Construction of Control Systems with High Potential of Robust Stability in the Case of Catastrophe Elliptical Umbilic

Mamyrbek A. Beisenbi
L.N Gumilyov Eurasian National University, Department of System Analysis and Control, 2 Satpayev St., Astana, 010008, Republic of Kazakhstan.

Salamat T. Suleimenova
L.N Gumilyov Eurasian National University, Department of System Analysis and Control, 2 Satpayev St., Astana, 010008, Republic of Kazakhstan.

Vladimir V. Nikulin
State University of New York, Department of Electrical and Computer Engineering, Binghamton, New York 13902-600, USA.

Dana K. Satybaldina
L.N Gumilyov Eurasian National University, Department of System Analysis and Control, 2 Satpayev St., Astana, 010008, Republic of Kazakhstan.

Abstract
This article is cover the study of control systems with a high potential of robust stability is formed in class of three-parameter structurally stable mapping (elliptical umbilic) for objects with m inputs and n outputs.

Research of robust stability of control systems based on the construction of Lyapunov function. Lyapunov’s function is formed as vector-function, antigradient of which is given by velocity vector’s components. Stability region of steady states of the system is obtained in the form of simple inequations according to uncertain parameters of control object and selected controller parameters.

Keywords: Lyapunov’s function, robust stability, elliptical umbilic, three-parameter structurally stable mapping.

Introduction
The development process of some technical device pose a deep problem in science and technology. System can declined from given program motion and execute other motion under real performance. One of the reasons for these actions may be the availability of small deviations from the initial state of a given movement or the availability of small forces unaccounted under modeling systems. One of the main problems that take place already during the construction of the system is stabilization problem of their movements.

This research focuses on current issues to the construction of robust stability of dynamic objects control systems with uncertain parameters with approach to the construction of the control system in the class of structurally stable mappings allowing increase of the potential of robust stability of the system. Research of robust stability is based on a new approach resulting from the geometric interpretation of Lyapunov’s theorem [1, 2]. Lyapunov’s function is synthesized as vector-function antigradient of which is given by velocity vector’s components in the form of tensor.

Consider the control system with m input and n output, described by the equation of state

\[ x = Ax + Bu \]
\[ y = Cx \] (1)

Where \( x(t) \in \mathbb{R}^n \) – control object’s state vector; \( u(t) \in \mathbb{R}^m \) – scalar function of control actions; \( A \in \mathbb{R}^{nxn} \) - control object’s matrix with uncertain parameters, \( B \in \mathbb{R}^{nxm} \) – control matrix.

Law of control is given in the form of a sum of three-parameter structurally stable mappings (catastrophe elliptical umbilic)
System (1) is written in the expanded form as:

\[ x_i = -b_{1i}x_i^3 + b_{2i}x_i x_i^2 - b_{1i}k_{12}(x_i^2 + x_j^2) + (a_{1i} + b_{1i}k_{1i})x_i + (a_{12} + b_{12}k_{2j})x_j + a_{13}x_j + \ldots + a_{1n}x_n, \]

\[ x_j = -b_{2j}x_j^3 + b_{2j}x_j x_j^2 - b_{2j}k_{12}(x_i^2 + x_j^2) + (a_{2j} + b_{2j}k_{1j})x_j + (a_{22} + b_{22}k_{2j})x_j + a_{23}x_j + \ldots + a_{2n}x_n, \]

\[ x_k = -b_{3k}x_k^3 + b_{3k}x_k x_k^2 - b_{3k}k_{12}(x_i^2 + x_j^2) + a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n, \]

\[ x_l = -b_{4l}x_l^3 + b_{4l}x_l x_l^2 - b_{4l}k_{12}(x_i^2 + x_j^2) + a_{41}x_1 + a_{42}x_2 + \ldots + a_{4n}x_n, \]

\[ \begin{array}{c}
\vdots \\
\end{array} \]

