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A Numerical Investigation on Variable Shape Parameter Schemes in a Meshfree Method Applied to a Convection-Diffusion Problem

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Abstract

It is known that all RBF-based meshfree methods suffer from a lack of reliable judgment on the choice of shape parameter. Several attempts have been proposed and tested with specific classes of PDEs and their effectiveness has been widely documented. Nevertheless, it has not well been agreed on which choice remains optimal. As a consequence, it is very much up to and very often down to the *'ad-hoc'* decision of the user. In this work we firstly numerically investigate the quality of each adaptive RBF-shape parameter approaches presented in literature by applying them to the same type of problem. Secondly, we proposed a new form of shape-parameter scheme taking into consideration both linear and exponent nature of the shape's value itself. The classical Kansa meshless method is implemented where the solutions' quality is carefully monitored and compared against one another based on both matrix condition number, and the results' accuracy.

Keywords: Radial Basis Functions (RBF), Shape Parameter, Collocation Meshless Method

INTRODUCTION

Considered as another comparably-new research area, finding numerical solutions to a given partial differential equations (PDEs) by means of meshfree methods are now receiving more and more attention. The main feature is its simplicity and the dependence of mesh/grid connection and topology making it much simpler to implement. The whole class of meshfree/meshless method can be categorized into three classes; weak forms, strong forms, and mixed, all nicely documented in Liu and Gu (2005). Each of these has its own advantages/disadvantages depending on several factors involved including domain geometry, governing equations, boundary/initial conditions, computer arithmetic, etc. Amongst those being proposed and developed nowadays, one of the well-known meshfree method is that called 'RBF-collocation' or sometimes called 'Kansa's method' (Kansa, 1990), where it uses a set of global approximation function to approximate the field variables on both the domain and the boundary when solving PDEs. This is of the following expression;

$$u(\mathbf{x}) = \sum_{j=1}^{N} c_j \varphi_j(r) = \sum_{j=1}^{N} c_j \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) \tag{1}$$

Over some given scattered data nodes $X=\{\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_N\}\subset\Omega$, Ω is the problem domain. The Radial Basis Functions (RBF) , φ , are commonly found as multivariate functions whose values are dependent only on the distance from the origin and commonly assumed to be strictly positive definite. This means that $\varphi(\mathbf{x})=\varphi(r)\in\mathbb{R}$ with $\mathbf{x}\in\mathbb{R}^n$ and $r\in\mathbb{R}$; or, in other words, on the distance from a point of a given set $\{\mathbf{x}_j\}$, and $\varphi(\mathbf{x}-\mathbf{x}_j)=\varphi(r_j)\in\mathbb{R}$ where can normally defined as follows;

$$r=\left\|\mathbf{x}-\mathbf{x}^{\Theta}\right\|_2=\sqrt{(x_1-x_1^{\Theta})^2+(x_2-x_2^{\Theta})^2+...+(x_n-x_n^{\Theta})^2} \tag{2}$$

For some fixed points $\mathbf{x}\in\mathbb{R}^n$. Nevertheless, in this work, $r_j=\left\|\mathbf{x}-\mathbf{x}_j\right\|_2$ is the Euclidean distance and the radial basis function, φ , is chosen to be the Multiquadric type as firstly proposed by Hardy (1971), defined as;

$$\varphi(r,\varepsilon)=\sqrt{\varepsilon^2+r^2} \tag{3}$$

Where ε is the so-called 'shape parameter' and is known to play a crucial role in determining the quality of the final results and have always been an open topic for decades. Hardy (1971) suggests that by fixing the shape at $\varepsilon=1/(0.815d)$, where $d=\left(\frac{1}{N}\right)\sum_{i=1}^{N} d_i$, and d_i is the distance from the node to its nearest neighbor, good results should be anticipated. Also, in the work of Franke and Schaback (1998) where the choice of a fixed shape of the form $\varepsilon = \frac{0.8\sqrt{N}}{D}$ where D is the diameter of the smallest circle containing all data nodes, can also be a good alternative.

Some recent attempts to pinpoint the optimal value of ε involve the work of Zhang, Song, Lu, and Liu (2000) where they demonstrated and concluded that the optimal shape parameter is problem dependent. In 2002, Wang and Lui (2002) pointed out that by analyzing the condition number of the collocation matrix, a suitable range of derivable values of ε can be found. Later in 2003, Lee, Liu, and Fan (2003) suggested that the final numerical solutions obtained are found to be less affected by the method when the approximation in equation (1) is applied locally rather than globally.

