International Journal of Applied Engineering Research ISSN 0973-4562 Volume 12, Number 16 (2017) pp. 5990-5994

Buckling Analysis of the Structures by Single Imposed Constraint Method

Prof. Dr. V. Agapov

Moscow State University of Civil Engineering (MGSU), 26, Yaroslavskoje highway, Moscow, 129337, Russia.

[1]*Orcid: 0000-0002-1749-5797*

Abstract

An iterative method for determining of the minimum critical parameter, which allows to reduce the buckling analysis of structure to its static analysis and thereby avoiding the eigenvalues problem solution, is proposed. The convergence of the method is substantiated. An algorithm of the minimal critical parameter determination by proposed method is described and the results of buckling analysis of structures of various types with the aid of developed by author program is presented.

Keywords: Equation of motion; Building structures; Finite elements; Buckling

INTRODUCTION

Several generations of scientists worked productively over the problem of the deformable systems stability (see [1], [2], [3] and [4]). Many different exact and approximate methods for determining of critical loads and buckling modes have been developed. Currently the finite element method (FEM) is used successfully for the buckling analysis (see [5], [6], [7] and [8]). At that the stability analysis is lead to the linear eigenvalue problem solution:

$$\left[K-\lambda K_{\sigma}\right]\{u\}=0\,, \qquad (1)$$

where $[K]$ – the linear stiffness matrix, $[K_{\sigma}]$ – initial stresses matrix, λ – eigenvalue and $\{u\}$ – eigenvector.

The problem (1) is quite time-consuming. In addition, for practical purposes, in most cases it is sufficient the knowledge of the minimum critical parameter. Therefore, some authors (see [4] and [6]) solve the equation (1) by direct iteration of one vector representing the simplest kind of power method for determining of the eigenvectors (see [9]). But as mentioned method leads to the maximum eigenvalue of the matrix iterated, then the equation (1) is first converted to the form:

$$\left[-\left(1/\lambda\right)[E]+[K]^{-1}[K_{\sigma}]\right]\{u\}=0\,, \qquad (2)$$

where $[E]$ – unit matrix.

Then the vector sequence is being built:

$$\{u\}^{(s)}=[K]^{-1}[K_{\sigma}]\{u\}^{(s-1)}\,. \qquad (3)$$

The desired value is obtained from the relationship:

$$1/\lambda_{\mathbf{min}}=\lim_{s\to\infty}\frac{u_i^{(s)}}{u_i^{(s-1)}}\,. \qquad (4)$$

The number s is determined by the required accuracy of calculating of the critical parameter. The algorithm for the problem solving with the use of the relations (2) - (4) turns out simple enough, but it includes a matrix $[K]$ inversion operation and multiplying of $[K]^{-1}$ and $[K_{\sigma}]$ matrices. For large orders of matrices, these operations can be quite time consuming, which is a drawback of this approach. In this paper the method of buckling analysis is proposed, which don't require the eigenvalue problem solution.

THE METHOD OF DETERMINING OF THE MINIMUM CRITICAL LOAD PARAMETER

For a given value of the load parameter λ the equilibrium equations of structure can be written as:

$$\left[K-\lambda K_{\sigma}\right]\{u\}=\{p\}, \qquad (5)$$

where $\{p\}$ - the vector of nodal loads.

Let's denote

$$[R]=\left[K-\lambda K_{\sigma}\right]. \qquad (6)$$

Equation (5) takes the form:

$$[R]\{u\}=\{p\}, \qquad (7)$$

or

$$\begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,k} & \cdots & r_{1,n-1} & r_{1,n} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,k} & \cdots & r_{2,n-1} & r_{2,n} \\ . & . & \cdots & . & \cdots & . & . \\ r_{i,1} & r_{i,2} & \cdots & r_{i,k} & \cdots & r_{i,n-1} & r_{i,n} \\ . & . & . & . & \cdots & . & . \\ r_{n-1,1} & r_{n-1,2} & \cdots & r_{n-1,k} & \cdots & r_{n-1,n-1} & r_{n-1,n} \\ r_{n,1} & r_{n,2} & \cdots & r_{n,k} & \cdots & r_{n,n-1} & r_{n,n} \end{bmatrix} \{u\} = \{P\} . \quad (7^*)$$

The coefficients $r_{i,k}$ (i=1,2...,n; k=1,2,...,n) in the equation (7) have the usual meaning of the displacement method equation system coefficients: $r_{i,k}$ is the reaction of i-th imposed constraint at unit displacement of k-th constraint.

