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# Duality Theorem For Infinite Hankel Transform

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## Abstract

This paper presents a new property called the Duality Theorem for the Infinite Hankel Transform. Infinite Hankel Transform or simply Hankel Transform is an integral transform related to the renowned Fourier Transform that finds applications in electromagnetics, heat power and a host of other applications. A formal derivation of the Duality Theorem corresponding to the Infinite Hankel Transform is given which was hitherto not mentioned or derived in the literature. The usage of Duality Theorem helps in finding the time-domain function from the Hankel Transform domain and vice versa thereby reducing considerable labour and computation time, if a known Hankel Transform pair is available and the same has been exemplified with suitable illustrative examples.

**Keywords:** kernel, duality, transform, bessel.

## Introduction

Hankel Transform or the Infinite Hankel Transform was invented by the mathematician Hermann Hankel. It is also referred to as the Fourier – Bessel Transform [1]. The Infinite Hankel Transform of a function $f(x), 0 < x < \infty$, is defined as

$$F(s) = H[f(x)] = \int_0^\infty f(x).xJ_n(sx)dx \ldots \quad (1)$$

where, $J_n(sx)$ is the Bessel function of the first kind of order $n$. Also, $F(s)$ is the Hankel Transform of order $n$ of the function $f(x)$. Thus, if $F(s)$ is defined as the Hankel Transform of order $n$, then, $F(s)$ can also be denoted as $F_n(s)$. Thus, we can Eq. (1) alternatively as

$$F_n(s) = H[f(x), n] = \int_0^{\infty} f(x).\, xJ_n(sx)dx \ \ldots \quad (2)$$

In Eq. (1), the term $xJ_n(sx)$ is called the kernel of the Infinite Hankel Transform. It is also referred to as the Analysis Equation of Infinite Hankel Transform [2]. It is evident from Eq. (1) or (2) that the Infinite Hankel transform expresses any given function $f(x)$ as the weighted sum of an infinite number of Bessel functions of the first kind $J_n(sx)$. The Bessel functions in the sum are all of the same order $n$, but differ in a scaling factor $s$ along the $x -$ axis. The necessary coefficient $F_n$ of each Bessel function in the sum, as a function of the scaling factor $s$ constitutes the transformed function. It is also an orthogonal transform like the renowned Fourier Transform [3].

Like the Fourier and Laplace Transforms, the Hankel Transform is also a linear and invertible transform. Thus, it is possible to recover $f(x)$ from $F(s)$ and this is obtained by the Inverse Hankel Transform or Inverse Infinite Hankel Transform of $F(s)$, and this is defined as

$$f(x) = H^{-1}[F(s)] = \int_0^{\infty} F(s).\, sJ_n(sx)ds \ \ldots \quad (3)$$

Eq. (3) is also called the Synthesis Equation of the Infinite Hankel Transform [4]. Also, $f(x)$ is continuous as it is defined in the interval $(0, \infty)$, and is piecewise continuous and of bounded variation in every finite subinterval in $(0, \infty)$, and

$$\int_0^{\infty} |f(x)| x^{\frac{1}{2}}\, dx < \infty \ \ldots \quad (4)$$

Further, to have Eq. (1) hold good, it is assumed that $n \geq \frac{1}{2}$ [5].

$J_n(x)$ is the solution of the following Bessel's differential equation of order $n$, viz.,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \ \ldots \quad (5)$$

$J_n(x)$is expressed in series form and integral form by the following equations

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!} (x/2)^{n+2r} \ \ldots \quad (6)$$

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin\theta)d\theta \ \ldots \quad (7)$$

It is to be noted that $f(x)$ and $F(s)$ form an Infinite Hankel Transform pair [6].

Hankel Transform is related to the Fourier Transform. Most of the properties of the Infinite Hankel Transform are similar to those of the Infinite Fourier Transform, but some differences do persist [7].

## Relationship Between Hankel Transform and Fourier Transform

The Infinite Hankel Transform is very closely related to the Infinite Fourier Transform through circularly – symmetric functions. Consider a two – dimensional function $\mu(x,y)$ in the spatial or temporal domain that exhibits circular symmetry. A two – dimensional function is said to exhibit circular symmetry if $\mu(\rho\cos\alpha, \rho\sin\alpha) = f(\rho,\alpha)$ is independent of $\alpha$ [8]. The two – dimensional Fourier Transform of $\mu(x,y)$ is given by

$$\mathrm{E}(\xi,\varsigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y).\exp[-i(x\,\xi, y\varsigma)]dx\,dy \;\; \ldots \tag{8}$$