RADIAL BASIS FUNCTION COLLOCATION MESHFREE METHOD

RBF collocation scheme for PDEs

For the methodology of RBF-collocation meshless method for numerically solving PDEs, it begins with considering a linear elliptic partial differential equation with boundary conditions, where $g(\mathbf{x})$ and $f(\mathbf{x})$ are known. We seek $u(\mathbf{x})$ from;

$$Lu(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \text{ in } \Omega \tag{4}$$

$$Mu(\mathbf{x}) = g(\mathbf{x}), \qquad \mathbf{x} \text{ on } \partial\Omega \tag{5}$$

where $\Omega \in \mathbb{R}^d$, $\partial\Omega$ denotes the boundary of domain Ω , L and M are the linear elliptic partial differential operators and operating on the domain Ω and boundary domain $\partial\Omega$, respectively. For Kansa's method, it represents the approximate solution $u(\mathbf{x})$ by the interpolation, using an RBF interpolation as expressed in equation (1). We can see that N linear dependent equations are required for solving N unknowns of c_j . Substituting $u(\mathbf{x})$ into equation (4) and equation (5), we obtain the system of equations as follows;

$$L\left(\sum_{j=1}^{N_I} c_j \varphi\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) = \sum_{j=1}^{N_I} c_j L\varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) = f(\mathbf{x}_i), \quad i = 1,\dots,N_I \tag{6}$$

$$M\left(\sum_{j=N_I+1}^{N} c_j \varphi\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) = \sum_{j=N_I+1}^{N} c_j M\varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) = g(\mathbf{x}_i), \quad i = N_I+1,\dots,N \tag{7}$$

Above equations, we choose N collocation points on both domain Ω and boundary domain $\partial\Omega$, and divide it into N_I interior points and N_B boundary points $(N = N_I + N_B)$. Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$ denotes the set of collocation points, $I = \{I_1, \dots, I_{N_I}\}$ denotes the set of interior points and $B = \{B_1, \dots, B_{N_B}\}$ the set of boundary points. The centers $\mathbf{x}_j$ used in equation (6) and equation (7) are chosen as collocation points. The previous substituting yields a system of linear algebraic equations which can be solved for seeking coefficient $\mathbf{c}$ by rewriting equation (6) and equation (7) equation in matrix form as;

$$\mathbf{Ac} = \mathbf{F} \tag{8}$$

where $\mathbf{A} = \begin{bmatrix} \mathbf{A}_L \\ \mathbf{A}_M \end{bmatrix}$

$(\mathbf{A}_L)_{ij} = L\varphi\left(\left\|\mathbf{x}_i - \mathbf{x}_j\right\|\right)$, $\mathbf{x}_i \in I,\ \mathbf{x}_j \in X$,
$i = 1,2,\dots,N_I, \quad j = 1,2,\dots,N$

$(\mathbf{A}_M)_{ij} = M\varphi\left(\left\|\mathbf{x}_i - \mathbf{x}_j\right\|\right)$, $\mathbf{x}_i \in B,\ \mathbf{x}_j \in X$,
$i = N_I+1,\dots,N, \quad j = 1,2,\dots,N,$

and $\mathbf{F} = \begin{bmatrix} f(\mathbf{x}_i) \\ g(\mathbf{x}_i) \end{bmatrix}$

$f(\mathbf{x}_i); \mathbf{x}_i \in I, \quad i = 1,2,\dots,N_I,$
$g(\mathbf{x}_i); \mathbf{x}_i \in B, \quad i = N_I+1,\dots,N.$

This yellow-heighted part should be written as a matrix form equation (8), the coefficient $\mathbf{c}'s$ are computed from the following system;

$$\mathbf{c} = \mathbf{A}^{-1}\mathbf{F} \tag{9}$$

Therefore, the matrix $\mathbf{c}$ is substituted into equation (1) and the approximate solution of $u(\mathbf{x})$ can be determined by;

$$u(\mathbf{x}) = \sum_{j=1}^{N} c_j \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) \tag{10}$$

Being one of the most popular choices of meshless method, its existence, uniqueness, and convergence have been nicely documented in Micchelli (1986) and Madych and Nelson (1990).