Solving the equation (7) by the method of consistent elimination of unknowns, you can bring it to the form:

$$\begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,i} & \cdots & r_{1,n-1} & r_{1,n} \\ . & . & \cdots & . & \cdots & . & . \\ 0 & 0 & \cdots & r_{i,i}^{(i-1)} & \cdots & r_{i,n-1}^{(i-1)} & r_{i,n}^{(i-1)} \\ . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & r_{n-1,n-1}^{(n-2)} & r_{n-1,n}^{(n-2)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & r_{n,n}^{(n-1)} \end{bmatrix} \{u\} = \{P\}. \quad (8)$$

Elimination of the i- th unknown in the system of equations (7) physically means the removal of the i-th superimposed constraint (the constraints imposed on a given system to form the main system of displacement method are having in view). Thus, coefficients, $r_{2,2}^{(1)}$, $r_{2,3}^{(1)}$ etc. refer to a system with a one constraint removed, which is the same to a system with n-1 imposed constraints, the coefficients $r_{i,i}^{(i-1)}, r_{i,i+1}^{(i-1)}$ relate to a system with i-1 removed constraints etc. Note that coefficient $r_{n,n}^{(n-1)}$ is a reaction to a unit displacement in a system with a single superimposed constraint.

The criterion of instability is the equality of the determinant of the matrix in the left part of equation (8) to zero:

$$\det \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,i} & \cdots & r_{1,n-1} & r_{1,n} \\ . & . & \cdots & . & \cdots & . & . \\ 0 & 0 & \cdots & r_{i,i}^{(i-1)} & \cdots & r_{i,n-1}^{(i-1)} & r_{i,n}^{(i-1)} \\ . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & r_{n-1,n-1}^{(n-2)} & r_{n-1,n}^{(n-2)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & r_{n,n}^{(n-1)} \end{bmatrix} = 0, \quad (9)$$

or

$$\prod_{i=1}^{n} r_{i,i}^{(i-1)} = 0 \quad (10)$$

The equality to zero of any of multipliers in left part of (10) is sufficient condition of instability. Let's use as instability criteria the relation

$$r_{n,n}^{(n-1)} = 0 . \quad (11)$$

It's possible to find the critical parameter from relation (11). Let's show, that under certain conditions criterion (11) leads to the minimal critical parameter finding.

For finding of $r_{n,n}^{(n-1)}$ it's necessary to impose the single constraint onto given structure, to give the unit displacement in direction of this constraint and to analyze the resulting system by one or another method. If to use for this purpose the displacement method the analysis is reduced to the solving of the equations system (5) for the structure with single imposed constraint.

Let's write the equations system (5) in the form:

$$[K]\{u\} = \lambda[K_\sigma]\{u\} + \{P\} = \{D\} + \{P\}, \quad (12)$$

Where

$$\{D\} = \lambda[K_\sigma]\{u\}. \quad (13)$$

Let's consider the next iterative process:

$$[K]\{u\}^{(j)} = \lambda[K_\sigma]\{u\}^{(j-1)} + \{P\} = \{D\}^{(j-1)} + \{P\}. \quad (14)$$

The convergence condition of iterative process (14) is the positive definiteness of matrices $[K - \lambda K_\sigma]$ and $[K + \lambda K_\sigma]$ as noted in [10]. The first matrix is positive defined at load parameter less then minimal critical parameter

[4], and second – in view of its compliance to tension load. It's known [11], that minimal critical parameter of the system with one imposed constraint lies between first and second parameters of given system. Thus the iterative process (13) for the system with single imposed constraint converges only in the region of minimal critical parameter of given system.

Vector {D}, included in the formula (14) is called as a vector of additional load. Let us prove the following theorem.

Theorem. At analysis of deformable system by the method of additional load the latter can be represented as:

$$\{D\}^{(j)} = \{D_1\}\lambda + \{D_2\}\lambda^2 + ... + \{D_j\}\lambda^j, \quad (15)$$

where *j*- approximation number.

