An EOQ Inventory Model for Deteriorating Items with Linear Demand, Salvage Value and Partial Backlogging

Pandit Jagatananda Mishra
Research Scholar, Department of Mathematics, Ravenshaw University, Cuttack-753003, Odisha, India

Trailokyanath Singh
Department of Mathematics, C. V. Raman College of Engineering, Bhubaneswar-752205, Odisha, India

Hadibandhu Pattanayak
Department of Mathematics, Institutes of Mathematics and Applications, Andharua, Bhubaneswar-751003, Odisha, India

Abstract
In this paper, an EOQ (economic order quantity) model is developed for a deteriorating item when the demand pattern is considered as a linear function of time. Shortages are allowed and partially backlogged. The salvage value is included into deteriorated units. The backlogging rate is variable and dependent on the waiting time for the next replenishment. The objective of the model is to determine the value of the shortage point, cycle length and order quantity in order to minimize the average total cost. The results obtained in this paper are illustrated with the help of a numerical example and sensitivity analysis.

Keywords: EOQ, Inventory, Linear Demand Pattern, Partial Backlogging, Salvage Value

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Introduction
In the recent three decades, various researches have been done on inventory problems for deteriorating items such as fashion goods, electronic components, hi-fi equipments, medicines, drugs, blood banks, chemicals, volatile liquids and radioactive substances. Deterioration is a phenomenon by which items may decay, damage, spoilage, evaporate, loss of utility or loss of marginal value that results in decreasing usefulness from the original one. The deterioration rate of inventory in stock during the storage period constitutes an important factor which has attracted the researchers. The research on inventory was begun with Whitin [1] who studied the deterioration of fashion goods after the prescribed time of storage. Later, Ghare and Schrader [2] first studied the consumption of deteriorating items which was closely related to the negative exponential function of time. They stated the deterioration of inventory with the help of a linear differential equation where inventory level and demand rate as function of time. Shah and Jaiswal [3] presented an order level inventory model for deteriorating items considering the demand as function of time. Aggarwal [4] re-established their model by rectifying the errors of their work. In all these models, the deterioration rate as well as the demand pattern was taken as constants, replenishment was infinite and no shortage in inventory was allowed. Earlier, in the classical EOQ (Economic Order Quantity) model, the demand rate of an item was constant. Many models on inventory were developed in the inventory literature considering the demand rate as constant. But in the real life situation, demand rate of any product is always in a dynamic state. Therefore, many researchers are motivated for developing the models with time dependent demand. Silver and Meal [5], the earliest researchers who developed the modified EOQ with time-varying demand. However, Donaldson [6], the first researcher who gave a fully analytical treatment to the problem of inventory replenishment with a linearly time-dependent demand. Many researchers like Silver [7], Ritchie [8], Mitra et al. [9], Singh and Pattanayak [10] and Singh and Pattnayak [11] made the valuable contribution in this direction. The possibilities of shortage and deterioration in inventory were not considered in all models. Dave and Patel [12] developed an inventory model for deteriorating items with time-proportional demand. Sachan [13] extended their model by considering the backlogging option. The assumption of constant deterioration rate was relaxed by Covert and Philip [14] who used a with two-parameter Weibull distribution to represent the distribution of time to deterioration. Further, Philip [15] generalized their model by considering a three-parameter Weibull distribution. Goyal and Giri [16] studied recent trends in modeling deteriorating inventory. Li et al. [17] also studied the review literature on deteriorating inventory.

In real life situation, shortages may occur in inventory model. Items like fashionable goods, hi-fi equipments and clothes, customers may prefer to wait for back orders when shortage occurs. This type of shortage is called the external shortage. In this case, the customer’s order is not fulfilled due to the limited resources. The internal shortage occurs when the demand is uncertain. The shortage can result in back order costs, profit loss, delay cost etc. During a shortage period, the
backlogging rate is variable and dependent on the waiting time for the next replenishment. Abad [18] studied a lot-sizing inventory model for deteriorating items with variable rate of deterioration, allowing shortages and backlogging. Chang and Dye [19] developed an EOQ model with varying demand and partial backlogging by considering stock out cost. Consequently, the opportunity cost due to lost sales was considered in the model. Researches in this direction came from Papachristos and Skouri [20], Ouyang et al. [21] and Singh et al. [22] etc.

Most of the articles addressed assume that the deterioration of a unit is a complete loss to the inventory system and that these deteriorated units have no sale value. However, in practice, the vendor can offer a reduced unit cost to his buyer for the deteriorated stock. Jaggi and Aggarwal [23] developed an EOQ model for deteriorating items with salvage values. Mishra and Shah [24] formulated a mathematical model for items which deteriorate with respect to time. In their paper the salvage value is incorporated into the deteriorated units. Annadurai [25] studied an optimal replenishment policy considering shortages and salvage value. Recently, Singh and Pattinayak [26] studied the EOQ model for deteriorating items considering linear demand and partial backlogging.