\[ x_{n-1} = -b_{n-1,n-1}x_{n-1}^3 + 3b_{n-1,n-1}x_{n-1}^2 - b_{n-1,n-1}k_{12}(x_i^2 + x_j^2) + a_{n-1,1}x_1 + a_{n-1,2}x_2 + \ldots + a_{n-1,n}x_n + (a_{n-1,n} + b_{n-1,n}k_{n,n})x_n, \]

\[ x_n = -b_{n,n}x_n^3 + 3b_{n,n}x_n x_n^2 - b_{n,n}k_{12}(x_i^2 + x_j^2) + a_{n,1}x_1 + a_{n,2}x_2 + \ldots + (a_{n,n} + b_{n,n}k_{n,n})x_n, \]

(3)

Steady states condition of the system (3) are defined by solution of the equations:

\[ \begin{array}{c}
\begin{aligned}
-b_{1i}x_i^3 + 3b_{1i}x_i x_i^2 - b_{1i}k_{12}(x_i^2 + x_j^2) + (a_{1i} + b_{1i}k_{1i})x_i \\
+ (a_{12} + b_{12}k_{2j})x_j + a_{13}x_j + \ldots + a_{1n}x_n = 0 \\
-b_{2j}x_j^3 + 3b_{2j}x_j x_j^2 - b_{2j}k_{12}(x_i^2 + x_j^2) + (a_{2j} + b_{2j}k_{1j})x_j + (a_{22} + b_{22}k_{2j})x_j + a_{23}x_j + \ldots + a_{2n}x_n = 0 \\
-b_{3k}x_k^3 + 3b_{3k}x_k x_k^2 - b_{3k}k_{12}(x_i^2 + x_j^2) + a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n + (a_{3n} + b_{3n}k_{n,n})x_n = 0 \\
\vdots \\
-b_{n-1,n-1}x_{n-1}^3 + 3b_{n-1,n-1}x_{n-1} x_{n-1}^2 - b_{n-1,n-1}k_{12}(x_i^2 + x_j^2) + a_{n-1,1}x_1 + a_{n-1,2}x_2 + \ldots + a_{n-1,n}x_n + (a_{n-1,n} + b_{n-1,n}k_{n,n})x_n = 0 \\
-b_{n,n}x_n^3 + 3b_{n,n}x_n x_n^2 - b_{n,n}k_{12}(x_i^2 + x_j^2) + a_{n,1}x_1 + a_{n,2}x_2 + \ldots + (a_{n,n} + b_{n,n}k_{n,n})x_n = 0 \\
\end{aligned}
\end{array} \]

(4)

Stationary states of the system (4) are defined as:

\[ x_{1j} = 0, x_{2j} = 0, \ldots, x_{nj} = 0 \]

(5)

Other stationary states will be determined by solution of the equations:

\[ -b_{ni}k_{i,j+1}x_i + a_{ni} + b_{ni}k_{i} = 0, \quad x_{ji} = 0 \quad \text{for} \quad i \neq j, i = 1, n \]

(6)

The solution of equation (6) will be written as follows:

\[ x_{ji} = \frac{a_{ni}}{b_{ni}k_{i,j+1}} + \frac{k_{i}}{k_{i,j+1}} x_{ji} = 0 \quad \text{for} \quad i \neq j, i = 1, n \]

(7)

Stability of the stationary states

Research the robust stability of the stationary state (7) based on the method of Lyapunov’s function. State equations (3) can be represented in the deviations with respect to the stationary state (7). For this we may use formal description allowing representation of the equation (3) in the deviations with respect to the stationary state. The state equation (3) with respect to the stationary state will be written as:

\[ \begin{array}{c}
\begin{aligned}
-x_{1j} = -b_{1j}x_{1j}^3 + 3b_{1j}x_{1j} x_{1j}^2 - b_{1j}k_{12}(x_i^2 + x_j^2) + 6a_{1j} + b_{1j}k_{1j}x_{1j} - (a_{1i} + b_{1i}k_{1i})x_i + (a_{12} + b_{12}k_{2j})x_j + a_{13}x_j + \ldots + a_{1n}x_n \\
-x_{2j} = -b_{2j}x_{2j}^3 + 3b_{2j}x_{2j} x_{2j}^2 - b_{2j}k_{12}(x_i^2 + x_j^2) + (a_{2i} + b_{2i}k_{1i})x_i + (a_{22} + b_{22}k_{2j})x_j + a_{23}x_j + \ldots + a_{2n}x_n \\
\vdots \\
-x_{n-1,j} = -b_{n-1,n-1}x_{n-1,j}^3 + 3b_{n-1,n-1}x_{n-1,j} x_{n-1,j}^2 - b_{n-1,n-1}k_{12}(x_i^2 + x_j^2) + a_{n-1,1}x_1 + a_{n-1,2}x_2 + \ldots + a_{n-1,n}x_n + (a_{n-1,n} + b_{n-1,n}k_{n,n})x_n \\
-x_{n,j} = -b_{n,n}x_{n,j}^3 + 3b_{n,n}x_{n,j} x_{n,j}^2 - b_{n,n}k_{12}(x_i^2 + x_j^2) + a_{n,1}x_1 + a_{n,2}x_2 + \ldots + (a_{n,n} + b_{n,n}k_{n,n})x_n \\
\end{aligned}
\end{array} \]

(8)
The components of the vector of antigradient from Lyapunov’s vector function $V(x) = (V_1(x), V_2(x), \ldots)$ taken as:

$$\frac{\partial V_i(x)}{\partial x_1} = b_{i1}k_{1i}x_1^2 - 2b_{i1}x_1x_i - 3\frac{a_{1i} + b_{1i}k_i}{k_i}x_i + (a_{1i} + b_{1i}k_i)x_i$$

$$\frac{\partial V_i(x)}{\partial x_2} = b_{i2}x_2^2 - b_{i2}x_1x_i + b_{i1}k_{1i}x_i^2 - 3\frac{a_{1i} + b_{1i}k_i}{k_i}x_i + (a_{12} + b_{12}k_i)x_2$$

$$\frac{\partial V_i(x)}{\partial x_3} = -a_{i1}x_3, \quad \frac{\partial V_i(x)}{\partial x_4} = -a_{i4}x_4, \ldots, \quad \frac{\partial V_i(x)}{\partial x_{i-1}} = -a_{i,i-1}x_{i-1}, \quad \frac{\partial V_i(x)}{\partial x_n} = -a_{in}x_n.$$
Total time derivative from components of Lyapunov’s vector-function is equal to:

\[
d\frac{V_1}{V_1}(x) - = \left[ b_{12}k_{12}x_1^2 - 2b_{11}x_2x_1^2 - 3 \frac{a_{11} + b_{11}k_1}{k_{12}} x_2x_1 + (a_{11} + b_{11}k_1)x_1 \right]^2 - \\
- \left[ b_{12}x_1^2 - 2b_{11}x_1^2 + b_{11}k_{12}x_2^2 - 3 \frac{a_{11} + b_{11}k_1}{k_{12}} x_1x_2 + (a_{12} + b_{11}k_2)x_2 \right]^2 - \\
-a_{13}x_3^2 - ... - a_{1n}x_n^2
\]

\[
d\frac{V_2}{V_2}(x) = - \left[ b_{22}k_{12}x_1^2 - 2b_{22}x_2x_1^2 + (a_{21} + b_{22}k_1)x_1 \right]^2 - \\
- \left[ b_{22}x_1^2 - 2b_{22}x_2x_1^2 + b_{22}k_{12}x_2^2 - (a_{22} + b_{22}k_2)x_2 \right]^2 - a_{23}x_3^2 - ... - a_{2n}x_n^2
\]