Proving. Let the additional load of (*j*-1) approximation has been found in the form:

$$\{D\}^{(j-1)} = \{D_1\}\lambda + \{D_2\}\lambda^2 + ... + \{D_j\}\lambda^{j-1} = \lambda[K_\sigma][K]^{-1}\{P\} + \lambda^2([K_\sigma][K]^{-1})^2\{P\} + ... + \lambda^{j-1}([K_\sigma][K]^{-1})^{j-1}\{P\}, \quad (16)$$

where

$$\{D_l\} = ([K_\sigma][K]^{-1})^l\{P\}, \; l = 1,2,...,j-1. \quad (17)$$

Then the displacements of *j*-th approximation according to (14) will be equal to:

$$\{u\}^{(j)} = [K]^{-1}\left(\{P\} + \{D\}^{(j-1)}\right) = [K]^{-1}(\{P\} + \lambda[K_\sigma][K]^{-1}\{P\} + ... + \lambda^{j-1}([K_\sigma][K]^{-1})^{j-1}\{P\}), \quad (18)$$

and the additional load of *j*-th approximation is equal to:

$$\{D\}^{(j)} = \lambda[K_\sigma][K]^{-1}(\{P\} + \lambda[K_\sigma][K]^{-1}\{P\} + ... + \lambda^{j-1}([K_\sigma][K]^{-1})^{j-1}\{P\}). \quad (19)$$

Designating $([K_\sigma][K]^{-1})^j\{P\} = \{D_j\}$ we come to the relation (15). Thus, the formula (15) is true, if the assumption (16) is true. Putting λ = 0, we get:

$$\{u\}^{(0)} = [K]^{-1}\{P\}; \quad (20)$$

$$\{D\}^{(1)} = \lambda[K_\sigma][K]^{-1}\{P\}. \quad (21)$$

Formula (21) is the beginning of the sequence (15), the validity of which, in general, was proved above.

On the basis of formula (15) we can draw the following important conclusion: at determining of the additional load of *j*-th approximation it is sufficient to find its j-th member.

Direct application of formula (12) to the system with a superimposed constraint may cause some computational difficulties, since in this case it is necessary to form the stiffness matrix for a given system and for a system with a superimposed constraint. It's more convenient to find the main reaction in the direction of a single superimposed constraint from the force method equation:

$$\delta_{n,n} r_{n,n}^{(n-1)} + \Delta_{n,p}^{(add)} = 1, \quad (22)$$

where $\delta_{n,n}$, $r_{n,n}^{(n-1)}$ and $\Delta_{n,p}^{(add)}$ have the usual meaning, and $\Delta_{n,p}^{(add)}$ represent the displacement in the main system, caused by the additional load. But the basic system at using of equation (22) coincides with the given one, so all the calculations associated with determining of factors $\delta_{n,n}$, $r_{n,n}^{(n-1)}$ and $\Delta_{n,p}^{(add)}$ should be carried out in a given system.

When calculating the additional load displacements in each successive cycle are taken from the previous and the initial – equal to zero. Thus, on the basis of (15) $\Delta_{n,p}^{(add)}$ can be represented as:

$$\Delta_{n,p}^{(add)} = \Delta_1\lambda + \Delta_2\lambda^2 + ... + \Delta_m\lambda^m \quad (23)$$

where $\Delta_1, \Delta_2, ..., \Delta_m$ - some numerical coefficients, *m*-number of approximations.

At the analysis of the structure by the described method the reaction of imposed constraint is determined by a polynomial:

$$r_{n,n}^{(n-1)} = r_0 + r_1\lambda + r_2\lambda^2 + ... + r_m\lambda^m, \quad (24)$$

where $r_0, r_1, r_2, ..., r_m$ - some numerical coefficients.

Iterative process (14), with the aid of which the expression (24) was reached, converges and $r_{n,n}^{(n-1)} \gtreqless 0$ at load parameter λ less then $\lambda_{1,kp}^{(n-1)}$ for the system with one imposed constraint. Otherwise, the expression (24) diverges. Therefore, the minimum critical parameter of given system $\lambda_{kp,\min}$ must be the only real positive root of the equation

$$r_{n,n}^{(n-1)} = r_0 + r_1\lambda + r_2\lambda^2 + \ldots + r_m\lambda^m = 0 \quad (25)$$

at $m \to \infty$. In practice, the number m is determined by the required accuracy of determining of the critical load.

The buckling form is characterized by the displacement vector components $\{u\}_\lambda$ of the system with a superimposed constraint at $r_{n,n}^{(n-1)} = 0$, i.e. at $\lambda = \lambda_{cr,\min}$. In accordance with (22) and (23), this vector can be represented as:

$$\{u\}_\lambda = \{u\}_0 + \lambda_{cr,\min}\{u\}_1 + \lambda^2{}_{cr,\min}\{u\}_2 + \ldots + \lambda^m{}_{cr,\min}\{u\}_m \quad (26)$$

In formula (26) $\{u\}_0$ is the vector of nodal displacements of structure caused by unit displacement of imposed constraint at λ=0, $\{u\}_1$ - vector of nodal displacements of structure caused by additional load of first approximation at λ=1, $(u)_2$ - vector of nodal displacements of structure caused by additional load of second approximation at λ=1 and so on.