In this study, an EOQ inventory model for deteriorating items has been developed considering the linear demand pattern and variable deterioration rate. Shortages are allowed and partially backlogged. Items such as paddy, food stuffs, fruits and vegetables etc. whose deterioration rate increase with time is considered as linear demand pattern. The backlogging rate is inversely proportional to the waiting time for the next replenishment. The salvage value is incorporated into these deteriorating items. It is observed that inclusion of the salvage value results in a reduced total cost of the inventory system. Furthermore, we have used the numerical example by minimizing the total cost by simultaneously optimizing the shortage period and the length of cycle. Finally, we have studied the sensitivity analysis of the various parameters on the effect of the optimal solution.

Assumptions

The following assumptions are made in developing the model.

(i) The inventory system involves only one item and the planning horizon is infinite.

(ii) The replenishment occurs instantaneously at an infinite rate.

(iii) The deteriorating rate \( \theta(t) = \theta, \quad 0 \leq \theta < 1 \), is a variable deterioration and there is no replacement or repair of deteriorated units during the period under consideration.

(iv) The demand rate, \( D(t) = \begin{cases} a + bt, & I(t) > 0, \\ D_0, & I(t) \leq 0, \end{cases} \) where \( a > 0, b > 0 \) and \( a \) is initial demand.

(v) During the shortage period, the backlogging rate is variable and is dependent on the length of the waiting time for the next replenishment. The longer the waiting time is, the smaller the backlogging rate would be. Hence, the proportion of customers who would like to accept backlogging at time \( t \) is decreasing with the waiting time \( (T - t) \) waiting for the next replenishment. To take care of this situation we have defined the backlogging rate to be \( \frac{1}{1 + \delta(T - t)} \) when inventory is negative. The backlogging parameter \( \delta \) is a positive constant \( 0 \leq t \leq T \).

Notations

The following notations have been used in developing the model.

(i) \( C_1 \): holding cost, $/per unit/per unit time.

(ii) \( C_2 \): cost of the inventory item, $/per unit.

(iii) \( C_3 \): ordering cost of inventory, $/per order.

(iv) \( C_4 \): shortage cost, $/per unit/per unit time.

(v) \( C_5 \): opportunity cost due to lost sales, $/per unit.

(vi) \( C_v \): salvage value parameter, where \( 0 \leq C_v < 1 \), associated with deteriorated units during the cycle time.

(vii) \( t_1 \): time at which shortages start.

(viii) \( T \): length of each ordering cycle.

(ix) \( W \): the maximum inventory level for each ordering cycle.

(x) \( S \): the maximum amount of demand backlogged for each ordering cycle.

(xi) \( Q \): the economic order quantity for each ordering cycle.

(xii) \( I(t) \): the inventory level at time \( t \).

(xiii) \( t_1^* \): the optimal solution of \( t_1 \).

(xiv) \( T^* \): the optimal solution of \( T \).

(xv) \( Q^* \): the optimal economic order quantity.

(xvi) \( W^* \): the optimal maximum inventory level.

(xvii) \( T^* \): the minimum average total cost per unit time.

Mathematical Model

We consider the deteriorating inventory model with linear demand. Replenishment occurs at time \( t = 0 \) when the inventory level attains its maximum, \( W \). From \( t = 0 \) to \( t_1 \), the inventory level reduces due to demand and deterioration. At time \( t_1 \), the inventory level achieves zero, then shortage is allowed to occur during the time interval \( [t_1, T] \), and all of
the demand during shortage period \([t_1, T]\) partially backlogged. 
As the inventory level reduces due to demand rate as well as deterioration during the inventory interval \([t_1, T]\), the differential equation representing the inventory status is governed by
\[
\frac{dI_1(t)}{dt} + \theta(t)I_1(t) = -D(t), \quad 0 \leq t \leq t_1,
\]
where \(\theta(t) = \theta t\) and \(D(t) = a + bt\).

The integrating factor is \(IF = e^{\frac{\theta t}{2}}\).

The solution of Equation (1) using the condition \(I_1(0) = 0\) is
\[
I_1(t) = a\left(t + \frac{\theta t_1^3}{6}\right) + b\left(\frac{t^2}{2} + \frac{\theta t_1^4}{8}\right) -
\]
\[
a\left(t + \frac{\theta t_1^3}{6}\right) - b\left(\frac{t^2}{2} + \frac{\theta t_1^4}{8}\right)e^{\frac{\theta t}{2}}, \quad 0 \leq t \leq t_1,
\]
(by neglecting the higher power of \(\theta\) as \(0 < \theta << 1\)).

Maximum inventory level for each cycle is obtained by putting the boundary condition \(I_1(0) = W\) in Equation (2).