The system is known to provide solution if and only if the matrix $\mathbf{A}$ is non-singular, its inverse exists. This aspect is related directly to its condition number, as can be defined as;

$$\Lambda_\delta(\mathbf{A}) = \|\mathbf{A}\|_\delta \|\mathbf{A}^{-1}\|_\delta, \qquad \delta = 1, 2, \infty \tag{11}$$

In the RBF-meshless method, it is well-known that this condition number is strongly affected by the magnitude of the shape parameter ε and the number of nodes involved, N.

Application to Convection-Diffusion PDE

Consider the governing partial differential equation of convection diffusion problems expressed as;

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \beta\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = R \tag{12}$$

where α, β are the diffusive term and convective term, respectively and R describes source or sinks of the quantity α. In steady state case, the convection–diffusion equation (12) is reduced to the following steady state form;

$$-\alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \beta\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = R,$$

where $(x, y) \in \Omega \subset \mathbb{R}^d$ (13)

In this practice, its boundary condition is of Dirichlet type as follows:

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{14}$$

In order to implement the RBF-collocation meshless procedure, the above governing equation can be approximated using the summation and becomes;

$$\sum_{j=1}^{N} c_j \left[\begin{array}{l} -\alpha\left(\dfrac{\partial^2\varphi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right)}{\partial x^2} + \dfrac{\partial^2\varphi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right)}{\partial y^2}\right) \\ +\beta\left(\dfrac{\partial\varphi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right)}{\partial x} + \dfrac{\partial\varphi\left(\|\mathbf{x}_i - \mathbf{x}_j\|\right)}{\partial y}\right) \end{array} \right] = R, \tag{15}$$

for $i = 1, \ldots, N_I$ and;

$$\sum_{j=1}^{N} c_j \varphi\left(\mathbf{x}_i - \mathbf{x}_j\right) = 0 \tag{16}$$

for $i = N_I + 1, \ldots, N$.

This system can be written in the form of;

$$\sum_{j=1}^{N} c_j \left[-\alpha\left[\nabla^2\varphi\right]_{ij} + \beta\left[\nabla\varphi\right]_{ij}\right] = R, \quad i = 1, \ldots, N_I \tag{17}$$

$$\sum_{j=1}^{N} c_j \varphi_{ij} = 0, \quad i = N_I + 1, \ldots, N \tag{18}$$

This system can be generated in matrix form and the approximate solutions then can be obtained by substituting the coefficients c_j, obtained by solving the above matrix form, in equation (8).

VARIABLE PARAMETER SCHEMES AND THE PROPOSED FORM

It is known that the effectiveness of the RBF-collocation method can well be influenced by the choice of the shape parameter. Many researchers have agreed that using variable shape parameters provides superior in solution quality (See Sarra and Sturgill (2009) and references herein). The result of allowing the shape parameter to vary locally is that each column of the interpolation matrix, matrix $\mathbf{A}$ in (8), no longer contains constant entries leading to lower condition number. Several strategies have been proposed for providing reliable numerical solution accuracy and they are revisited as shown in Table 1 .

The variable abbreviated as **V1** is clearly in an exponential manner and was used in Kansa (1990) before its further modified version was later invented in Kansa and Carlson (1992), noted as **V2**. In their work, it was demonstrated that if $\varepsilon^2_{\min}$ and $\varepsilon^2_{\max}$ varied by several orders of magnitude, then a very satisfactory result quality can well be expected. Later, a linear form of variable shape parameter was proposed and applied to both interpolation and some benchmark partial differential equations by Sarra (2005) and Sarra and Sturgill (2009), as noted in Table 1 by **V3** and **V4** respectively. Here, the command 'rand' is the MATLAB function that returns uniformly distributed pseudo-random numbers on the unit interval. It was proven in their work that the variable shape outperformed the fixed value of parameter especially when the scheme includes the information about the minimum distance of a center to its nearest neighbor, h_n, with also a user input value μ. In terms or the condition number produced by equation (11), it was also found to be considerable smaller over most of the average shape range.

Table 1: Variable shaper parameter schemes used in literature.

