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Vague Prime Ideals In Γ-Semirings

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Abstract

The concept of vague prime ideals of Γ-semiring with membership and non-membership functions taking values in the unit interval [0, 1] of real numbers is introduced. All vague prime ideals of a Γ-semiring are determined by establishing a one-to-one correspondence between vague prime ideals and the pair (P, α), where P is a prime ideal of a Γ-semiring and α is a prime element in [0, 1].

Key Words: Vague set, Vague characteristic set, Vague ideal, Vague prime ideal, Vague maximal ideal.

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Introduction

In 1965, Zadeh.L.A[15] introduced the study of fuzzy sets. Mathematically a fuzzy set on a set X is a mapping μ into [0, 1] of real numbers; for x in X, μ(x) is called the membership of x belonging to X. The membership function gives only an approximation for belonging but it does not give any information of not belonging. To avoid this, Gau.W.L. and Buehrer.D.J.[6] introduced the concept of vague sets. A vague set A of a set X is a pair of functions (t_A, f_A), where t_A and f_A are fuzzy sets on X satisfying $t_A(x) + f_A(x) \leq 1$, for all x in X. A fuzzy set t_A of X can be identified with the pair $(t_A, 1 - t_A)$. Thus the theory of vague sets is a generalization of fuzzy sets. Ranjit Biswas[13] initiated the study of vague Algebra by studying the properties related to vague groups and vague normal groups. Further Ramakrishna.N[12] and Eswarlal.T[5,15] continued the study of vague Algebra by studying the characterization of cyclic groups in terms of vague groups, vague normal groups, vague normalizer, vague centralizer, vague ideals, normal vague ideals, vague fields, vague vector spaces etc.

Swamy.K.L.N and Swamy.U.M[14] introduced and studied the notions of fuzzy ideals and fuzzy prime ideals of a ring with truth values in a complete lattice satisfying the infinite meet distributive law which generalizes the existing notions with truth values in the unit interval of real numbers. They proved that all fuzzy prime(maximal) ideals of a given ring are determined by establishing a one-to-one correspondence between fuzzy prime(maximal) ideals and the pair (P, α), where P is a prime(maximal) ideal of a ring and α is a prime element(dual atom) in the complete lattice. In fact Ramakrishna.N and Eswarlal.T[12] studied Boolean vague sets where the vague set of the universe X is defined by the pair of functions (t_A, f_A), where t_A and f_A are mappings from a set X into a Boolean Algebra satisfying the condition $t_A(x) \leq (f_A(x))'$, for all x in X, $(f_A(x))'$ is the complement of $f_A(x)$ in the Boolean algebra. Moreover Eswarlal.T[5] introduced Boolean vague ideals, Boolean vague prime ideals, Boolean vague maximal ideals of a ring. Further M.K.Rao[10] introduced the concept of Γ-semiring as a generalization of Γ-ring by Nobusawa.N[11] as well as semiring. The concepts of Γ-semirings and its sub Γ-semirings with a left(right) unity was studied by Luh.J[9] and M.K.Rao[10], further the ideals, prime ideals, semiprime ideals, k-ideals and h-ideals of a Γ-semiring, regular Γ-semiring were extensively studied by Kyuno.S[8] and M.K.Rao[10]. The properties of an ideal in semirings and Γ-semirngs were some what different from the properties of the usual ring ideals. Moreover the notion of Γ-semiring not only generalizes the notions of semiring and Γ-ring but also the notion of ternary semiring.

The authors in this paper are introduce and study vague prime ideals and vague maximal ideals of a Γ-semiring and established a one-to-one correspondence between vague prime ideals of a Γ-semiring and the pair (P, α), where P is a prime ideal of a Γ-semiring and α is a prime element in [0, 1]. If one confines only to the case [0, 1], no Γ-semiring possesses any vague maximal ideals, if [0, 1] has dual atoms, then Γ-semiring will possesses vague maximal ideals if and only if it possesses maximal ideals. Since [0, 1] does not contains dual atoms, instead of [0, 1] we consider a complete lattice L satisfying the infinite meet distributive law and hence we established a one-to-one correspondence between vague maximal ideals of a Γ-semiring and the pair (M, α), where M is a maximal ideal of a Γ-semiring and α is a dual atom in L.

Throughout this paper, R stands for Γ-semiring with zero. That is Let R and Γ be two additive commutative semigroups. Then R is called a Γ-semiring if there exists a mapping $R \times \Gamma \times R \rightarrow R$ image to be denoted by $a\alpha b$ for a, b $\in$ R and $\alpha \in \Gamma$ satisfying the following conditions.

1. $a\alpha(b + c) = a\alpha b + a\alpha c$
2. $(a + b)\alpha c = a\alpha c + b\alpha c$
3. $a(\alpha + \beta)c = a\alpha c + a\beta c$
4. $a\alpha (b\beta c) = (a\alpha b)\beta c$, $\forall$ a, b, c $\in$ R; $\alpha, \beta \in \Gamma$.

A Γ-semiring R is said to have a zero element if there exists an element $0 \in R$ such that $0 + x = x = x + 0$ and $0\gamma x = 0 = x\gamma 0$, $\forall x \in R, \gamma \in \Gamma$.

Preliminaries

In this section we recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1
An ideal P of R is said to be prime ideal of R, if for any two ideals A, B of R such that $A\Gamma B \subseteq P$ that implies $A \subseteq P$ or $B \subseteq P$.

Definition 2.2
An ideal M of R is said to be maximal ideal of a Γ-semiring R, if there exists an ideal N of R such that $M \subseteq N \subseteq R$ that implies either M = N or N = R.

Definition 2.3
Let X be any non-empty set. A mapping $\mu : X \to [0,1]$ is called a fuzzy subset of R.

Definition 2.4
A vague set A in the universe of discourse U is a pair (t_A, f_A), where $t_A : U \to [0, 1]$, $f_A : U \to [0, 1]$ are mappings such that $t_A(u) + f_A(u) \leq 1$, $\forall\ u \in U$. The functions t_A and f_A are called true membership function and false membership function respectively.

