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Fractal Index Method for Solving Generalized Fractional Riccati Differentional Equations

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Abstract

In this paper, we applied a new analytical geometrical method which so called fractal index method to find a new solution of the generalized fractional Riccati equation of arbitrary order, we also wrote the solution on the closed form as an infinte series and showed that the series is convergent within some closed disk. The fractional operators are taken in sense of the Riemann-Liouville operators.

Keywords: Fractional calculus; Fractional differential equations; Fractal index.

AMS Mathematics Subject Classification: 34A12.

INTRODUCTION

Fractional differential equations are viewed as alternative models to nonlinear differential equations. Varieties of them play important roles and tools not only in mathematics but also in physics, dynamical systems, control systems and engineering to create the mathematical modeling of many physical phenomena. Furthermore, they employed in social science such as food supplement, climate and economics. Fractional differential equations concerning the Riemann-Liouville fractional operators or Caputo derivative have been recommended by many authors [1-5]. Transform is a significant technique to solve mathematical problems. Many useful transforms for solving various problems were appeared in open literature such as wave transformation, Laplace transform, the Fourier transform, the Bücklund transformation, the integral transform, the local fractional integral transforms and the fractional complex transform and Mellin transform [6-10].

One of the most tools in the theory of fractional calculus is viewed by the Riemann-Liouville operators. It imposes advantages of fast convergence, higher stability and higher accuracy to derive different types of numerical algorithms. In this note, we shall deal with scalar linear time-space fractional differential equations. The time and the space are taken in sense of the Riemann-Liouville fractional operators. Also, This type of differential equation arises in many interesting applications [11-16]. Several researchers have studied fractional dynamic equations generalizing the diffusion or wave equations in terms of R-L or Caputo time fractional derivatives, and their fundamental solutions have been represented in terms of the Mittag-Leffler (M-L) functions and their generalizations.

In this paper we using fractal index method [10] to impose a new solution for Riccati equation of arbitrary order. The fractional operators are taken in sense of the Riemann-Liouville operators.

FRACTIONAL CALCULUS

The idea of the fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) was planted over 300 years ago. Abel in 1823 investigated the generalized tautochrone problem and for the first time applied fractional calculus techniques in a physical problem. Later Liouville applied fractional calculus to problems in potential theory. Since that time the fractional calculus has haggard the attention of many researchers in all area of sciences [1-5].

This section concerns with some preliminaries and notations regarding the fractional calculus.

Definition 2.1 *The fractional (arbitrary) order integral of the function* f *of order* $\alpha > 0$ *is defined by*

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau.$$

When $a = 0,$ we write $I_a^\alpha f(t) = f(t) * \phi_\alpha(t),$ where $(*)$ denoted the convolution product (see [16]), $\phi_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, t > 0$ and $\phi_\alpha(t) = 0, t \le 0$ and

$\phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2 *The fractional (arbitrary) order derivative of the function* f *of order* $0 \le \alpha < 1$ *is defined by*

$$D_a^\alpha f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

Remark 2.1 From Definition 2.1 and Definition 2.2, $a = 0$, we have

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \mu > -1; 0 < \alpha < 1$$

and

$$I^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \mu > -1; \alpha > 0.$$

The Leibniz rule is

$$D_a^\alpha[f(t)g(t)] = \sum_{k=0}^{\infty}\binom{\alpha}{k} D_a^{\alpha-k} f(t) D_a^k g(t) = \sum_{k=0}^{\infty}\binom{\alpha}{k} D_a^{\alpha-k} g(t) D_a^k f(t),$$

where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1-k)}.$$

Definition 2.3 *The Caputo fractional derivative of order* $\mu > 0$ *is defined, for a smooth function* $f(t)$ *by*

$${}^c D^\mu f(t) := \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(\zeta)}{(t-\zeta)^{\mu-n+1}} d\zeta,$$

where $n = [\mu]+1$, (the notation $[\mu]$ stands for the largest integer not greater than μ). Note that there is a relationship between Riemann-Liouville differential operator and the Caputo operator

$$D_a^\mu f(t) = \frac{1}{\Gamma(1-\mu)} \frac{f(a)}{(t-a)^\mu} + {}^c D_a^\mu f(t);$$

and they are equivalent in a physical problem (i.e., a problem which specifies the initial conditions) see [17] and [18].

