\Phi\left(x\sqrt{\frac{2(1-\rho_i)}{\rho_i}}+y\sqrt{\frac{\rho_i}{1-\rho_i}}\right)-$$
$$-\exp\left(-2\sqrt{2}yx\right)\Phi\left(x\sqrt{\frac{2(1-\rho_i)}{\rho_i}}-y\sqrt{\frac{\rho_i}{1-\rho_i}}\right),\quad i=1,2,$$

where $\rho_1=\max(0,2\eta_{\min}-1)$, $\rho_2=\max(0,2\kappa_{\min}-1)$, $d_1=\sqrt{\eta_{\max}-1}$, $d_2=\sqrt{\kappa_{\max}-1}$.

Differentiating the expression (23) produces

$$w_i(x)=\frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left[-\left(\frac{y}{\sqrt{2}}-x+\frac{z_1}{2}\right)^2\right]\Psi_i(y)\times \quad (24)$$
$$\times\left[\left(\sqrt{2}y-2x+z_1\right)V_i(x,y)+\omega_i(x,y)\right]dy,$$

$$\omega_i(x,y)=2\sqrt{2}y\exp\left(-2\sqrt{2}yx\right)\Phi\left(x\sqrt{\frac{2(1-\rho_i)}{\rho_i}}-y\sqrt{\frac{\rho_i}{1-\rho_i}}\right).$$

By substituting Eqs. (23), (24) in Eq. (22), we find the asymptotically exact (with increasing z_1 and z_2) expression for the distribution function $F(h)$ of the absolute maximum of the random field.

In order to establish the borders of applicability of the asymptotically exact formulas (22)-(24) for the finite values of z_i, $i=1,2$, we carry out the statistical computer simulation of the value of the absolute maximum of the generalized Wiener random field $L(\eta,\kappa)$ with the mathematical expectation (16) and the covariance function (17) under $z_1=z_2=z$, $g=0$. During the simulation, based on the independent Gaussian random numbers n_{lk} with the zero mathematical expectations and the unity dispersions, the samples

$$X_{ij}=L(i\Delta,j\Delta)=\Delta\sum_{l=1}^{i}\sum_{k=1}^{j}n_{lk}+$$
$$+z\Delta^2\left[2\min(i,\{1/\Delta\})\min(j,\{1/\Delta\})-ij\right],\quad i,j=1,2,\ldots$$

of the random field are formed with the discretization step $\Delta=0.005$ by the η and κ variables. The value X_m of the absolute maximum of the field $L(\eta,\kappa)$ is found as the greatest value X_{ij} for all $i\in[\{\eta_{\min}/\Delta\},\{\eta_{\max}/\Delta\}]$, $j\in[\{\kappa_{\min}/\Delta\},\{\kappa_{\max}/\Delta\}]$. The experimental values for the probabilities $\alpha(h)$ and $F(h)$ for the various thresholds h are determined as the relative frequencies of the variable X_m exceeding and not exceeding the thresholds, respectively.

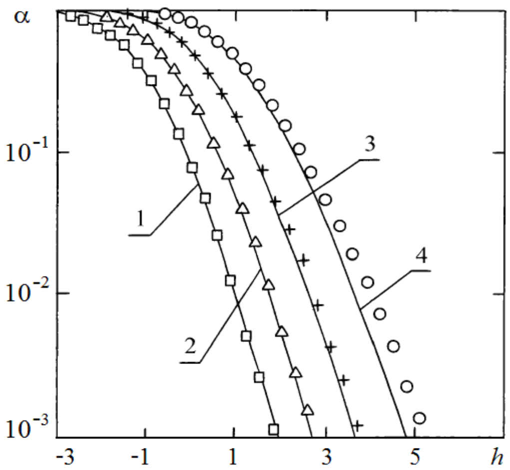

Figure 4: The probability of the absolute maximum of the generalized Wiener random field exceeding the threshold under $\eta_{\min}=\kappa_{\min}=1/2$, $\eta_{\max}=\kappa_{\max}=2$.

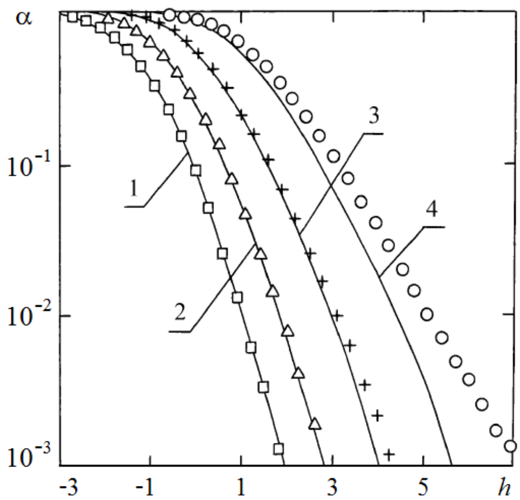

Figure 5: The probability of the absolute maximum of the generalized Wiener random field exceeding the threshold under $\eta_{\min}=\kappa_{\min}=1/2$, $\eta_{\max}=\kappa_{\max}=3$.

In Fig. 4 the experimental values for the probability $\alpha(h)=1-F(h)$ are presented obtained during the processing of no less than 10^4 realizations of the random field under $\eta_{\min}=\kappa_{\min}=1/2$, $\eta_{\max}=\kappa_{\max}=2$ and in Fig. 5 – under $\eta_{\min}=\kappa_{\min}=1/2$, $\eta_{\max}=\kappa_{\max}=3$. Here, by continuous lines the corresponding theoretical dependences $\alpha(h)$ are also drawn calculated by the formulas (22)-(24). The confidence limits coincide with the corresponding ones in Fig. 1. Curves 1 and squares correspond to $z=2$, curves 2 and triangles – to $z=1.5$, curves 3 and crosses – to $z=1$, curves 4 and circles – to $z=0.5$. From Figs. 2, 3 and other simulation results, it follows that the theoretical formulas (22)-(24) start successfully approximating the experimental data under $z_i\geq 0.5$, $i=1,2$.

CONCLUSION

To calculate the characteristics of the generalized Wiener random fields absolute maximum values, there can be used the limiting laws of the absolute maximum distribution obtained for the case of the unlimited increase in the threshold and when Wiener field is statistically equivalent to the sum of the two independent Wiener random processes. The asymptotic formulas for the characteristics thus determined turn out to be more accurate than the common ones, if the threshold values are finite. While comparing the introduced expressions with the experimental data produced during the simulation in a number of particular cases we come to the conclusion that that they successfully describe the true distributions in terms of a wide range of the random fields parameters values.

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