Abbreviation used in this work	Reference	Formulation of ε for j[th]-element
V1	Kansa (1990)	$\varepsilon_j = \left[\varepsilon_{\min}^2 \left(\frac{\varepsilon_{\max}^2}{\varepsilon_{\min}^2} \right)^{\frac{j-1}{N-1}} \right]^{\frac{1}{2}}$, $j=1,2,...,N$
V2	Kansa and Carlson (1992)	$\varepsilon_j = \varepsilon_{\min} + \left(\frac{\varepsilon_{\max} - \varepsilon_{\min}}{N-1} \right) j \quad j=0,1,2,...,N-1$
V3	Sarra (2005)	$\varepsilon_j = \varepsilon_{\min} + \left(\varepsilon_{\max} - \varepsilon_{\min} \right) \times rand(1,N)$
V4	Sarra and Sturgill (2009)	$\varepsilon_j = \frac{\mu}{h_n} \left[\varepsilon_{\min} + \left(\varepsilon_{\max} - \varepsilon_{\min} \right) \times rand(1,N) \right]$

In this work, we proposed a new form of variable shape parameter where both linear and exponential manners are taken into consideration, expressed as in equation (19).

$$\varepsilon_j = (1-\zeta) \left[\varepsilon_{\min}^2 \left(\frac{\varepsilon_{\max}^2}{\varepsilon_{\min}^2} \right)^{\zeta} \right]^{\frac{1}{2}} + \zeta \left[\varepsilon_{\min} + \left(\varepsilon_{\max} - \varepsilon_{\min} \right) \zeta \right] \tag{19}$$

Where $\zeta = \frac{j-1}{N-1}$ and $j=1,2,...,N$. The automatically self-adjusted parameter ζ is introduced and act as a weight function leading ε_j to equal to the exponential manner when $j=1$. This weight then sets ε_j to become its linear mode when $j=N$. This proposed variable shape is referred to as **V5** throughout the work.

NUMERICAL RESULTS AND GENERAL DISCUSSION

To demonstrate how effective the variables **V1**, **V2**, **V3**, **V4** and **V5** can be, let us consider a 2D convection-diffusion problem in steady state, as given and studied in Gu and Liu (2006). The governing equation is expressed as follows;

$$L(u) = \mathbf{v}^T \cdot \nabla u - \nabla^T \left(\mathbf{D} \nabla u \right) + \beta u - q(\mathbf{x}) = 0 \tag{20}$$

The computational domain is taken to be $(x,y) \in \Omega = [0,1] \times [0,1]$, and the coefficients are $\mathbf{D} = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}$, $\mathbf{v} = \{3-x, 4-y\}$, and $\beta = 1$ in which γ is a given

constant of diffusion coefficient. The boundary conditions are set with $u=0$ on all four sides. The exact solution for this problem is given by;

$$u(x,y) = \sin(x) \left(1 - e^{-\frac{2(1-x)}{\gamma}} \right) y^2 \left(1 - e^{-\frac{3(1-y)}{\gamma}} \right) \tag{21}$$

Throughout the experiment, all solutions obtained are measured using the

following error norms, with N being the total number of computational nodes involved;

$$\text{Relative error} = \frac{\left| u_{num} - u_{ext} \right|}{\left| u_{ext} \right|} \tag{22}$$

Root Mean Square (RMS) error = $\sqrt{\dfrac{\sum_{i=1}^{N}(u_{num}-u_{ext})^2}{N}}$ (23)

In order to cover as wide aspect as possible, a large number of experiments were carried out but only some is presented here. Table 2 – Table 4 show the optimal shape parameter value obtained at different levels of diffusion coefficients (γ), the number of nodes (N), and the range of $(\varepsilon_{\min}, \varepsilon_{\max})$. At relative high value of $\gamma = 10$, it is found that the proposed form of variable shape yielded the same level of solutions quality as those obtained from other forms gathered from literatures. This is still the case even when the number of nodes and $(\varepsilon_{\min}, \varepsilon_{\max})$ range are increasing. It is interesting to see that the proposed variable shape is still numerically compatible with the others even when the problem becomes more convective-dominated, $\gamma = 1.0 - 0.1$, where the same order of error norms magnitudes are found as seen in Table 3 and Table 4.

Table 2: Solution quality comparison at different numbers of nodes (100 and 289), and the value of $\varepsilon_{\min}$ and $\varepsilon_{\max}$ for $\gamma = 10$.