Therefore,
\[
W = I_1(0) = a\left(t_1 + \frac{\theta t_1^3}{6}\right) + b\left(\frac{t_1^2}{2} + \frac{\theta t_1^4}{8}\right).
\]
(3)

During the shortage interval \([t_1, T]\), the demand at time \(t\) is partially backlogged at the fraction \(\frac{1}{1+\delta(T-t)}\). Therefore,

the differential equation governing the amount of demand backlogged is
\[
\frac{dI_2(t)}{dt} = -\frac{D_0}{1+\delta(T-t)}, \quad t_1 \leq t \leq T.
\]
(4)

with the boundary condition \(I_2(t_1) = 0\).

The solution of Equation (4) is
\[
I_2(t) = \frac{D_0}{\delta}\left[\ln\left(1+\delta(T-t_1)\right)
- \ln\left(1+\delta(T-t)\right)\right], \quad t_1 \leq t \leq T.
\]
(5)

Maximum amount of demand backlogged per cycle is obtained by putting \(t = T\) in Equation (5). Therefore,
\[
S = -I_2(T) = \frac{D_0}{\delta}\ln\left[1+\delta(T-t_1)\right].
\]
(6)

Hence, the economic order quantity per cycle is

\[
Q = W + S = a\left(t_1 + \frac{\theta t_1^3}{6}\right) +
\]
\[
b\left(\frac{t_1^2}{2} + \frac{\theta t_1^4}{8}\right) = \frac{D_0}{\delta}\ln\left[1+\delta(T-t_1)\right].
\]
(7)

The inventory holding cost per cycle is

\[
HC = C_1\int_0^h I_1(t)dt = C_1\int_0^h \left[ a\left(t + \frac{\theta t_1^3}{6}\right) +
\right.
\]
\[
 b\left(\frac{t^2}{2} + \frac{\theta t_1^4}{8}\right)e^{\frac{\theta t}{2}}\right]dt
\]
\[
= C_1\left[ a\left(t_1 + \frac{\theta t_1^3}{6}\right) + b\left(\frac{t_1^3}{3} + \frac{\theta t_1^4}{15}\right)\right],
\]
(by neglecting the higher power of \(\theta\) as \(0 < \theta << 1\)).

The deterioration cost per cycle is

\[
DC = \frac{D_0}{\delta}\left[W - \int_0^h D(t)dt\right]
\]
\[
= \frac{D_0}{\delta}\left[W - \int_0^h (a + bt)dt\right] = \frac{D_0}{\delta}\left[\frac{at_1^3}{6} + \frac{bt_1^4}{8}\right].
\]
(9)

The ordering cost is

\[
OC = C_3.
\]
(10)

The shortage cost per cycle is

\[
SC = C_4\left[-\int_0^h I_2(t)dt\right]
\]
\[
= -\frac{C_4D_0}{\delta}\left[\ln\left(1+\delta(T-t)\right) - \ln\left(1+\delta(T-t_1)\right)\right]dt
\]
\[
= C_4D_0\left[\frac{T-t_1}{\delta} - \frac{1}{\delta^2}\ln\left(1+\delta(T-t_1)\right)\right].
\]
(11)

The opportunity cost due to lost sales per cycle is

\[
OCLS = C_5\int_0^h \left[\frac{1}{1+\delta(T-t)}\right]dt
\]
\[
= C_5D_0\left[T - t_1 - \frac{1}{\delta}\ln\left(1+\delta(T-t_1)\right)\right].
\]
(12)

The salvage value of the deteriorated items is

\[
SV = C_5\int_0^h \theta(t)I_1(t)dt
\]
\[
= C_5\int_0^h \theta(t)\left[a\left(t_1 + \frac{\theta t_1^3}{6}\right) + b\left(\frac{t^2}{2} + \frac{\theta t_1^4}{8}\right)\right]
\]
\[
\]
6481
Now, *\( t \) and *\( T \) respectively. Next, by using *\( t \) and *\( T \), we can obtained the optimal economic order quantity and the minimum average total cost per unit time from Equations (7) and (14) respectively.

**Numerical Example**

In this section, we provide a numerical example to illustrate the above theory.

**Example 1:*** Let us take the parameter values of the inventory system as follows:

- *\( a = 12 \), \( b = 2 \), \( C_1 = 0.5 \), \( C_2 = 1.5 \), \( C_3 = 3 \), \( C_4 = 2.5 \),
- \( C_5 = 2 \), \( D_0 = 8 \), \( C_v = 0.1 \), \( \theta = 0.01 \) and *\( \delta = 2 \).

Solving Equations (16) and (17), we have the optimal shortage period *\( t'^* = 0.847139 \) unit time and the optimal length of ordering cycle *\( T'^* = 0.993363 \) unit time. Thereafter, we get the optimal order quantity *\( Q'^* = 11.9229 \) units and the minimum average total cost per unit time *\( TC'^* = 5.88315 \).

