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Abundant and Deficient Groups

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Abstract

Let n be a natural number and $\sigma(n)$ be the sum of all divisors of n. Then the number n is said to be abundant if $\sigma(n) > 2n$. Also, the number n is said to be deficient if $\sigma(n) < 2n$. In this paper, we extend the notion of abundant and deficient numbers to finite groups and we study its property. In addition, we provide some theorems and examples of abundant and deficient groups.

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1- Introduction

Number theory is about properties of the natural numbers, integers, or rational numbers. Number theory has long been a favorite subject for students and teachers of mathematics. In this times are not covered by any one application of number theory. Number theory is one of the oldest disciplines dating thousands of years back. The study of numbers has been in progress for as long as many other important mathematical fields.

It might be argued that elementary number theory began with Pythagoras and his followers. The following is a list of some scientists who have worked on the development of the number theory.

Pythagoras 500 BC, Euclid 250 BC, Eratosthenes 196 BC, Diophantice 250 BC, Fermat (1601-1665), Euler (1707-1783), Gauss (1777-1855), Lagrange (1736-1813), Goldbach (1690-1764), Legendre (1752-1833), Cauchy (1789-1857), Dirichlet (1805-1859).

At the beginning of the elementary number theory used in astrology. The ancient Greeks had too much attention to the number because for each word or name was associated with a number. Or the ancient Greeks thought that if the number of the names of the two couples are treated amicable are very good friends, especially in mate selection. The natural numbers can be divided into three types. In the other words, positive integers are classified as abundant, perfect or deficient. For more details we refer the reader to [2, 9, 10, 16, 17].

Also, an arithmetic function f is called multiplicative if $f(m\times n) = f(m) \times f(n)$ whenever m and n are relatively prime positive integers.

Now, we provide some general theorems and examples of abundant and deficient groups. We shall show that there are infinitely many abundant and deficient groups. At the end of the paper, we compare the number of deficient groups with the number of abundant groups. We now turn to definitions and elementary results.

Definition 1.1.
An arithmetic function is a function that is defined for all positive integers. For a natural number n let us denote

$$\sigma(n) = \sum_{d \mid n} d.$$

Theorem 1.2.
The function σ is multiplicative. That is if n and m are two natural numbers satisfying (n, m) = 1, then

$$\sigma(mn) = \sigma(m)\sigma(n). \text{ ([3])}$$

Definition 1.3.
The number n is said to be abundant if $\sigma(n) > 2n$. For example, 12 is an abundant number because

$$\sigma(12)=\sum_{d \mid 12} d =1+2+3+4+6+12=28 > 24=2\times 12.$$

The integer 12 is the first abundant number. The first trace abundant numbers are: 12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, 102, 104, 108, 112, 114, 120. The smallest odd abundant number is 945.

Theorem 1.4.
There are infinitely many deficient numbers. ([5])

Theorem 1.5.
Every multiple of a perfect number or an abundant number is abundant. ([5])

Definition 1.6.
The number n is said to be deficient if $\sigma(n) < 2n$. For example, 2 is a deficient number because

$$\sigma(2)=\sum_{d \mid 2} d = 1+2=3< 4=2\times 2.$$

The first trace deficient numbers are: 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27.

Theorem 1.7.
There are infinitely many prime numbers. ([9])

Corollary 1.8.
There are infinitely many deficient numbers. ([9])

Definition 1.9.
The number n is said to be perfect if $\sigma(n) = 2n$. For example, 6 is a perfect number because $\sigma(6) =\sum_{d \mid 6} d = 1+2+3+6=12=2\times 6$. The smallest perfect number is 6.

Notice 1.10.
Euclid proved that a number of the form $2^{p-1}(2^p - 1)$, where both p and $(2^p - 1)$ are primes, is perfect. Euler proved the converse, i.e. that any even perfect number has the form specified by Euclid.

It is interesting that 2000 passed before a new large success in the research of perfect numbers. L. Euler was successful to convert the Euclid's theorem for even perfect numbers.

Theorem 1.11.
(Euclid, Euler) An even number is perfect if and only if it has the form $(2^p - 1)\, M_p$, where M_p is the Mersenne prime.

In 2001, T. Leinster [7] extended the notion of perfect numbers to finite groups. He called a finite group is perfect (FPG) if its order is equal to the sum of the orders of all normal subgroups of the group. In the other words, G is called perfect group if $\sigma(G) = \sum_{N \trianglelefteq G} |N| = 2|G|$. On the other hand, he developed and studied a group-theoretic analogue of perfect numbers. A finite group is said to be a perfect group or an immaculate group or a Leinster group if the sum of the orders of its normal subgroups equals twice the order of the group itself. Clearly, a finite cyclic group C_n is a Leinster group if and only if its order n is a perfect number. In fact, the abelian Leinster groups are precisely the finite cyclic groups whose orders are perfect numbers. More information on this and the related concepts can be found in the works of A. K. Das [14], M. Tarnauceanu [12, 15], T. D. Medts and A. Maroti [6], etc. For more details we refer the reader to [1, 4, 8].