Shape Type	N = 100					
	$(\varepsilon_{\min}, \varepsilon_{\max}) = (1,10)$			$(\varepsilon_{\min}, \varepsilon_{\max}) = (0.1,20)$		
	Opt Value	RMS	Relative Error	Opt Value	RMS	Relative Error
V1	1.00E+00	7.55E-08	0.008637222	1.00E+00	7.55E-08	0.008637222
V2	1.09E+00	1.16E-07	0.013233971	1.09E+00	1.16E-07	0.013233971
V3	1.04E+00	9.26E-08	0.01059299	1.04E+00	9.26E-08	0.01059299
V4	9.05E-01	4.42E-08	0.005054461	9.05E-01	4.42E-08	0.005054461
V5	1.00E+00	7.55E-08	0.008637222	1.00E+00	7.55E-08	0.008637222
Shape Type	N = 289					
	$(\varepsilon_{\min}, \varepsilon_{\max}) = (1,10)$			$(\varepsilon_{\min}, \varepsilon_{\max}) = (0.1,20)$		
	Opt Value	RMS	Relative Error	Opt Value	RMS	Relative Error
V1	1.53E+00	4.01E-09	0.000459006	1.34E+00	5.62E-09	6.43E-04
V2	1.56E+00	7.99E-09	0.000914221	1.21E+00	7.83E-09	0.000895872
V3	1.26E+00	5.78E-09	0.000661003	1.43E+00	6.27E-09	0.000717719
V4	1.81E+00	1.31E-08	0.001494512	1.76E+00	1.29E-08	0.001472447
V5	1.40E+00	3.38E-09	0.000387093	1.51E+00	6.20E-09	0.000709044

Table 3: Solution quality comparison at different numbers of nodes (100 and 289), and the value of ε_{min} and ε_{max} for $\gamma = 1$.

Shape Type	N = 100					
	$(\varepsilon_{min}, \varepsilon_{max}) = (1, 10)$			$(\varepsilon_{min}, \varepsilon_{max}) = (0.1, 20)$		
	Opt Value	RMS	Relative Error	Opt Value	RMS	Relative Error
V1	1.00E+00	1.25E-05	0.031834723	7.64E-01	6.12E-06	0.015527228
V2	1.09E+00	1.73E-05	0.044034906	9.04E-01	8.46E-06	0.02146592
V3	1.13E+00	1.99E-05	0.050421885	7.89E-01	6.17E-06	0.01565911
V4	9.36E-01	9.62E-06	0.024412497	7.57E-01	9.35E-06	0.023723049
V5	1.00E+00	1.25E-05	0.031834723	8.10E-01	6.49E-06	0.016476151
Shape Type	N = 289					
	$(\varepsilon_{min}, \varepsilon_{max}) = (1, 10)$			$(\varepsilon_{min}, \varepsilon_{max}) = (0.1, 20)$		
	Opt Value	RMS	Relative Error	Opt Value	RMS	Relative Error
V1	1.38E+00	6.13E-07	1.55E-03	1.61E+00	2.73E-06	0.001912115
V2	1.50E+00	7.26E-07	0.001841492	1.48E+00	4.15E-06	0.002034224
V3	1.55E+00	7.47E-07	0.001896039	1.66E+00	4.40E-06	0.00243597
V4	1.63E+00	1.24E-06	3.15E-03	1.21E+00	4.22E-06	0.002429868
V5	1.48E+00	5.58E-07	0.001415846	1.63E+00	2.80E-06	0.001881917

Table 4: Solution quality comparison at different numbers of nodes (100 and 289), and the value of $\varepsilon_{\min}$ and $\varepsilon_{\max}$ for $\gamma = 0.1$.

Shape Type	N = 100					
	$(\varepsilon_{\min}, \varepsilon_{\max}) = (1,10)$			$(\varepsilon_{\min}, \varepsilon_{\max}) = (0.1, 20)$		
	Opt Value	RMS	Relative Error	Opt Value	RMS	Relative Error
V1	1.00E+00	0.03945149	22.20265065	8.97E-01	0.037139	20.90127
V2	1.09E+00	0.044200016	24.87504347	9.04E-01	0.038185	21.4897
V3	1.09E+00	0.044190864	24.86989265	9.33E-01	0.036503	20.54336
V4	9.73E-01	0.039405804	22.17693963	1.01E+00	0.04048	22.78149
V5	1.00E+00	0.03945149	22.20265065	9.76E-01	0.039476	22.21635
Shape Type	**N = 289**					
	$(\varepsilon_{\min}, \varepsilon_{\max}) = (1,10)$			$(\varepsilon_{\min}, \varepsilon_{\max}) = (0.1, 20)$		
	Opt Value	RMS	Relative Error	Opt Value	RMS	Relative Error
V1	1.76E+00	0.00132098	0.74342579	1.73E+00	0.001265	0.711933
V2	1.81E+00	0.001412886	0.795149013	1.97E+00	0.001523	0.856849
V3	1.89E+00	0.001408979	0.792950412	1.88E+00	0.001402	0.789286
V4	1.80E+00	0.001507024	0.848128631	1.81E+00	0.001246	0.701123
V5	1.86E+00	0.001417853	0.797944544	1.79E+00	0.001324	0.74495