Definition 2.5
The interval $[t_A(u), 1 - f_A(u)]$ is called the vague value of u in A and it is denoted by $V_A(u)$ i.e., $V_A(u) = [t_A(u), 1 - f_A(u)]$.

Definition 2.6
A vague set A is contained in the other vague set B, $A \subseteq B$ if and only if $V_A(u) \leq V_B(u)$ i.e., $t_A(u) \leq t_B(u)$ and $1 - f_A(u) \leq 1 - f_B(u)$, $\forall\ u \in U$.

Definition 2.7
Two vague sets A and B are equal written as A = B, if and only if $A \subseteq B$ and $B \subseteq A$ i.e., $V_A(u) \leq V_B(u)$ and $V_B(u) \leq V_A(u)$, $\forall\ u \in U$.

Definition 2.8
The union of two vague sets A and B with respective truth membership and membership functions t_A, f_A ; t_B, f_B is a vague set C, written as $C = A \cup B$, whose truth membership and false membership functions are related to those of A and B by $t_C = \max\{t_A, t_B\}$ and $1 - f_C = \max\{1 - f_A, 1 - f_B\} = 1 - \min\{f_A, f_B\}$.

Definition 2.9
The intersection of two vague sets A and B with respective truth membership and membership functions t_A, f_A : t_B, f_B is a vague set C, written as $C = A \cap B$, whose truth membership and false membership functions are related to those of A and B by $t_C = \min\{t_A, t_B\}$ and $1 - f_C = \min\{1 - f_A, 1 - f_B\} = 1 - \max\{f_A, f_B\}$.

Definition 2.10
A vague set A of a set U with $t_A(u) = 0$ and $f_A(u) = 1$, $\forall$ u $\in$ U is called zero vague set of U.

Definition 2.11
A vague set A of a set U with $t_A(u) = 1$ and $f_A(u) = 0$, $\forall$ u $\in$ U is called unit vague set of U.

Definition 2.12
A vague set A of R is called a constant vague set if $V_A(x) = V_A(y)$, $\forall$ x, y $\in$ R.

Definition 2.13
Let $\chi = (t_\chi, f_\chi)$ be a vague set of R. For any subset S of R , the characteristic function of S taking values in [0, 1] of a vague set $\chi_S = (t_{\chi_S}, f_{\chi_S})$ by

$$V_{\chi_S}(x) = \begin{cases} [1,1] \ if \ x \in S \\ [0,0] \ if \ x \notin S \end{cases}$$

$$\text{i.e., } t_{\chi_S}(x) = \begin{cases} 1 \ if \ x \in S \\ 0 \ if \ x \notin S \end{cases} \ and \ f_{\chi_S}(x) = \begin{cases} 0 \ if \ x \in S \\ 1 \ if \ x \notin S \end{cases}$$

Then χ_S is called the vague characteristic set of S in [0, 1].

Definition 2.14 [3]
A vague set A = (t_A, f_A) of R is said to be left(right) vague ideal of R if the following conditions are true:

For all x, y $\in$ R; $\gamma \in \Gamma$,
$V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$ and $V_A(x\gamma y) \geq V_A(y)$ ($\geq V_A(x)$)
i.e., (i). $t_A(x + y) \geq \min\{t_A(x), t_A(y)\}$,
$1 - f_A(x + y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$ and
(ii). $t_A(x\gamma y) \geq t_A(y)$ ($\geq t_A(x)$),
$1 - f_A(x\gamma y) \geq 1 - f_A(y)$ ($\geq 1 - f_A(x)$).

Definition 2.15
A vague set A = (t_A, f_A) of R is both left vague ideal and right vague ideal of R then A is called a vague ideal of R.

Definition 2.16
Let A and B are two vague sets of R Then the product of A and B, denoted by AΓB is defined by

$V_{A\Gamma B}(x) = \sup\{\min\{V_A(y), V_B(z)\}/ x = y\gamma z$, where y, z $\in$ R; $\gamma \in \Gamma\}$, $\forall$ x $\in$ R.
i.e., $t_{A\Gamma B}(x) = \sup\{\min\{t_A(y), t_B(z)\}/ x = y\gamma z$, where y, z $\in$ R; $\gamma \in \Gamma\}$
$1 - f_{A\Gamma B}(x) = \sup\{\min\{1- f_A(y), 1- f_B(z)\}/ x = y\gamma z$, where y, z $\in$ R; $\gamma \in \Gamma\}$.

Notation

Let I[0, 1] denote the family of all closed sub intervals of [0, 1]. I_1= [a_1, b_1] and I_2= [a_2, b_2] are two elements of I[0, 1]. We call $I_1 \geq I_2$, if $a_1 \geq a_2$ and $b_1 \geq b_2$, with the order in I[0, 1] is a lattice with the operations min. or inf and max. or sup given by min{I_1, I_2} = [min{a_1, a_2}, min{b_1, b_2}] and max{I_1, I_2} = [max{a_1, a_2}, max{b_1, b_2}]. Also we denote $I_1 + I_2 = [a_1 + a_2, b_1 + b_2]$.

Vague Prime Ideals of Γ–Semirings

In this section we introduce the concept of vague prime ideals of R and we proved, if A=(t_A, f_A) is a vague prime ideal of R, then $V_A(0)$=[1, 1] and A takes only two values. Further we established a one-to-one correspondence between vague prime ideals of R and prime ideals of R.

Definition 3.1

A non constant vague ideal A of R is called a vague prime ideal of R if for any two vague ideals B, C of R such that BΓC $\subseteq$ A implies B $\subseteq$ A or C $\subseteq$ A.

Example 3.2

Let R be the set of natural numbers with zero and let Γ be the set of positive even integers. Then R, Γ are commutative semi groups.

Define the mapping R × Γ × R → R by aαb usual product of a, α, b $\forall$ a, b $\in$ R; α $\in$ Γ.

Then R is a Γ-semiring.