According to the force method idea

$$\{u\}_0 = \{\bar{u}\}_0 r_0 , \quad (27)$$

where $\{\bar{u}\}_0$ - displacement vector caused by unit force acting in given system in direction of single imposed constraint and

$$\{u\}_j = \{\bar{u}\}_j + \{\bar{u}\}_0 r_j = \{\bar{u}\}_j - \frac{\Delta_{n,p}^{(j)}}{\delta_{n,n}}\{u\}_0, \; j = 1,2,\ldots,m, \quad (28)$$

где $\{\bar{u}\}_j$ - displacement vector in given system caused by additional load of j-th approximation at λ=1 and $\Delta_{n,p}^{(j)}$ - displacement in direction of imposed constraint in given system caused by the load of *j*-th approximation at λ=1.

The above-described iterative method of buckling analysis is implemented in developed by author computer program PRINS. The program was tested on a big amount of tasks. Some of them is described below.

THE SAMPLES OF BUCKLING ANALYSIS BY THE SINGLE IMPOSED CONSTRAINT METHOD

Sample 1. Fig. 1 shows the stiffened one-sidedly compressed rectangular plate. The plate was calculated at the next source data: modulus of elasticity E=2×10^7 kN/sm^2 ; Poisson ratio μ=0,3; thickness of plate h=1 sm; inertia moment of bars I_b =2,25 sm^4 ; cross section area of bars A= 3 sm^2 . The theoretical value of minimum critical force for this plate equal to 40.26 kN [13]. The result obtained by program PRINS is $P_{kp,1\min} = 39,5$ kN. The discrepancy is 1.88%.

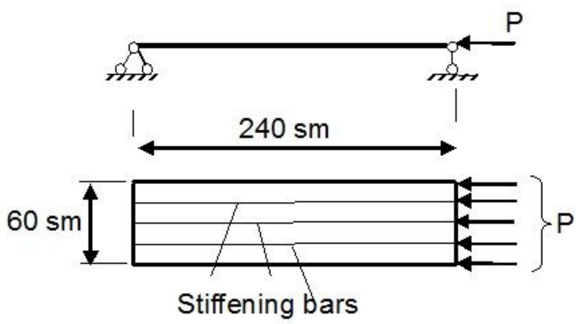

Figure 1: The stiffening plate under one-sidedly compression

*Sample 2***.** Fig. 2 shows the shell, for which the critical pressure $q_{cr,\min} = 1,36$ MPa and the buckling form, characterized by the formation of eight waves along a circle, were found with the aid of program PRINS. The parameters of shell: L= 100 sm; R= 50 sm;

h= 0,5 sm; E=2x10^7 kN/sm^2 ;μ=0,3. The theoretical value of the critical pressure is 1.42 MPa [13] (the shell is clamped at the ends). The discrepancy is 3.58%.

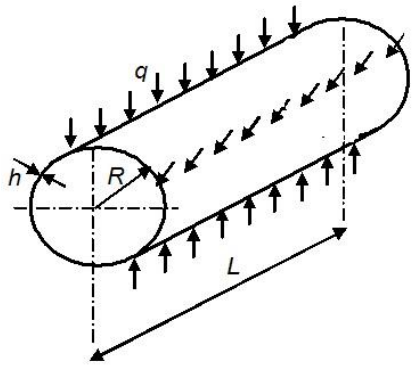

Figure 2: Clamped cylindrical shell under lateral pressure

Sample 3. The buckling analysis of the frame, shown on fig.3, was performed with the program PRINS[14]. The characteristics of the frame: E=2x10^7 kN/sm^2 ; h=600 sm; I=1200 sm^4 *;* n=1,66.

The critical parameter $P_{cr,\min}$ = 419 kN and corresponding buckling form were found. The calculated value is in good agreement with the theoretical result ($P_{cr,\min}$ = 418 kN[3]). Buckling form is shown in Fig. 3 by dotted line.