Now, we introduce some examples and theorems of abundant and deficient groups.

2- Abundant and Deficient Groups

Lemma 2.1.
Let G_1 and G_2 are coprime then $\sigma(G_1 \times G_2) = \sigma(G_1)\sigma(G_2)$. This means the function σ is multiplicative. ([7])

Definition 2.2.
A finite group G is said to be abundant group if $\sigma(G) = \sum_{N \trianglelefteq G} |N| > 2|G|$.

Example 2.3. (Cyclic Groups)
Let C_n be the cyclic group of order n. Then C_n has one normal subgroup of order d for each divisor d of n, and no others, so $\sigma(C_n) = \sigma(n)$ and C_n is abundant group just when n is abundant number. For example, C_{945} is an abundant group because we have

$$\sigma(C_{945}) = \sigma(C_{27\times5\times7}) = \sigma(C_{27})\times\sigma(C_5)\times\sigma(C_7) \underline{\underline{\sigma(C_n)=\sigma(n)}} \sigma(27\times5\times7) = 1920 > 1890 = 2\times945.$$

Example 2.4.(Dihedral Groups)
Let E_{2n} be the dihedral group of order 2n. If n is even number, then $\sigma(E_{2n}) \geq 1 + n + n + 2n > 4n = 2|E_{2n}|$. Hence E_{2n} is abundant group.

Definition 2.5.
A finite group G is said to be deficient group if $\sigma(G) = \sum_{N \trianglelefteq G} |N| < 2|G|$.

Example 2.6. (Cyclic Groups)
Let C_n be the cyclic group of order n. Then C_n has one normal subgroup of order d for each divisor d of n, and no others, so $\sigma(C_n) = \sigma(n)$ and C_n is deficient group just when n is deficient number.

Theorem 2.7.
If m>0, n>1 are natural numbers, then we have

$$\sigma(mn) \geq m\sigma(n). \text{ ([11])}$$

Theorem 2.8.
If m>1, n>1 are natural numbers, then we have

$$\sigma(mn) \geq \sigma(m) + \sigma(n). \text{ ([13])}$$

Theorem 2.9.
Let p be a prime. Then we have $\sigma(p^k) = \frac{p^{k+1}-1}{p-1}$, $k \in \mathbb{N}$.

Theorem 2.10.

Let C_n be the cyclic group of order n. Then $C_{2^{m-1}(2^m-1)}$ is an abundant group, where m is a positive integer and 2^m-1 is composite.

Proof.

We have

$$\sigma(C_{2^{m-1}(2^m-1)}) = \sigma(2^{m-1}(2^m-1)) = \sigma(2^{m-1})\sigma(2^m-1). \quad (*)$$

But we know that

$$\begin{cases} \sigma(2^{m-1}) = 2^m - 1 \\ \sigma(2^m-1) > 2^m - 1 + 1 = 2^m \end{cases}. \qquad (**)$$

$$\xrightarrow{(*,**)} \sigma(C_{2^{m-1}(2^m-1)}) = \sigma(2^{m-1}(2^m-1)) > (2^m-1)(2^m) = 2(2^m-1)(2^{m-1}) = 2|C_{2^{m-1}(2^m-1)}|.$$

Notice 2.11.

In number theory, we observed any positive multiple of an abundant numbers is an abundant number. Now, we have modeled this theorem in finite group theory.

Theorem 2.12.

Let C_n be the cyclic group of order n. Then $C_n \times C_m$ is abundant group, where C_n is abundant group and (n, m) = 1, $m \in \mathbb{N}$.

Proof.

It is sufficient to prove, $\sigma(C_n \times C_m) > 2|C_n \times C_m|$. Let C_n is abundant group. But we have

$$\sigma(C_n \times C_m) \underset{\sigma \text{ is multiplicative}}{=\!=} \sigma(C_n) \times \sigma(C_m) > 2n \times \sigma(C_m) = 2nm = 2|C_n \times C_m|.$$

This completes the proof.

Corollary 2.13.

There are infinitely many odd abundant groups.

Proof.

We know that $C_{n=945}$ is an odd abundant group. There are infinitely many m such that (945, m) = 1, where m is odd positive integer. The proof was done.

Corollary 2.14.

There are infinitely many even abundant groups.

Proof 1.

We know that $C_{n=12}$ is an even abundant group. There are infinitely many m such that (12, m) =1, where m is positive integer.