Let A = (t_A, f_A), where t_A : R → [0,1] and f_A : R → [0,1] defined by

$$t_A(x) = \begin{cases} 0.9 \; if \; x \; is \; even \; or \; 0 \\ 0.3 \; if \; x \; is \; odd \end{cases} \quad \text{and} \quad f_A(x) = \begin{cases} 0.1 \; if \; x \; is \; even \; or \; 0 \\ 0.6 \; if \; x \; is \; odd \end{cases}$$

Clearly A is a vague ideal of R.

Now we consider two vague ideals B= (t_B, f_B) and C= (t_C, f_C) of R as

$$t_B(x) = \begin{cases} 0.6 \; if \; x = 0 \\ 0.5 \; if \; x \; is \; even \\ 0.4 \; if \; x \; is \; odd \end{cases} ; \quad f_B(x) = \begin{cases} 0.4 \; if \; x = 0 \\ 0.3 \; if \; x \; is \; even \\ 0.6 \; if \; x \; is \; odd \end{cases}$$

and

$$t_C(x) = \begin{cases} 0.7 \; if \; x = 0 \\ 0.6 \; if \; x \; is \; even \\ 0.1 \; if \; x \; is \; odd \end{cases} ; \quad f_C(x) = \begin{cases} 0.1 \; if \; x = 0 \\ 0.2 \; if \; x \; is \; even \\ 0.5 \; if \; x \; is \; odd \end{cases}$$

It was shown that BΓC $\subseteq$ A.

Further it was verified that $B \not\subset A$ but $C \subseteq A$.
It gives that $B\Gamma C \subseteq A \Rightarrow C \subseteq A$.
Hence by definition:3.1, A is a vague prime ideal of R.

Lemma 3.3
Let I be an ideal of R. If the vague set $A = (t_A, f_A)$ of R is defined as

$$V_A(x) = \begin{cases} [1,1] & if\ x \in I \\ [\alpha, \beta] & otherwise \end{cases}, \ 0 \le \alpha < \beta \le 1,\ x \in R.$$

Then A is a left(right) vague ideal of R.

Proof
Let $x, y \in R$; $\gamma \in \Gamma$.
If $x, y \in I$, then $x + y, x\gamma y \in I$.
So, $V_A(x + y) = [1, 1] = \min\{V_A(x), V_A(y)\}$ and
$V_A(x\gamma y) = [1, 1] = V_A(y)$ ($= V_A(x)$).
If $x, y \notin I$, then $V_A(x) = [0, 0]$ and $V_A(y) = [0, 0]$.
So, $V_A(x + y) \ge [0, 0] = \min\{V_A(x), V_A(y)\}$ and
$V_A(x\gamma y) \ge [0, 0] = V_A(y)$ ($= V_A(x)$).
Hence A is a left(right) vague ideal of R.

Theorem 3.4
Let I be an ideal of a R. Then the characteristic function χ_I of I is a vague prime ideal of R if and only if I is a prime ideal of R.

Proof
Suppose χ_I is a vague prime ideal of R.
Let G, H be two ideals of R such that $G\Gamma H \subseteq I$.
We prove that $G \subseteq I$ or $H \subseteq I$.
Suppose if possible $G \not\subset I$ and $H \not\subset I$.
That implies there exists $a \in G$ such that $a \notin I$ and $b \in H$ such that $b \notin I$, for some $a, b \in R$.
Define two vague sets $A = (t_A, f_A)$ and $B = (t_B, f_B)$ of R as

$$V_A(x) = \begin{cases} [1,1] & if\ x \in G \\ [0,0] & if\ x \notin G \end{cases} \text{ and } V_B(x) = \begin{cases} [1,1] & if\ x \in H \\ [0,0] & if\ x \notin H \end{cases}$$

Clearly A and B are vague ideals of R.
Let $x \in R$.
Now, $V_{A\Gamma B}(x) = \sup\{\min\{V_A(y), V_B(z)\}/ x = y\gamma z$, where $y, z \in R; \gamma \in \Gamma\}$
= either [0, 0] or [1, 1]
$\le V_{\chi_I}(x)$.
Then $A\Gamma B \subseteq \chi_I$.
This implies $A \subseteq \chi_I$ or $B \subseteq \chi_I$.

If A $\subseteq \chi_I$, then [1, 1]= $V_A(a) = V_{\chi_I}(a)$ = [0, 0]
This is impossible.
Therefore G $\subseteq$ I or H $\subseteq$ I.
Similarly if B $\subseteq \chi_I$, we get G $\subseteq$ I or H $\subseteq$ I.
Hence I is a prime ideal of R.
Conversely suppose that I is a prime ideal of R.
Let A and B be two vague ideals of R such that AΓB $\subseteq$ I.
Suppose if possible that A $\not\subset \chi_I$ and B $\not\subset \chi_I$.
That implies $V_A(x) > V_{\chi_I}(x)$ and $V_B(y) > V_{\chi_I}(y)$, for some x, y ∈ R.
$\Rightarrow V_{\chi_I}(x) = V_{\chi_I}(y)$ = [0, 0].
$\Rightarrow$ x, y $\notin$ I.
$\Rightarrow$ xγy $\notin$ I, where γ ∈ Γ.
Now, $V_{A\Gamma B}(x\gamma y) = \sup\{\min\{V_A(x), V_B(y)\}\}$
$> \sup\{\min\{V_{\chi_I}(x), V_{\chi_I}(y)\}\}$
= [0, 0]
$= V_{\chi_I}(x\gamma y)$
So, AΓB $\supset \chi_I$.
This is a contradiction.
Hence χ_I is a vague prime ideal of R.

Theorem 3.5
If A is a vague prime ideal of a R, then $V_A(0)$ = [1, 1].

Proof
On the contrary suppose $V_A(0)$ < [1, 1].
Since A is non constant, there exists a ∈ R such that $V_A(a) < V_A(0)$.
Define vague sets B and C on R as

$$V_B(x) = \begin{cases} [1,1] & if\ V_A(x) = V_A(0) \\ [0,0] & otherwise \end{cases}$$ and $V_C(x) = V_A(0)$, ∀ x ∈ R.