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 12, Number 16 (2017) pp. 5990-5994

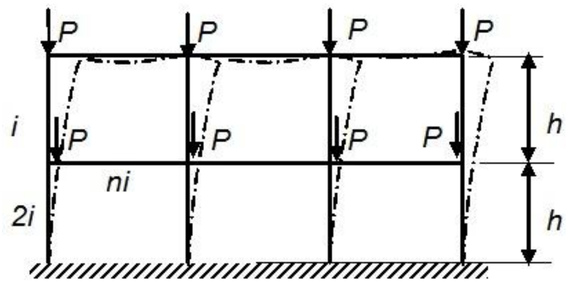

Figure 3: Buckling analysis of the frame

The results obtained for the different types of structures confirm the effectiveness of the proposed method.

CONCLUSION

Summarizing all written above with respect to the proposed method and algorithm for computing of the minimum critical parameter and the corresponding buckling form, we note that it became possible to avoid such operations as a solution of linear eigenvalue problem, conversion and multiplication of matrices of higher order. It allows to hope that the proposed method and developed on its base computer program will be used in the stability analysis of complex structures as an alternative or addition to existing methods. However, in some cases, the proposed method may be the only possible one, for example, in the problems of bifurcation buckling analysis when the initial displacements have place. The quadratic eigenvalue problem arise in these cases. This problem is described by equation (see [15])

$$\left[K_0 + \Delta\lambda K_{G_1} + \Delta\lambda^2 K_{G_2}\right]\{dq\} = 0 \, , \qquad (29)$$

where K_{G_1} and K_{G_2} - the matrices, which depend on initial forces and displacements.

The methods for equation (29) solving are unknown. But it can be solved by proposed method with some modifications.

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

REFERENCES

[1] Euler L., Metod nachojdenija krivych linii, obladajuchich svoistvami maksimuma ili minimuma (Metodus Inveniendi Lineas CurvasMaximi Minimive propertietate goudentes). Russkoe izd.(Russian edition) M.:GTTI, 1934.- 600 p

[2] Timoshenko S.P., Ustoichivost uprugich system (Stability of elastic systems) M.:GITTL, 1955.- 568 p

[3] Kornouchov N.V., Prochnost I ustoichivost sterjnevych system (Strength and stability of rod system) M.:Stroiizdat, 1949.- 376 p

[4] Smirnov A.F. Ustoichivost I kolebanija soorujenii (Stability and vibration of structures) M:GTJI, 1958.- 571 p

[5] K.J.Bathe., Finite Element Procedures: Prentice Hall, Inc., 1037 p.; 1996.

[6] Carson W.C., Newton R.E. Plate buckling analysis using a fully compatible finite element AIAA Journal.- 1969.-V. 7, № 3.-pp 527-529

[7] Gallagher R., Gellatly R., Mollett K. At al. A discrete element procedure for thin shell instability analysis AIAA Journal.-1967.-V. 5, № 1.-pp 138-145

[8] Zienkiewicz O.C., Taylor R.L. The Finite Element for Solid and Structural Mechanics. Sixth edition McGraw-Hill, 631 p.; 2005.

[9] Wilkinson J.H., The algebraic eigenvalue problem Oxford University Press, New York, 1965

[10] Gofman S.M., Iteracionnye processy v stroitelnoy mechanike//Voprosy matematiki I mechaniki (Iteration processes in structural mechanics//The questions of mathematics and mechanics). Tashkent:TashNIIJT, 1959.-Issue IX.- pp 17-59

[11] Leiderman Y.R, Ustoichivost I kolebanija ramnych konstrukcii(Stability and vibration of frame structure Tashkent, AN UzSSR, 1955.- 220 p

[12] Prochnost, ustoichivost, kolebanija. Spravochnik v trech tomach (Strength, stability, vibration. Handbook in three volumes) M:Mashinostroenie, 1968.-V.3.- 567 p

[13] Grigoluc E.I., Kabanov V.V. Ustoichivost obolochek (Stability of shells) M.:Nauka, 1978.-359 p

[14] Agapov V.P. , Issledovanie prochnosti prostranstvennyh konstrukcii v linejnoi i nelinejnoi postanovkah s ispolzovaniem vychislitelnogo kompleksa “PRINS” (Strength analysis of the three-dimensional structures with computer program PRINS). In: Prostranstvennye konstrukcii zdanii i soorujenii (issledovanie,raschet,proektirovanie, primenenie). Sb.statei, vyp. 11. MOO «Prostranstvennye konstrukcii », pod red. V.V.Shugaeva I dr. Moscow; 2008. pp 57-67

[15] Agapov V.P., Osnovnye sootnoschenija MKE v staticheskich I dinamicheskich raschetach geometricheski nelineinych konstrukcii//Stroitelnaja mechanika I raschet soorujenii (Structural mechanics and building analysis). - 1984. - N 5. - Pp. 43-47.