\[
d\frac{V_{n-1}}{V_{n-1}}(x) = -a_{n-1,1}x_1^2 - a_{n-1,2}x_2^2 - ... - \left[ b_{n-1,n-1}k_{n-1,n}x_{n-1}^2 - 2b_{n-1,n-1}x_{n-1} \right]^2 - \\
- \left[ b_{n-1,n-1}k_{n-1,n}x_{n-1}^2 - b_{n-1,n-1}x_{n-1}x_{n-1} - 3 \frac{a_{n-1,n-1} + b_{n-1,n-1}k_{n-1}}{k_{n-1,n}} x_{n-1}x_{n-1} \right]^2 - \\
- \left( a_{n-1,n-1} - b_{n-1,n-1}k_{n-1}x_{n-1} \right)x_n + b_{n-1,n-1}x_n^2
\]

\[
d\frac{V_n}{V_n}(x) = -a_{n,1}x_1^2 - a_{n,2}x_2^2 - ... - \left[ b_{m,n}k_{m,n}x_{n-1}^2 + (a_{n,n} + b_{m,n})x_{n-1} \right]^2 - \\
- \left[ b_{m,n}x_n^3 - b_{m,n}x_n^2 + b_{m,n}k_{n-1,n}x_n^2 - (a_{n,n} + b_{m,n})x_n \right]^2
\]

By components of the vector gradient of Lyapunov’s vector-function there may be built the components of Lyapunov’s vector-function in the following form

\[
V_1(x) = - \frac{1}{3} b_{11}k_{12}x_1^3 - \frac{3}{2} \frac{a_{11} + b_{11}k_1}{k_{12}} x_2x_1^2 + \frac{1}{2} (a_{11} + b_{11}k_1)x_1^2 - \\
- \frac{2}{3} b_{11}x_1^2 + \frac{1}{4} b_{11}x_1^4 - \frac{1}{2} b_{11}x_1^2x_1^2 + \frac{1}{3} b_{11}k_{12}x_2 - \frac{3}{2} \frac{a_{11} + b_{11}k_1}{k_{12}} x_1x_2 \] - \\
- \frac{1}{2} (a_{11} + b_{11}k_1)x_2 - \frac{1}{2} a_{13}x_3^2 - ... - \frac{1}{2} a_{1n}x_n^2
\]