Since B is a characteristic function of $R_A = \{x \in R / V_A(x) = V_A(0)\}$, we have B is a vague ideal of R.
Also, C is a constant function, it is a vague ideal of R.
Now, $V_B(0)$ = [1, 1] > $V_A(0)$ that implies B $\not\subset$ A and
$V_C(a) = V_A(0) > V_A(a)$ that implies C $\not\subset$ A.
Let x ∈ R.
$V_{B\Gamma C}(x) = \sup\{\min\{V_B(y), V_C(z)\} / x = y\gamma z$, where y, z ∈ R; γ ∈ Γ}.
We have two cases
Case(i). If $V_B(y)$ = [0, 0]
Then $V_A(y) < V_A(0)$
So, $\min\{V_B(y), V_C(z)\} = \min\{[0, 0], V_A(0)\} = [0, 0] \le V_A(x)$.
Case(ii). If $V_B(y)$ = [1, 1]

Then $V_A(y) = V_A(0)$
So, $\min\{V_B(y), V_C(z)\} = \min\{[1, 1], V_A(0)\} = V_A(0) = V_A(y) \leq V_A(y\gamma z) = V_A(x)$.
From two cases $B\Gamma C \subseteq A$.
Since A is vague prime ideal of R, we have $B \subseteq A$ or $C \subseteq A$.
This is a contradiction.
Hence $V_A(0) = [1, 1]$.

Theorem 3.6
If A is a vague prime ideal of R, then A takes only two values.

Proof
Suppose A is a vague prime ideal of R.
From theorem-3.4, $V_A(0) = [1, 1]$.
Let $a, b \in R$ such that $V_A(a) < [1, 1]$ and $V_A(b) < [1, 1]$.
We prove that $V_A(a) = V_A(b)$.
Part(i). Define two vague sets $B = (t_B, f_B)$ and $C = (t_C, f_C)$ on R as

$$V_B(x) = \begin{cases} [1,1] & \text{if } x \in \langle a \rangle \\ [0,0] & \text{otherwise} \end{cases}$$ and $V_C(x) = V_A(a), \forall\, x \in R$.

Now B and C are vague ideals of R.
We have $V_B(a) = [1, 1] > V_C(x) = V_A(a)$, that implies $B \not\subset A$.
Let $x \in R$.
$V_{B\Gamma C}(x) = \sup\{\min\{V_B(y), V_C(z)\}/ x = y\gamma z, \text{ where } y, z \in R; \gamma \in \Gamma\}$.
If $y \notin \langle a \rangle$, then $V_B(y) = [0, 0]$.
So, $\min\{ V_B(y), V_C(z)\} = \min\{0, V_C(z)\} = 0 \leq V_A(y\gamma z) = V_A(x)$.
If $y \in \langle a \rangle$, then $V_B(y) = [1, 1]$.
So, $\min\{ V_B(y), V_C(z)\} = \min\{1, V_A(a)\} = V_A(a) \leq V_A(y) \leq V_A(y\gamma z) = V_A(x)$.
From these two cases $B\Gamma C \subseteq A$.
Since A is vague prime ideal of R, we have $B \subseteq A$ or $C \subseteq A$.
But $B \not\subset A$.
We have $C \subseteq A$.
Now, $V_A(b) \geq V_C(b) = V_A(a)$.
i.e., $V_A(b) \geq V_A(a)$.
Part(ii). Define two vague sets $G = (t_G, f_G)$ and $H = (t_H, f_H)$ on R as

$$V_G(x) = \begin{cases} [1,1] & \text{if } x \in \langle b \rangle \\ [0,0] & \text{otherwise} \end{cases}$$ and $V_H(x) = V_A(b), \forall\, x \in R$.

As similarly in part(i), we get $V_A(a) \geq V_A(b)$.
From part(i) and part(ii), $V_A(a) = V_A(b)$.
Hence A takes only two values.

Theorem 3.7

If A is a vague prime ideal of R, then $R_A = \{ x \in R / V_A(x) = V_A(0)\}$ is a prime ideal of R.

Proof

Suppose A is a vague prime ideal of R.
We have R_A is an ideal of R [3].
Let P and Q be two ideals of R such that $P\Gamma Q \subseteq R_A$.
Suppose if possible $P \not\subset R_A$ and $Q \not\subset R_A$.
$\Rightarrow$ there exists $a \in P$ such that $a \notin R_A$ and $b \in Q$ such that $b \notin R_A$, for some $a, b \in R_A$.
Define two vague sets $B = (t_B, f_B)$ and $C = (t_C, f_C)$ on R as

$$V_B(x) = \begin{cases} V_A(0) & if\ x \in P \\ [0,0] & otherwise \end{cases} \text{ and } V_C(x) = \begin{cases} V_A(0) & if\ x \in Q \\ [0,0] & otherwise \end{cases}$$

So, B and C are vague ideals of R.
Let $x \in R$.
Now, $V_{B\Gamma C}(x) = \sup\{\min\{V_B(y), V_C(z)\} / x = y\gamma z$, where $y, z \in R; \gamma \in \Gamma\}$.
We consider the case where $\min\{V_B(y), V_C(z)\} > 0$.
Then $V_B(y) = V_C(z) = V_A(0)$.
That implies $y \in P$ and $z \in Q$.
Now $x = y\gamma z \in P\Gamma Q \subseteq R_A$.
That implies $V_A(x) = V_A(0)$.
There fore $V_{B\Gamma C}(x) \le V_A(0) = V_A(x)$.
So, we get $B\Gamma C \subseteq A$.
Since A is a vague prime ideal of R, we have $B \subseteq A$ or $C \subseteq A$.
Suppose $B \subseteq A$.
Now, $a \notin R_A \Rightarrow V_A(a) \neq V_A(0)$.
i.e., $V_A(a) < V_A(0)$.
Also $V_B(a) = V_A(0) > V_A(a)$
That implies $B \not\subset A$.
This is a contradiction.
There fore $P \subseteq R_A$.
Similarly if $C \subseteq A$, we get $Q \subseteq R_A$.
Hence R_A is a prime ideal of R.