Figure 1 illustrates the error plots measured when using each form of variable shapes. All plots reveal the same trend containing fluctuations at the beginning of $\varepsilon_{\min}$ before reaching the optimal point and starts to grow exponentially. The results from **V4** is not shown here since they behave similarly to those observed from **V3**. In the figure, it should be noted that the number of nodes considered is as high as 361, at $\gamma = 5.0$ and $(\varepsilon_{\min}, \varepsilon_{\max}) = (1.0, 20)$. In terms of the condition number of the interpolation matrix, equation (11), Figure 2 shows the same plots where the condition number is seen to decrease around the optimal value of ε. The larger ε causes $\Lambda_{\delta}(\mathbf{A})$, as expected, to decrease meaning that the numerical process is farther from being affected by the 'singularity' problem.

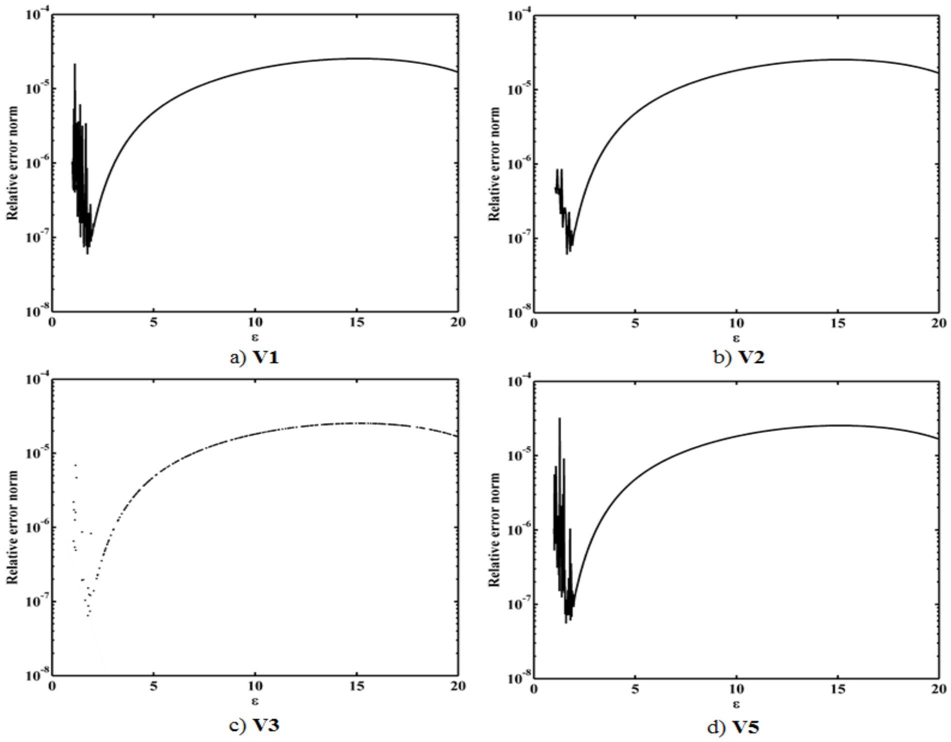

Figure 1: Relative error norms obtained using N = 361 nodes, γ = 5.0 for each type of variable shape parameter: **V1, V2, V3**, and **V5**.