THE FRACTAL INDEX METHOD

To understanding the fractional complex transform. Consider a plane with fractal structure shown in Fig. 1. The shortest path between two points is not a line and we have

$$ds_E = k ds^\alpha, \tag{1}$$

where ds_E is the actual distance between two points *A* and *B* (the green curve in Fig. 1), ds is the line distance between two points (the red line in Fig. 1), α is the fractal dimension and *k* is a constant.

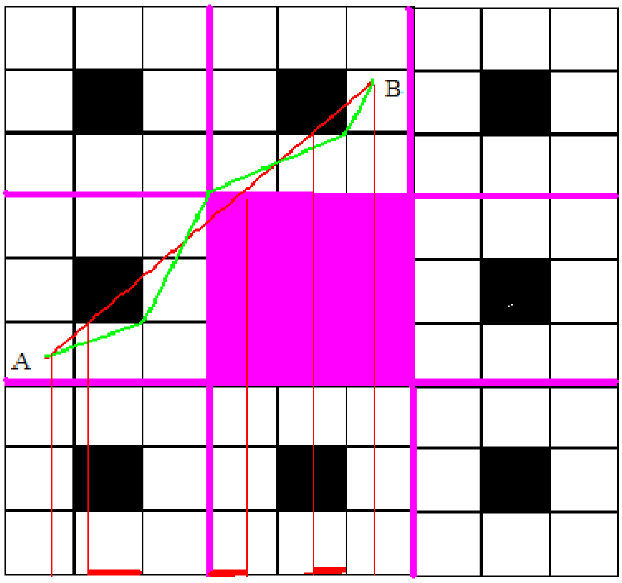

Figure 1. The distance between two points in a discontinuous space.

Projection the ds_E (the green curve) into horizontal direction yields Canton-like sets, and its length can be expressed as

$$\Delta_x AB = k_x x^{\alpha_x} \tag{2}$$

where α_x are the fractal dimensions of the Canton-like sets in the horizontal direction, k_x is a constant. Eq. (1) means the following transform $s_E = k s^\alpha$, this idea leads to the fractional complex transform, the fractal curve "*AB*" in Fig. 1 is projected to Cantor-like sets in horizontal direction. From Fig. 1, we have

$$\Delta_x AB = \cos\theta ds_E \tag{3}$$

or

$$\Delta_x AB = \frac{dx}{ds} ds_E \tag{4}$$

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where θ is the slope angle of straight line *AB*. From the relations Eqs. (2) and (4), we have

$$k_x dx^{\alpha_x} = k\frac{dx}{ds}ds^{\alpha}$$

or

$$d\,x^{\alpha_x} = \frac{k}{k_x}\frac{dx}{ds}ds^{\alpha} = \sigma\frac{dx}{ds}ds^{\alpha}$$

where $\sigma = \frac{k}{k_x}$ and so called the fractal index, therefore, we have the following chain rule for fractional calculus $\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \sigma\frac{\partial u}{\partial s}\frac{\partial^{\alpha}s}{\partial t^{\alpha}}$.

FRACTIONAL RICCATI EQUATION

Consider the following generalized fractional Riccati equation of arbitrary order

$$D_x^{\alpha}\psi(x) = \lambda_1 + \lambda_2\psi(x) + \lambda_3\psi^2(x) \tag{5}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. To calculate the fractal index for the Eq. (5), we assume the transform $X = x^{\alpha}$ and the solution can be expressed in a series of the form

$$\psi(X) = \sum_{m=0}^{\infty}\psi_m X^m, \quad \psi(0) = 1 \tag{6}$$

where ψ_m are constants. Substitute Eq. (6) into Eq. (5) and by applying the fractal index method we impose

$$\frac{\partial}{\partial X}\sum_{m=0}^{\infty}\theta_{\alpha m}\psi_m X^m = \lambda_1 + \lambda_2\sum_{m=0}^{\infty}\psi_m X^m + \lambda_3\left(\sum_{m=0}^{\infty}\psi_m X^m\right)\left(\sum_{m=0}^{\infty}\psi_m X^m\right) \tag{7a}$$

$$\sum_{m=0}^{\infty}\frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha-\alpha)}\psi_m X^{m-1} = \lambda_1 + \lambda_2\sum_{m=0}^{\infty}\psi_m X^m + \lambda_3\left(\sum_{m=0}^{\infty}\psi_m X^m\right)\left(\sum_{m=0}^{\infty}\psi_m X^m\right) \tag{7b}$$

where

$$\theta_{\alpha m} = \frac{\Gamma(1+m\alpha)}{m\Gamma(1+m\alpha-\alpha)}.$$

Comparing the coefficients of X^0, we have

$$\frac{\Gamma(1+\alpha)}{\Gamma(1)}\psi_1 = \lambda_1 + \lambda_2\psi_0 + \lambda_3\psi_0^2;$$

but $\psi_0 = 1$ so terms of degree m different of X^0

$$\frac{\Gamma(1+m\alpha+\alpha)}{\Gamma(1+m\alpha)}\psi_{m+1} = \lambda_2\psi_m + \lambda_3\sum_{k=0}^{m}\psi_k\psi_{m-k}$$