Theorem 3.8

Let $A = (t_A, f_A)$ be a vague set of R. Suppose that

1. $V_A(0) = [1, 1]$
2. A takes only two values say [1, 1] and $[\alpha, \beta]$, $0 \le \alpha \le \beta \le 1$.
3. $R_A = \{x \in R / V_A(x) = V_A(0)$ is a prime ideal of R,

then A is a vague prime ideal of R.

Proof

Let $A = (t_A, f_A)$ be a vague set of R.
Let $x, y \in R$.
Suppose $x, y \in R_A$.
Then $x + y, x\gamma y \in R_A, \gamma \in \Gamma$.
So, $V_A(x + y) = [1, 1] = \min\{V_A(x), V_A(y)\}$ and
$V_A(x \gamma y) = [1, 1] = V_A(y)$ $(=V_A(x))$
Suppose $x, y \notin R_A$.
Then $x + y, x\gamma y \notin R_A, \gamma \in \Gamma$.
So, $V_A(x + y) = [\alpha, \beta] = \min\{V_A(x), V_A(y)\}$ and
$V_A(x \gamma y) = [\alpha, \beta] = V_A(y)$ $(=V_A(x))$
If at least one of x, y is not in R_A.
Then $V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$ and
$V_A(x \gamma y) \geq V_A(y)$ $(\geq V_A(x)), \gamma \in \Gamma$.
Hence A is a vague ideal of R
Let $B = (t_B, f_B)$ and $C = (t_C, f_C)$ be two vague ideals of R such that $B\Gamma C \subseteq A$.
Suppose if possible $B \not\subset A$ and $C \not\subset A$.
$\Rightarrow V_B(x) > V_A(x)$ and $V_C(y) > V_A(y)$, for some $x, y \in R$.
$\Rightarrow x, y \notin R_A$.
$\Rightarrow x\gamma y \notin R_A, \gamma \in \Gamma$.
Now, $V_{B\Gamma C}(x\gamma y) = \sup\{\min\{V_B(x), V_C(y)\}$
$\geq \min\{V_B(x), V_C(y)\}$
$> \min\{V_A(x), V_A(y)\}$
$= [\alpha, \beta]$
$= V_A(x\gamma y)$
This is a contradiction.
So, either $B \subseteq A$ or $C \subseteq A$.
Hence A is a vague prime ideal of R.

Theorem 3.9
let $A = (t_A, f_A)$ be a vague set of R Then A is a vague prime ideal of R if and only if

1. $V_A(0) = [1, 1]$
2. A takes only two values say [1, 1] and $[\alpha, \beta]$, $0 \leq \alpha \leq \beta \leq 1$.
3. $R_A = \{x \in R / V_A(x) = V_A(0)$ is a prime ideal of R.

Proof
Proof is clear from theorems 3.5, 3.6, 3.7, 3.8.

Definition 3.10
An element $1 \neq \alpha \in [0, 1]$ is called prime element if for any two elements $a, b \in [0, 1]$ such that $\min\{a, b\} \leq \alpha$ implies $a \leq \alpha$ or $b \leq \alpha$.

The following theorem determines all vague prime ideals of R by establishing a one-to-one correspondence between vague prime ideals of R and the pair (P, α), where P is a prime ideal of R and α is a prime element in [0, 1].

Theorem 3.11
Let I be a prime ideal of R and α be a prime element in [0, 1]. Let P = (t_P, f_P) be a vague set of R defined by

$$V_P(x) = \begin{cases} [1,1] & if \ x \in I \\ [\alpha, \beta] & otherwise \end{cases}$$

where β ∈ {α, 1}. Then P is a vague prime ideal of R.

Conversely for any vague prime ideal P of R there exists a prime ideal I and a prime element α of [0, 1] such that

$$V_P(x) = \begin{cases} [1,1] & if \ x \in I \\ [\alpha, \beta] & otherwise \end{cases}$$

where β is a prime element of [0, 1] or β = 1.

Proof
Let I be a prime ideal of R and α be a prime element in [0, 1].
Clearly P is a vague ideal of R.
Also P is non constant vague ideal of R because α ≠ 1.
Let A and B be two vague ideals of R such that $A\Gamma B \subseteq P$.
Suppose if possible $A \not\subset P$ and $B \not\subset P$.
$\Rightarrow V_A(x) > V_P(x)$ and $V_B(y) > V_P(y)$, for some x, y ∈ R.
So, $V_P(x) = V_P(y) = [\alpha, \beta]$.
$\Rightarrow x, y \notin I$.
$\Rightarrow x\gamma y \notin I, \gamma \in \Gamma$.
Case(i). If β = α
Then $V_{A\Gamma B}(x\gamma y) = \sup\{\min\{V_A(x), V_B(y)\}\}$
$\geq \min\{V_A(x), V_B(y)\}$
$> \min\{V_P(x), V_P(y)\}$
$= [\alpha, \alpha]$
$= V_P(x\gamma y)$
This is a contradiction.
Therefore $A \subseteq P$ or $B \subseteq P$.
Hence P is a vague prime ideal of R.
Case(ii). If β = 1
Then $V_{A\Gamma B}(x\gamma y) = \sup\{\min\{V_A(x), V_B(y)\}\}$
$\geq \min\{V_A(x), V_B(y)\}$
$> \min\{V_P(x), V_P(y)\}$