\[
V_2(x) = - \frac{1}{3} b_{22}k_{12}x_1^3 + \frac{1}{2} (a_{21} + b_{22}k_1)x_1^2 + \frac{1}{4} b_{22}x_2^4 - \frac{2}{3} b_{22}x_2x_1^3 - \\
- \frac{1}{2} b_{22}x_1^2 + \frac{1}{3} b_{22}k_{12}x_2 - \frac{1}{2} (a_{22} + b_{22}k_2)x_2^2 - \frac{1}{2} a_{23}x_3^2 - ... - \frac{1}{2} a_{2n}x_n^2
\]
\[ V_{n-1}(x) = -\frac{1}{2}a_{n-1}x_1^2 - \ldots - \frac{1}{2}a_{n-1,n-2}x_{n-2}^2 + \frac{1}{3}b_{n-1,n-1}k_{n-1,n-1}x_{n-1}^3 - \]
\[ \frac{2}{3}b_{n-1,n-1}x_n x_{n-1}^3 - \frac{3}{2}a_{n-1,n-1} + b_{n-1,n-1}k_{n-1,n-1}x_n x_{n-1}^2 - \]
\[ \frac{1}{2}b_{n-1,n-1}x_n^2 x_{n-1}^2 + \]
\[ \frac{1}{2}(a_{n-1,n-1} + b_{n-1,n-1}k_{n-1,n-1})x_n^2 - \]
\[ \frac{3}{2}a_{n-1,n-1} + b_{n-1,n-1}k_{n-1,n-1}x_n x_{n-1}^2 + \]
\[ \frac{1}{3}b_{n-1,n-1}k_{n-1,n-1}x_n^3 + \frac{1}{4}b_{n-1,n-1}x_n^4 - \]
\[ \frac{1}{2}(a_{n-1,n} + b_{n-1,n-1}k_n)x_n^2 - \]
\[ \frac{1}{2}(a_{n,n-1} + b_{n-1,n}k_{n-1,n})x_n^2 + \]
\[ \frac{1}{4}(b_{n,n-1} + b_{n-1,n}k_{n-1,n})k_{n-1,n}x_n^4 + \]
\[ \frac{1}{2}(b_{n-1,n-1} + b_{n+1,n}k_{n+1,n})x_n x_{n-1}^3 + \]
\[ \frac{1}{2}(b_{n,n-1} + b_{n+1,n}k_{n+1,n})x_n^2 x_{n-1}^2 + \]
\[ \frac{3}{2} \left( \frac{a_{n-1,n} + b_{n-1,n-1}k_n}{k_{n-1,n}} \right) x_n x_{n-1} \left( x_n + x_{n-1} \right) - \]
\[ \frac{1}{2} \left( \frac{b_{n-1,n} + b_{n,n-1}k_n}{k_{n-1,n}} \right) x_n x_{n-1} \left( x_n + x_{n-1} \right) - \]
\[ \frac{1}{2} \left( \frac{b_{n,n-1} + b_{n-1,n}k_{n-1,n}}{k_{n-1,n}} \right) x_n x_{n-1} \left( x_n + x_{n-1} \right) - \]
\[ \frac{1}{2} \left( \frac{b_{n-1,n} + b_{n,n-1}k_n}{k_{n-1,n}} \right) x_n x_{n-1} \left( x_n + x_{n-1} \right) - \]
\[ \frac{1}{2} \left( \frac{b_{n,n-1} + b_{n-1,n}k_{n-1,n}}{k_{n-1,n}} \right) x_n x_{n-1} \left( x_n + x_{n-1} \right) - \]
\[ (10) \]
Lyapunov’s function (10) can locally be represented in quadratic form. The matrix of quadratic form can be calculated as the Hessian matrix from Lyapunov’s vector function (9):

\[ \frac{\partial^2 V(x)}{\partial x_i \partial x_j} = (2a_{11} + b_{11}k_1 + 1)(b_{11} + b_{22})k_i + [a_{11} + a_{21} - (a_{31} + \ldots + a_{n,1})] \]

\[ \frac{\partial^2 V(x)}{\partial x_2 \partial x_2} = (2a_{22} + b_{22}k_1 + 3(a_{21} + b_{22}k_1)^2 - 1)(b_{11} + b_{22})k_2 - [a_{12} + a_{22} + \ldots + a_{n,2}] \]

\[ \frac{\partial^2 V(x)}{\partial x_2 \partial x_j} = 0 \text{, for } j \neq 2 \]

\[ \frac{\partial^2 V(x)}{\partial x_i \partial x_{n-1}} = \left(2a_{i,n-1} + b_{i,n-1}k_{n-1} + 1\right)b_{n-1,n-1}k_{n-1} + \left[a_{i,n-1} + a_{n-1,i} - (a_{i,n-1} + \ldots + a_{n-2,n-1})\right] \]

\[ \frac{\partial^2 V(x)}{\partial x_i \partial x_j} = 0 \text{, for } j \neq n-1 \]

\[ \frac{\partial^2 V(x)}{\partial x_n \partial x_n} = \left(2a_{n,n} + b_{n,n}k_n + 3(a_{n,n} + b_{n,n}k_n)^2 - 1\right)b_{n-1,n-1}k_{n-1}k_n - \left[a_{n} + a_{2n} + \ldots + a_{nn}\right] \]

\[ \frac{\partial^2 V(x)}{\partial x_n \partial x_j} = 0 \text{, for } j \neq n \]