Proof 2.

Let n be a natural number, where n ≥ 4. Therefore, we know that $n! = 12k$. Then n! is an even abundant number. By $\sigma(C_n) = \sigma(n)$, we have $C_{n!}$ is an even abundant group.

Proof 3.

Another proof of the above theorem is as follows:

There are infinitely many m such that $2^m - 1$ is composite. (1)

We have

$$(x^m - 1) = (x - 1)(x^{m-1} + \ldots + 1).$$

Let us suppose that m = r×s ≥ 4. Therefore, we have

$$(2^m - 1) = (2^r - 1)(2^{rs-r} + \ldots + 2^r + 1) \rightarrow \begin{cases} r \geq 2 & \rightarrow \quad 2^r - 1 \neq 1 \\ s \geq 2 \rightarrow (2^{rs-r} + \ldots + 2^r + 1) \neq 1 \end{cases}.$$

By theorem 2.10, this completes the proof.

Proof 4.

Let C_n be the cyclic group of order n. Let n = 3×2^a, a ≥ 2. Then $C_{3\times 2^a}$ is abundant group because

$$\sigma(C_n) = \sigma(C_{3\times 2^a}) = \sigma(C_3)\,\sigma(C_{2^a}) = \sigma(3)\,\sigma(2^a) = \underbrace{4(2^{a+1} - 1) > 2|C_{3\times 2^a}| = 3 \times 2^{a+1}}_{(1)}.$$

The inequality (1) is true because $2^{a+1} > 4$. In addition, we have $3 \times 2^{a+1} - 3 + 2^{a+1} - 1 > 3 \times 2^{a+1} \rightarrow (3+1) \times (2^{a+1} - 1) > 3 \times 2^{a+1}$.

Proof 5.

Let C_n be the cyclic group of order n. Let n = 3^k×2^a, a ≥ 2, k ≥ 1. Then $C_{3^k\times 2^a}$ is an abundant group because

$$\underbrace{\left(\frac{3^{k+1} - 1}{3 - 1}\right)\left(\frac{2^{a+1} - 1}{2 - 1}\right) > 3^k \times 2^{a+1}}_{(*)}.$$

The inequality (*) is true because

$$3^{k+1}\times 2^{a+1} - 3^{k+1} - 2^{a+1} + 1 > 3^k \times 2^{a+2}.$$

This is equivalent to $3^k \times 2^{a+1} > 3^{k+1} + 2^{a+1} - 1$. Assume that m = 3^k and n = 2^{a+1}. Therefore, we have m ≥ 3 and n ≥ 8. The relation m×n > 3m+n-1 is equal to (n-3) × (m-1)-2 > 0. Hence, this inequality is always true.

Corollary 2.15.
There exist infinitely many abundant groups.

Theorem 2.16.
There exist infinitely many deficient groups.

Proof 1.
We know that C_p is odd deficient group, where p is an odd prime. On the other hand, there are infinitely many prime and $\sigma(C_p) = \sigma(p) = p + 1 < 2p$.

Proof 2.
Let C_n be the cyclic group of order n. Assume that $n=p^a$, where p is prime and $a \in \mathbb{N}$.

It is easy to prove that $n=p^a$ is deficient number because $2p^a - 1 < 2p^{a+1}$. Therefore, we have the following relationship.

$$\sigma(n = p^a) = \frac{p^{a+1}-1}{p-1} < 2p^a = 2n.$$

Theorem 2.17.
Let C_n be the cyclic group of order n, where p is an odd prime. Then $C_{(2^a p)}$

i) is abundant group if $p < 2^{a+1} - 1$.

ii) is deficient group if $p > 2^{a+1} - 1$.

Proof.

i) It is sufficient that, $C_{(2^a p)} > 2|C_{(2^a p)}|$. But we have $\sigma(C_n) = \sigma(n)$. Therefore, we have

$$\sigma(C_{(2^a p)}) = \sigma(2^a p) = \sigma(2^a)\sigma(p) = (2^{a+1} - 1)(p + 1) = 2^{a+1}p + \underbrace{2^{a+1} - (p + 1)}_{>0} > 2^{a+1}p = 2|C_{(2^a p)}|.$$

ii) The proof is similar to the previous section.

Notice 2.18.
In this paper, we denote by EAN (EAG) the set of all even abundant numbers (even abundant group). Also, we denote by OAN (OAG) the set of all odd abundant numbers (odd abundant group). In addition, we denote by AN (DN) the set of all abundant numbers (abundant numbers). In this paper, it is proved that there are infinitely many abundant and deficient groups. Therefore, we have $AG \sim \mathbb{N} \sim DG$. (In this paper, we use of the symbol $\sim$ for equipotence relation)

Definition 2.19.