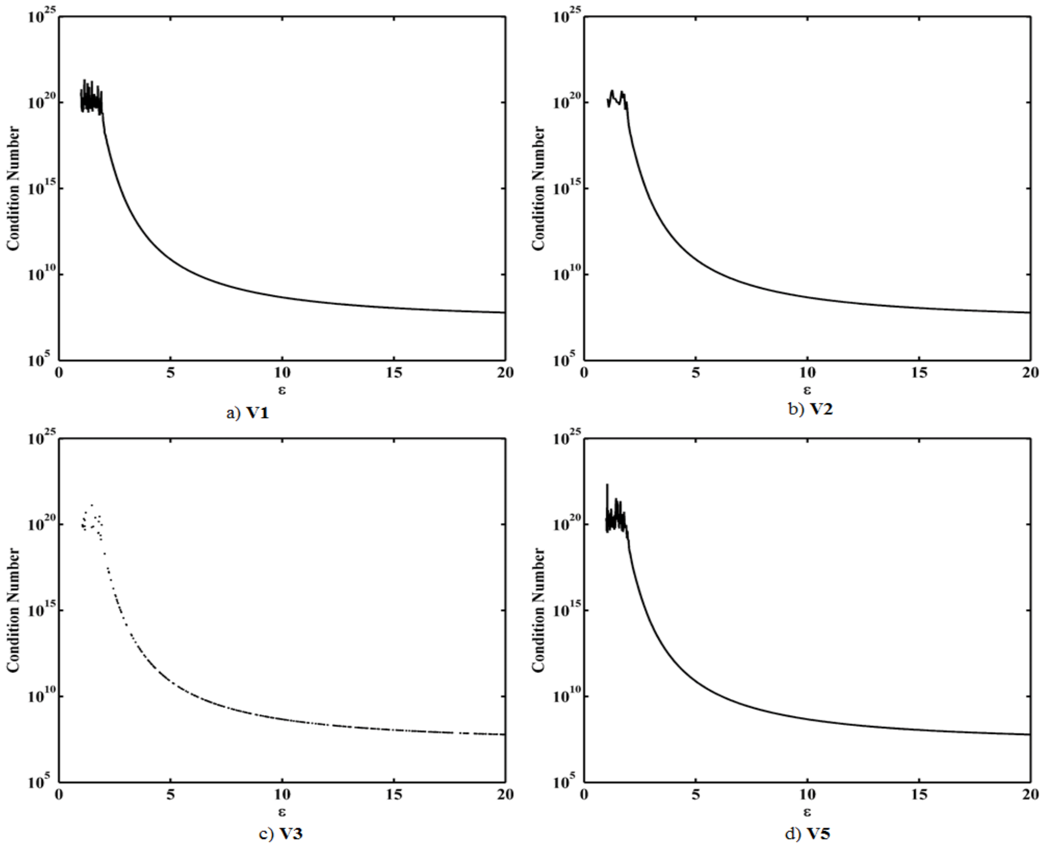

Figure 2: Condition numbers obtained using N = 361 nodes, γ = 5.0 for each type of variable shape parameter: **V1, V2, V3,** and **V5**.

CONCLUSIONS

There are three main objectives of this numerical study. Firstly, we have successfully applied a type of meshless known as 'Kansa's or RBF-collocation' method to the convective-diffusive type of problem. Secondly, different types/forms of variable multiquadric shape parameters proposed and widely used in literature were gathered and tested out with the same type of PDEs. Lastly, a new form of variable shape parameter containing both linear and exponential manners is proposed and implemented to the problem. Several useful findings obtained from this work are as follows;

- It is demonstrated that the RBF-collocation method can well be another alterative numerical tool for solving convection-diffusion type of PDEs.
- The variable shape parameter proposed in this work is proven numerically to perform as well as those available in literature even when the problem becomes more convective dominant, and, therefore, deserves further investigation.
- Observed from all forms under investigation, the optimal values of shape parameter ε is seen to be between 0.9 – 1.50 in most cases.
- The range of values of ε, i.e. $(\varepsilon_{\min}, \varepsilon_{\max})$, is seen not to significantly influence the accuracy of the final results whereas it is actually the number of nodes (N) involved that affects the computation the most.
- All forms of shape parameters investigated here resulted in the same trend of condition number and all computations were not significantly affected by the singularity problem.

In the case where the problem is more convective-dominated (i.e. $\gamma \ll 1$), nevertheless, the situation can be totally different. At this range it is well-known that the problem of instability can well ruin the whole simulation and, therefore, it should be more sensible to take into consideration some local feature such as the *Peclet* number or the diffusive coefficient itself. This prompts the main objective of our future investigation.

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