Lyapunov’s function (10) may be represented in the quadratic form locally near of the stationary state.
\[ V(x) = \left( 2 \frac{a_{i1} + b_{i1} k_i}{k_i b_{i1}} + 1 \right) \left( b_{i1} + b_{22} \right) k_i + a_{i1} + a_{21} - \left( a_{i1} + \ldots + a_{n,1} \right) \right) x_1^2 + \]
\[ + \left( 2 \frac{a_{21} + b_{22} k_1}{k_2 b_{22}} + 3 \left( \frac{a_{21} + b_{22} k_1}{k_2 b_{22}} \right)^2 - 1 \right) \left( b_{12} + b_{22} \right) k_2 - \left( a_{12} + a_{22} + \ldots + a_{n,2} \right) \right) x_2^2 + \]
\[ + \ldots + \]
\[ + \left( 2 \frac{a_{n-1,n-1} + b_{n-1,n-1} k_{n-1}}{k_{n-1} b_{n-1,n-1}} + 1 \right) \left( b_{n-1,n-1} + b_{mn} \right) k_{n-1} + \]
\[ + a_{n-1,n-1} - \left( a_{n-1,n-1} + \ldots + a_{n-2,n-1} \right) \right) x_{n-1}^2 + \]
\[ + \left( 2 \frac{a_{n,n} + b_{mn} k_{n-1}}{k_{n} b_{mn}} + 3 \left( \frac{a_{n,n} + b_{mn} k_{n-1}}{k_{n} b_{mn}} \right)^2 - 1 \right) \left( b_{n-1,n-1} + b_{mn} \right) k_n - \]
\[ - \left( a_{1n} + a_{2n} + \ldots + a_{mn} \right) \right) x_n^2 \]

The criterions of stability of the stationary state (7) will be defined by positive definiteness of quadratic form (11) by the system of inequalities:
\[
\begin{align*}
2 \frac{a_{i1} + b_{i1} k_i}{k_i b_{i1}} + 1 & \left( b_{i1} + b_{22} \right) k_i + a_{i1} + a_{21} - \left( a_{i1} + \ldots + a_{n,1} \right) > 0 \\
2 \frac{a_{21} + b_{22} k_1}{k_2 b_{22}} + 3 \left( \frac{a_{21} + b_{22} k_1}{k_2 b_{22}} \right)^2 - 1 & \left( b_{12} + b_{22} \right) k_2 - \left( a_{12} + a_{22} + \ldots + a_{n,2} \right) > 0 \\
2 \frac{a_{n-1,n-1} + b_{n-1,n-1} k_{n-1}}{k_{n-1} b_{n-1,n-1}} + 1 & \left( b_{n-1,n-1} + b_{mn} \right) k_{n-1} + a_{n-1,n-1} + \left( a_{1n} + a_{2n} + \ldots + a_{mn} \right) > 0 \\
2 \frac{a_{n,n} + b_{mn} k_{n-1}}{k_{n} b_{mn}} + 3 \left( \frac{a_{n,n} + b_{mn} k_{n-1}}{k_{n} b_{mn}} \right)^2 - 1 & \left( b_{n-1,n-1} + b_{mn} \right) k_n - \left( a_{1n} + a_{2n} + \ldots + a_{mn} \right) > 0
\end{align*}
\]

The control system had built in class of three-parameter structurally-stable mappings for stationary state (7), is stable when you varying undetermined parameters of the object in the region (12).

**CONCLUSION**

In this paper developed a new approach for construction control systems with high potential of robustness. Structurally-stable mappings of the catastrophe theory, in particular, elliptical umbilic are used as the control laws. The use of the mappings helps to increase the range of robustness.

This paper describes a new idea – the increase of the range of robustness of control systems. Research of robust stability of control systems based on the construction of Lyapunov function. Lyapunov’s function is formed as vector-function, antigradient of which is given by velocity vector’s components. The results of this research can be used in the analysis and synthesis of methods of solving practical tasks related to robustness.
In result, it will help to increase the robust stability when designing control systems. Therefore, the quality of management of technological processes can be improved, which is especially desired when operating in presence of uncertainties and incomplete data.

REFERENCES