$= [\alpha, 1]$
$= V_P(x\gamma y)$
This is a contradiction.
Therefore $A \subseteq P$ or $B \subseteq P$.
Hence P is a vague prime ideal of R.
Conversely suppose that P is a vague prime ideal of R.
From theorem:3.9, $V_P(0) = [1, 1]$ and P takes only two values.
Let $I = \{x \in R / V_P(x) = [1, 1]\}$.
Then I is an ideal of R.
We prove that α, β are prime elements, where $\beta \neq 1$ and I is a prime ideal of R.
Suppose $\min\{a, b\} \leq \alpha$, where $a, b \in [0, 1]$.
Define vague sets A and B as $V_A(x) = [a, a]$ and $V_B(x) = [b, b]$, $\forall\ x \in R$.
Now, $V_{A\Gamma B}(x) = \sup\{\min\{V_A(y), V_B(z)\} / x = y\gamma z$, where $y, z \in R; \gamma \in \Gamma\}$.
$= \sup\{\min\{[a, a], [b, b]\}\}$
$= [\min\{a, b\}, \min\{a, b\}]$
$\leq [\alpha, \alpha]$
$\leq [\alpha, \beta]$
$\leq V_P(x)$
That implies $A\Gamma B \subseteq P$.
Since P is vague prime ideal, either $A \subseteq P$ or $B \subseteq P$.
That implies $a \leq \alpha$ or $b \leq \alpha$.
Therefore α is prime element.
Suppose $\min\{c, d\} \leq \beta$.
Define vague sets C and D as $V_C(x) = [\alpha, \max\{c, \beta\}]$ and $V_D(x) = [\alpha, \max\{d, \beta\}]$, $\forall\ x \in R$.
Now, $V_{C\Gamma D}(x) = \sup\{\min\{V_C(y), V_D(z)\} / x = y\gamma z$, where $y, z \in R; \gamma \in \Gamma\}$.
$= \sup\{\min\{[\alpha, \max\{c, \beta\}], [\alpha, \max\{d, \beta\}]\}\}$
$= [\alpha, \max\{\min\{c, d\}, \beta\}]$
$\leq [\alpha, \beta]$
$\leq V_P(x)$
That implies $C\Gamma D \subseteq P$.
Since P is vague prime ideal, either $C \subseteq P$ or $D \subseteq P$.
That implies $\max\{c, \beta\} \leq \beta$ or $\max\{d, \beta\} \leq \beta$.
Implies $c \leq \beta$ or $d \leq \beta$.
Therefore $\beta \neq 1$ is a prime element.
Now we will show that $I = \{x \in R / V_P(x) = [1, 1]\}$ is a prime ideal of R.
Let A and B be ideals of R such that $A\Gamma B \subseteq I$.
$\Rightarrow \chi_{A\Gamma B} \subseteq \chi_I$.
Also, $\chi_I \subseteq P$.
So, $\chi_{A\Gamma B} \subseteq P$.
Now we will show that $\chi_A \Gamma \chi_B \subseteq \chi_{A\Gamma B}$.
$V_{\chi_A \Gamma \chi_B}(x) = \sup\{\min\{V_{\chi_A}(y), V_{\chi_B}(z)\} / x = y\gamma z$, where $y, z \in R; \gamma \in \Gamma\}$.

If $x \in A\Gamma B$, then $V_{\chi_{A\Gamma B}}(x)$ = [1, 1].
We have $V_{\chi_A\Gamma\chi_B}(x) \leq [1, 1] = V_{\chi_{A\Gamma B}}(x)$
If $x \notin A\Gamma B$, then $V_{\chi_{A\Gamma B}}(x)$ = [0, 0].
Since $x \notin A\Gamma B$, we have $y \notin A$ or $z \notin B$, whenever $x = y\gamma z$, where $y, z \in R$; $\gamma \in \Gamma$.
Thus $V_{\chi_A\Gamma\chi_B}(x)$ = [0, 0] = $V_{\chi_{A\Gamma B}}(x)$.
Therefore $\chi_A\Gamma\chi_B \subseteq \chi_{A\Gamma B}$.
That implies $\chi_A\Gamma\chi_B \subseteq P$.
Since P is vague prime ideal of R, we have $\chi_A \subseteq P$ or $\chi_B \subseteq P$.
If $\chi_A \subseteq P$, then $A \subseteq I$ or if $\chi_B \subseteq P$, then $B \subseteq I$.
Hence I is a prime ideal of R.

Corollary 3.12
Let I be a prime ideal of R, $\alpha, \beta \in [0, 1]$ and $\alpha \leq \beta$. Define

$$V_P(x) = \begin{cases} [1,1] & if\ x \in I \\ [\alpha, \beta] & otherwise \end{cases}$$

Then P is a vague prime ideal of R if and only if β is either α or 1.

Proof
Suppose β is either α or 1.
Since every element in [0, 1] is prime element, we have P is a vague prime ideal of R by theorem 3.11.
Conversely suppose that $\beta \notin \{\alpha, 1\}$.
So, $\alpha < \beta < 1$.
Then there exists $\gamma, \delta \in [0, 1]$ such that $\alpha < \gamma < \beta < \delta$.
Define vague sets $A = (t_A, f_A)$ and $B = (t_B, f_B)$ as

$$V_P(x) = \begin{cases} [1,1] & if\ x \in I \\ [\alpha, \delta] & otherwise \end{cases} \quad \text{and} \quad V_P(x) = \begin{cases} [1,1] & if\ x \in I \\ [\gamma, \beta] & otherwise \end{cases}$$

Clearly A and B are vague ideals of R.
Also $A \not\subset P$ and $B \not\subset P$.
We have to show that $V_{A\Gamma B}(x) \leq V_P(x)$
Case(i): If $x \in I$, then $V_P(x)$ = [1, 1].
Hence $V_{A\Gamma B}(x) \leq V_P(x)$.
Case(ii): If $x \notin I$, then $y \notin I$ and $z \notin I$, whenever $x = y\eta z$, where $\eta \in \Gamma$.
Therefore $V_A(y) = [\alpha , \delta]$ and $V_B(z) = [\gamma, \beta]$.
Now, $V_{A\Gamma B}(x) = \sup \{\min\{V_A(y), V_B(z)\}\}$
$= \sup \{\min\{[\alpha , \delta], [\gamma, \beta]\}\}$
$= \sup\{[\min\{\alpha, \gamma\}, \min\{\delta, \beta\}]\}$
$= [\alpha, \beta]$.
From case(i) and case(ii), $V_{A\Gamma B}(x) \leq V_P(x)$.
We have $A\Gamma B \subseteq P$.

This is a contradiction to P is a vague prime ideal of R.
Hence β is either α or 1.