**Sensitivity Analysis**

We study now study the effects of changes in the values of the system parameters *\( a \), *\( b \), *\( C_1 \), *\( C_2 \), *\( C_3 \), *\( C_4 \), *\( C_5 \), *\( D_0 \), *\( C_v \), *\( \theta \) and *\( \delta \) on the optimal total cost and number of reorder. The sensitivity analysis is performed by changing each of the parameters by +50%, +10%, -10% and -50% taking one parameter at a time and keeping the remaining parameters unchanged.

The analysis is based on the Example 1 and the results are shown in Table 1. The following points are observed.

1) *\( t'^* \) and *\( T'^* \) decrease while *\( TC'^* \) increases with the increase in the value of the parameter *\( a \). Here all *\( t'^* \), *\( T'^* \) and *\( TC'^* \) are highly sensitivity to change in *\( a \).

2) *\( t'^* \) and *\( T'^* \) decrease while *\( TC'^* \) increases with the increase in the value of the parameter *\( b \). Here all *\( t'^* \), *\( T'^* \) and *\( TC'^* \) are lowly sensitivity to change in *\( b \).

3) *\( t'^* \) and *\( T'^* \) decrease while *\( TC'^* \) increases with the increase in the value of the parameter *\( C_1 \). Here all *\( t'^* \), *\( T'^* \) and *\( TC'^* \) are moderately sensitivity to change in *\( C_1 \).

4) *\( t'^* \) and *\( T'^* \) decrease while *\( TC'^* \) increases with the increase in the value of the parameter *\( C_2 \). Here all *\( t'^* \), *\( T'^* \) and *\( TC'^* \) are moderately sensitivity to change in *\( C_2 \).
5) $t^*_1$, $T^*$ and $TC^*$ increase with the increase in the value of the parameter $C_3$. Here all $t^*_1$, $T^*$ and $TC^*$ are highly sensitive to change in $C_3$.

6) $t^*_1$ and $TC^*$ increase while $T^*$ decreases in the value of the parameter $C_4$. Here all $t^*_1$, $T^*$ and $TC^*$ are lowly sensitivity to change in $C_4$.

7) $t^*_1$ and $TC^*$ increase while $T^*$ decreases in the value of the parameter $C_5$. Here all $t^*_1$, $T^*$ and $TC^*$ are lowly sensitivity to change in $C_5$.

8) $t^*_1$ and $TC^*$ increase while $T^*$ decreases in the value of the parameter $D_0$. Here all $t^*_1$, $T^*$ and $TC^*$ are moderately sensitivity to change in $D_0$.

9) $t^*_1$ and $T^*$ increase while $TC^*$ decreases in the value of the parameter $C_v$. Here all $t^*_1$, $T^*$ and $TC^*$ are insensitivity to change in $C_v$.

10) $t^*_1$ and $T^*$ decrease while $TC^*$ increases in the value of the parameter $\theta$. Here all $t^*_1$, $T^*$ and $TC^*$ are lowly sensitivity to change in $\theta$.

11) $t^*_1$ and $T^*$ increase while $TC^*$ decreases in the value of the parameter $\delta$. Here all $t^*_1$, $T^*$ and $TC^*$ are lowly sensitivity to change in $\delta$.

**Table 1. Sensitivity analysis**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% Change in parameter</th>
<th>$t^*_1$</th>
<th>$T^*$</th>
<th>$TC^*$</th>
<th>% Change in $TC^*$</th>
</tr>
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<tbody>
<tr>
<td>$a$</td>
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<td>$b$</td>
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**Conclusions**

In this study, an EOQ inventory model for deteriorating items has been developed considering the linear demand pattern and variable deterioration rate. Shortages are allowed and partially backlogged. Items such as paddy, food stuffs, fruits and vegetables etc. whose deterioration rate increase with time is considered as linear demand pattern. The backlogging rate is inversely proportional to the waiting time for the next replenishment. The salvage value is incorporated into these deteriorating items. It is observed that inclusion of the salvage value results in a reduced total cost of the inventory system. Furthermore, we have used the numerical example by minimizing the total cost by simultaneously optimizing the shortage period and the length of cycle. Finally, we have studied the sensitivity analysis of the various parameters on the effect of the optimal solution. There are a numerous directions in which this research can be extended. One possible extension stems from the demand. For instance, we may extend the linear demand into a more realistic time-varying demand that is function of time, selling price and others. Also, we could consider the effects of the variable deteriorations (two-parameter Weibull, three-parameter Weibull and Gamma distribution). Finally, we could generalize the model to stochastic fluctuating demand patterns, the economic production lot size model and allowing quantity discounts and inflation rate.

**References**