Let G_1 and G_2 are two finite groups. Then the pair of (G_1, G_2) is called amicable groups if

$$\sigma(G_1) = \sigma(G_2) = |G_1| + |G_2|. \text{ ([4])}$$

Proposition 2.20.
Let C_n be the cyclic group of order n. If C_n and C_m are two abundant groups then the pair of (C_n, C_m) is not amicable pair.

Proof.
Let the pair of (C_n, C_m) is an amicable pair, so

$$\begin{cases}\sigma(C_n) - |C_n| = |C_m| \\ \sigma(C_m) - |C_m| = |C_n|\end{cases} \xRightarrow{\sigma(C_n)=\sigma(n)} \begin{cases}\sigma(n) - n = m \\ \sigma(m) - m = n\end{cases}.$$

According to the assumptions of the theorem we have

$$\begin{cases}\sigma(C_n) > 2|C_n| \\ \sigma(C_m) > 2|C_m|\end{cases} \to \begin{cases}\sigma(n) > 2n \\ \sigma(m) > 2m\end{cases}.$$

Therefore, we have

$$\begin{cases}\sigma(n) - n = m > n \\ \sigma(m) - m = n > m\end{cases}.$$

This is a contradiction.

Proposition 2.21.
Let C_n be the cyclic group of order n. If the pair of (C_n, C_m) is an amicable pair and $m < n$ then n is deficient and m is abundant.

Proof.
Since the pair of (C_n, C_m) is an amicable pair of groups, so

$$\begin{cases}\sigma(C_n) - |C_n| = |C_m| \\ \sigma(C_m) - |C_m| = |C_n|\end{cases} \xRightarrow{\sigma(C_n)=\sigma(n)} \begin{cases}\sigma(n) - n = m \\ \sigma(m) - m = n\end{cases} \to \begin{cases}\sigma(n) = n + m \\ \sigma(m) = m + n\end{cases}. \quad (*)$$

By using (*) and $m < n$, we have

$$\begin{cases}\sigma(n) = m + n < n + n \\ \sigma(m) = n + m > m + m\end{cases} \to \begin{cases}\sigma(n) < 2n \\ \sigma(m) > 2m\end{cases}.$$

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References

[1] H. Khosravi, E. Faryad, Amicable Numbers and Groups II, Int. J. Math. Trends and Tech, vol 14(1), 2014, pp. 40-45.

[2] W. Borho, B. Zahlen, in Lebendige Zahlen, Math. Miniaturen, vol 1, Birkhäuser, Basel, 1981.

[3] H. Golmakani, H. Khosravi, Multiplicatively Perfect Groups, Biennial International Group Theory Conference, Istanbul, 2013, pp. 4-8.

[4] H. Khosravi, E. Faryad, Amicable Numbers and Groups I, I. Res. J. P. Algebra, vol 4 (10), 2014, pp. 593-598

[5] J. L. Schiffman, Odd Abundant Numbers, Appl. Prob. Trust, 2005, pp. 73-75.

[6] T. Medts, A. Maroti, Perfect Numbers and Finite Groups, Rend. Sem. Mat, Univ Padova, 2012.

[7] T. Leinster, Perfect Numbers and Groups, arXiv: math/0104012v1, 2001.

[8] S. J. Baishya, A. K. Das, Harmonic numbers and finite groups, European. Math. Soc, 2013, pp. 1-10.

[9] J. Sandor, S. D. Mitrinovic and B. Crstici, Handbook of Number Theory I, 2nd printing, Springer, 2006.

[10] W. Stein, Elementary Number Theory, Springer, 2009.

[11] J. Sandor, Some Arithmatic Inequalities, Bull. Number Theory Rel. Topics 11, 1987, pp. 149-161.

[12] D. M. Tom, Finite Groups Determined by an Inequality of the Orders of Their Subgroups, Bull. Belg. Math. Soc, 2008.

[13] J. L. Hunsucker, J. Nebb, Problem B 260 Fib. Quari. 11, 221 Solution by P.S. Bruckman, 1973, 223-224.

[14] A. K. Das, On Arithmatic Function of Finite Groups, Bull. Aust. Math. Soc, vol 75, 2007, pp. 45-58.

[15] D. M. Tom, M. Tarnauceanu, Finite Groups Determined by an Inequality of the Orders of Their Normal Subgroups, Sci. An. Univ, vol 57, 2011, pp. 229-238.

[16] Yan, Song Y., Number Theory for Computing, 2nd edition, Springer, Berlin, 2002.

[17] Burton, M. David, Elementary Number theory, 6th edition, McGraw-Hill, 2007.

[18] C. Pomerance, on the Distribution of Amicable Numbers II, Journal f. d. Angewandte Mathematik 325, 1981, pp. 183–188.