Vague Maximal Ideals of Γ– Semirings

We can observe [0, 1] does not have dual atoms and then R cannot have any vague maximal ideals, even though R may have plenty of maximal ideals. In this regard, we are replacing [0, 1] by a complete lattice L satisfying infinite meet distributive law. Throughout this section, L stands for a non-trivial complete lattice in which the infinite meet distributive law holds i.e., there is a partial order ≤ on L such that, for any S ⊆ L, infimum of S and supremum of S exist and these will be denoted by $\wedge_{s\in S} s$ and $\vee_{s\in S} s$ respectively. In particular, for any elements a, b ∈ L, inf{a, b} and sup{a, b} will be denoted by a∧b and a∨b, respectively, L is a distributive lattice with a least element 0 and a greatest element 1 and

$$a \wedge \left(\vee_{s\in S} s\right) = \vee_{s\in S}(a \wedge s).$$

We determine all vague maximal ideals of R by establishing a one-to-one correspondence between vague maximal ideals of R and the pair (M, α), where M is a maximal ideal of R and α is a dual atom of L.

Definition 4.1
A vague maximal ideal of R is a maximal element in the set of all non constant vague ideals of R under inclusion.

Definition 4.2
An element α ≠ 1 in L is called a dual atom, if there is no β in L such that α < β < 1. Clearly α is a dual atom if and only if α is a maximal element of L-{1}.

Example 4.3
Let M be an ideal of R and α ∈ L. Then

$$V_A(x) = \begin{cases} [1,1] & if\ x \in M \\ [\alpha,\alpha] & otherwise \end{cases}, \ \forall\ x \in R.$$

is not a vague maximal ideal of R.

Solution:
Let M be an ideal of R.
Then we have A is a vague ideal of R.
If α = 1, then A is not a vague maximal ideal of R.
If α ≠ 1, then α < 1.

Consider the vague set B = (t_B, f_B) as $V_B(x) = \begin{cases} [1,1] & if\ x \in M \\ [\alpha,1] & otherwise \end{cases}$, ∀ x ∈ R.

Then B is a non constant vague ideal of R.

Also, A ⊂ B and A ≠ B.

Hence A is not a vague maximal ideal of R.

Lemma 4.4

If A is a vague maximal ideal of R, then $V_A(0)$ = [1, 1].

Proof

Define a vague set B = (t_B, f_B) as

$$V_B(x) = \begin{cases} [1,1] & if\ V_A(x) = V_A(0) \\ V_A(0) & otherwise \end{cases}, \forall\ x \in R.$$

Clearly B is a vague ideal of R and A ⊆ B.

Since A is vague maximal ideal of R, we have A = B or B is constant.

Hence $V_A(0)$ = [1, 1].

Lemma 4.5

Let A = (t_A, f_A) be a vague ideal and B = (t_B, f_B) be a constant vague ideal of R. Then A ∪ B is a vague ideal of R.

Proof

Let x, y ∈ R; γ ∈ Γ.

1. $t_{A\cup B}(x + y) = \vee\{t_A(x + y), t_B(x + y)\}$
$\geq \vee\{\wedge\{t_A(x), t_A(y)\}, \wedge\{t_B(x), t_B(y)\}\}$
$= \vee\{\wedge\{t_A(x), t_A(y)\}, t_B(x)\}$
$= \wedge\{\vee\{t_A(x), t_B(x)\}, \vee\{t_A(y), t_B(x)\}\}$
$= \wedge\{\vee\{t_A(x), t_B(x)\}, \vee\{t_A(y), t_B(y)\}\}$
$= \wedge\{t_{A\cup B}(x), t_{A\cup B}(y)\}$

Similarly $1 - f_{A\cup B}(x + y) \geq \wedge\{1 - f_{A\cup B}(x), 1 - f_{A\cup B}(y)\}$

2. $t_{A\cup B}(x\gamma y) = \vee\{t_A(x\gamma y), t_B(x\gamma y)\}$
$\geq \vee\{t_A(y), t_B(y)\}$ $(\geq \vee\{t_A(x), t_B(x)\})$
$= t_{A\cup B}(y)$ $(= t_{A\cup B}(x))$

Similarly $1 - f_{A\cup B}(x\gamma y) \geq 1 - f_{A\cup B}(y)$ $(\geq 1 - f_{A\cup B}(x))$

Hence A ∪ B is a vague ideal of R.

Theorem 4.6

Let A = (t_A ,f_A) be a vague set of R. Then A is vague maximal ideal of R if and only if there exists a maximal ideal M of R and a dual atom α in L such that

$$V_A(x) = \begin{cases} [1,1] & if \ x \in M \\ [\alpha,1] & otherwise \end{cases}, \forall \ x \in R.$$

Proof

Suppose M is a maximal ideal of R and α is a dual atom in L.
We will prove that A is a vague maximal ideal of R.
Clearly A is a non constant vague ideal of R.
Let B be a non constant vague ideal of R such that $A \subseteq B$.
Since $A \subseteq B$, we have $M \subseteq \{x \in R / V_B(x) = [1, 1]\} \neq R$.
That implies $M = \{x \in R / V_B(x) = [1, 1]\}$.
Let $x \notin M$.
Then $[\alpha, 1] = V_A(x) \leq V_B(x) < [1, 1]$.
That implies $V_A(x) = V_B(x)$ because α is a dual atom.
Hence A is a vague maximal ideal of R.
Conversely suppose that A is a vague maximal ideal of R.
By lemma 4.4, $V_A(0) = [1, 1]$.
First we prove that A takes only two values.
Let $x, y \in R$ such that $V_A(x) < [1, 1]$ and $V_A(x) < [1, 1]$.
Consider $B = (t_A(x), f_A(x))$, then B is a constant vague set of R.
Moreover $A \subseteq A \cup B \subseteq (1, 0)$.
By lemma 4.5, $A \cup B$ is a vague ideal of R.
Since $A \cup B$ is a vague ideal of R and A is a vague maximal ideal of R, we have $A = A \cup B$ or $A \cup B = (1, 0)$.
Suppose $A \cup B = (1, 0)$.
This implies $V_{A \cup B}(z) = [1, 1], \forall \ z \in R$.
In particular if we put $z = x$, we will get $V_A(x) = [1, 1]$.
This is a contradiction.
Therefore $A = A \cup B$.
$V_A(z) = V_{A \cup B}(z) = \vee\{V_A(z), V_B(z)\}, z \in R$.
In particular if we put $z = y$, then we get $V_A(y) = \vee\{V_A(y), V_B(y)\}$.
i.e., $V_A(y) = \vee\{V_A(y), V_A(x)\}$.
That implies $V_A(y) \geq V_A(x)$.
By symmetry $V_A(x) \geq V_A(y)$.
Therefore $V_A(x) = V_A(y)$.
Thus A takes only one value other than [1, 1] say $[\alpha, \beta]$.
Let $M = \{x \in R / V_A(x) = V_A(0)\}$.
Then M is a ideal of R.
Since A is non constant, we have M is properly contained in R.
Therefore M is a proper ideal of R.
Thus for any $x \in R$,

$$V_A(x) = \begin{cases} [1,1] & if \ x \in M \\ [\alpha, \beta] & otherwise \end{cases}$$