Hence

$$\psi_{m+1} = \frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha+\alpha)}\left(\lambda_2\psi_m + \lambda_3\sum_{k=0}^{m}\psi_k\psi_{m-k}\right)$$

So we can write the solution in the following form

$$\psi(x^{\alpha}) = 1 + \frac{1}{\Gamma(1+\alpha)}(\lambda_1+\lambda_2+\lambda_3)x^{\alpha} + \sum_{m=1}^{\infty}\frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha+\alpha)}\left(\lambda_2\psi_m + \lambda_3\sum_{k=0}^{m}\psi_k\psi_{m-k}\right)x^{\alpha m} \tag{8}$$

Now the following two Lemma and Theorem show that the solution of Eq. (8) is convergent

Lemma 4.1 *Let* $0 < \alpha < 1$. *The sequence* ψ_m *is defined by* $\psi_0 = 1$ *and*

$$\psi_{m+1} = \frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha+\alpha)}\left(\lambda_1\psi_m + \lambda_2\sum_{k=0}^{m}\psi_k\psi_{m-k}\right).$$

There is apositive number c such that, for all $m = 0,1,2,...,$

$$\frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha+\alpha)} \le c.$$

Proof. *It is known that* $\Gamma(x) \le \Gamma(y)$ *when* $2 \le x \le y$. *Therefore, if* $m \ge m_0 > \frac{1}{\alpha}$ *then* $g_m := \frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha+\alpha)} \le 1$. *We can take*

$$c = \max\{g_0, ..., g_{m_0}\}$$

Using the lemma, we have

$$|\psi_{m+1}| \le c|\lambda_1||\psi_m| + c|\lambda_2|\sum_{k=0}^{m}|\psi_k||\psi_{m-k}| \le d\sum_{k=0}^{m}|\psi_k||\psi_{m-k}|,$$

where

$$d = c(|\lambda_1| + |\lambda_2|).$$

Theorem 4.2 *The solution of the recursion* $\phi_0 = 1$ *and* $\phi_{m+1} = d\sum_{k=0}^{m} \phi_k \phi_{m-k}$ is

$$\phi_m = \frac{\left(\frac{1}{2}\right)_m (4d)^m}{(m+1)!}.$$

Proof. *Define* $F(x) = \sum_{m=0}^{\infty} \phi_m x^m$. *Then*

$$\frac{F(x)-1}{x} = \sum_{m=0}^{\infty} \phi_{m+1} x^m$$

and

$$F(x)^2 = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \phi_k \phi_{m-k} x^m.$$

Therefore, $F(x)$ satisfies the equation

$$\frac{F(x)-1}{x} = dF(x)^2,$$

so

$$F(x) = \frac{1-\sqrt{1-4dx}}{2dx}.$$

By the binomial theorem,

$$F(x) = \frac{1}{2dx}\left(1 - \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n}{n!}(4dx)^n\right)$$

$$= -\frac{1}{2dx}\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)_n}{n!}(4dx)^n$$

$$= \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m (4d)^m}{(m+1)!} x^m.$$

Therefore,

$$|\psi_m| \le \phi_m = \frac{\left(\frac{1}{2}\right)_m (4d)^m}{(m+1)!} \le (4d)^m.$$

This implies that

$$\sum_{m=0}^{\infty} \left|\psi_m x^{m\alpha}\right| \le \sum_{m=0}^{\infty} \left(4dx^{\alpha}\right)^m$$

so $\sum_{m=0}^{\infty} \psi_m x^{m\alpha}$ converges if

$$|x|^{\alpha} < \frac{1}{4d}.$$

CONCLUSION

From above we conclude that the transform method of fractional differential equation affected on the exact solutions of fractional differential equations. This method has more advantages, it is direct and concise. Thus, we realize that the proposed method can be extended to solve many systems of nonlinear fractional partial differential equations. Moreover, the solution of the generalized fractional Reccati equation of any arbitrary order is analytic in its domain.

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