We will prove that M is a maximal ideal of R, α is a dual atom and $\beta = 1$.
If $\alpha = 1$, then $\beta = 1$.
This implies A is constant.
This is a contradiction.
Suppose $\alpha < 1$.
Let $a \in L$ such that $\alpha < a$.
Define a vague set C = (t_A, f_A) on R as $V_C(x) = \begin{cases} [1,1] & if \ x \in M \\ [a,1] & otherwise \end{cases}$.
Clearly C is a vague ideal of R.
Also, $V_A(x) \le V_C(x)$, $x \in R$.
Since $\alpha < a$, we have A is properly contained in C.
Since A is vague maximal ideal of R, we have C is constant.
So, we get a = 1.
Therefore α is a dual atom.
Suppose $\beta < 1$.
Consider $V_C(x) = \begin{cases} [1,1] & if \ x \in M \\ [\alpha,1] & otherwise \end{cases}$
Since $\beta < 1$, we have A is properly contained in C.
So, A is not a vague maximal ideal of R.
This is a contradiction.
Therefore $\beta = 1$.
Now we will show that M is a maximal ideal of R.
Let N be an ideal of R such that $M \subseteq N \subseteq R$, $N \neq R$.
Define a vague set D = (t_D, f_D) on R as $V_D(x) = \begin{cases} [1,1] & if \ x \in N \\ [\alpha,1] & otherwise \end{cases}$.
Clearly N is non constant.
Since $M \subseteq N$, we have $V_A(x) \le V_D(x) < [1, 1]$.
So, $A \subseteq D$.
Since A is vague maximal ideal of R, we have A = D.
That implies M = N.
Hence M is a maximal ideal of R.

In distributive lattices every dual atom is a prime element. So we have the following corollary.

Corollary 4.7
In R each maximal ideal is prime ideal. Then each vague maximal ideal of R is a vague prime ideal of R.

Proof
Let P be a vague maximal ideal of R.
Then by theorem 4.6, there exists a maximal ideal M of R and a dual atom α in L such that

$$V_P(x) = \begin{cases} [1,1] & if\ x \in M \\ [\alpha,1] & otherwise \end{cases}$$

Since every maximal ideal is a prime ideal and every dual atom is a prime element, we have P is a vague prime ideal of R.

Now we state the converse of the above corollary.

Corollary 4.8
In R each vague maximal ideal of R is a vague prime ideal of R. If L has a dual atom then each maximal ideal of R is prime ideal of R.

Proof
Let M be a maximal ideal of R and α be a dual atom in L.
Then by theorem 4.6, there exists a vague maximal ideal A of R such that

$$V_A(x) = \begin{cases} [1,1] & if\ x \in M \\ [\alpha,1] & otherwise \end{cases}$$

Since each vague maximal ideal is a vague prime ideal of R, we have A is a vague prime ideal of R.
Hence M is prime ideal of R.

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References

[1] Bhargavi.Y and Eswarlal.T, 2015, “Fuzzy Γ-semirings”, International Journal of Pure and Applied Mathematics, Vol. 98, No. 3, 339-349.
[2] Bhargavi.Y and Eswarlal.T, 2015, “Vague Γ-semirings”, Global Journal of Pure and Applied Mathematics, Vol. 11, No. 1, 117-127.

[3] Bhargavi.Y and Eswarlal.T, 2015, “Vague Ideals and Normal Vague Ideals in Γ-semirings”, International Journal of Innovative Research and Development, Vol. 4, Issue. 3, 1-8.
[4] Dutta.T.K and Biswas. B.K, 1994, “Fuzzy prime ideals of a semirings”, Bull. Malaysian Math. Soc., 17, 9-16.
[5] Eswarlal.T, 2009, “Boolean Vague Prime Ideals of Rings”, International journal of Computational Cognition, Vol. 7, No. 1, 70-75.
[6] Gau.W.L and Buehrer.D.J, 1993, “Vague Sets”, IEEE Transactions on systems, man and cybernetics, Vol. 23, No.2, 610-613.
[7] John N Mordeson and D.S.Malik, 1998, “Fuzzy Commutative Algebra”, World Scientific Publishing Co. Pvt. Ltd.
[8] Kyuno.S, 1981, “A Gamma ring with minimum conditions”, Tsukuba J. Math., 59(1), 47-65.
[9] Luh.J, 1968, “On primitive Γ-rings with minimal one-sided ideals”, Osaka J. Math., 5, 165-173.
[10] M.K.Rao, 1995, “Γ-semiring1”, Southeast Asian Bulletin of Mathematics, 19, 49-54.
[11] Nobusawa.N, 1978, “On generalization of the ring theory”, Osaka J. Math. 1, 185-190.
[12] Ramakrishna.N and Eswarlal.T, 2007, “Boolean Vague sets”, International journal of Computational Cognition, Vol. 5, No. 4, 50-53.
[13] Ranjit Biswas, June(2006), “Vague Groups”, International journal of Computational Cognition, Vol. 4, No. 2, 20-23.
[14] Swamy.U.M and Swamy.K.L.N, 1988, “Fuzzy prime ideals of rings”, J. Math. Anal. Appl., Vol.134, No.1, 94-103.
[15] Zadeh.L.A, 1965, “Fuzzy sets”, Information and Control 8, 338-353.