Expressing the above integral in polar coordinates, we get,

$$x = \rho\cos\alpha\,, y = \rho\sin\alpha\,, \xi = \tau\cos\mu\,, \varsigma = \tau\sin\mu \ldots \tag{9}$$

$$\mathrm{F}(\tau,\mu) = \frac{1}{2\pi}\int_0^{\infty}\rho\,d\rho\int_0^{2\pi} e^{-i\rho\tau\cos(\alpha-\mu)} f(\rho)d\alpha \ldots \tag{10}$$

$$\mathrm{F}(\tau,\mu) = \frac{1}{2\pi}\int_0^{\infty}\rho\,f(\rho)d\rho\int_0^{2\pi} e^{-i\rho\tau\cos\beta}d\beta,$$

where, we have substituted $\alpha - \mu = \beta$.

$$\mathrm{F}(\tau,\mu) = \int_0^{\infty}\rho\,f(\rho)d\rho\{\frac{1}{2\pi}\int_0^{2\pi} e^{-i\rho\tau\cos\beta}d\beta\} \ldots \tag{11}$$

But, from the theory of Bessel Functions, we have,

$$J_0(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\gamma\cos\beta}d\beta \;\; \ldots \tag{12}$$

Replacing $\gamma$ by $\rho\tau$ in the above equation yields

$$J_0(\rho\tau) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\rho\tau\cos\beta}d\beta \;\; \ldots \tag{13}$$

Hence, using Eq. (13) in Eq. (11), we get,

$$\mathrm{F}(\tau,\mu) = \int_0^{\infty}\rho f(\rho)J_0(\rho\tau)d\rho \;\; \ldots \tag{14}$$

Eq. (14) shows that $\mathrm{F}(\tau,\mu)$ is independent of $\mu$, so that we can write $\mathrm{F}(\tau)$, instead of $\mathrm{F}(\tau,\mu)$ [9]. Thus, the two – dimensional Fourier Transform of a circular symmetric function is, in fact, the Infinite Hankel Transform of order zero [10].

*Generalization: -*

The $N$ – dimensional Fourier Transform of a circularly symmetric function of $N$ variables is related to the Hankel Transform of order $\{\frac{N}{2} - 1\}$ [11]. If $f(\rho,\alpha)$ is dependent on $\alpha$, it can be expanded in the form of a Fourier Series,

$$f(\rho,\alpha) = \sum_{n=-\infty}^{+\infty} f_n(\rho)\; e^{+in\alpha} \ldots \tag{15}$$

Thus, we have,

$$\mathrm{F}(\tau,\mu) = \frac{1}{2\pi}\int_0^{\infty}\rho\,d\rho\;\int_0^{\infty} e^{-i\rho\tau\cos(\alpha-\mu)}\,f(\rho,\alpha)\,d\alpha \;\; \ldots \tag{16}$$

$$\mathrm{F}(\tau,\mu) = \sum_{n=-\infty}^{+\infty} F_n(\tau) e^{+in\mu} \quad \ldots \tag{17}$$

where,

$$f_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho,\alpha)\, e^{-in\alpha}\, d\alpha \quad \ldots \tag{18}$$

and

$$F_n(\tau) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{F}(\tau,\mu)\, e^{-in\mu}\, d\mu \quad \ldots \tag{19}$$

Substituting Eq. (16) into Eq. (19), making use of Eq. (15) and simplifying we get,

$$F_n(\tau) = \frac{1}{(2\pi)^2}\int_0^{2\pi} e^{-in\mu}\, d\mu \int_0^{2\pi} d\alpha \int_0^{\infty} f(\rho,\alpha) e^{+i\rho\tau\cos(\alpha-\mu)}\, \rho\, d\rho$$

$$F_n(\tau) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\mu}\, d\mu \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} e^{+i\rho\tau\cos(\alpha-\mu)} d\alpha \,(\sum_{m=-\infty}^{+\infty} f_m(\rho)\; e^{+im\alpha})$$

Putting $\alpha - \mu = \delta$ and carrying out the integration leads to

$$F_n(\tau) = \frac{1}{2\pi}\int_0^{\infty} \rho\, d\rho \int_0^{2\pi} e^{-in\delta}\, e^{+i\rho\tau\cos\delta}\, f_n(\rho)\, d\delta$$

$$F_n(\tau) = \int_0^{\infty} \rho f_n(\rho)\, \{\frac{1}{2\pi}\int_0^{2\pi} e^{-in\delta}\, e^{+i\rho\tau\cos\delta}\, d\delta\} d\rho$$

The inner integral is nothing but $J_n(\rho\tau)$. The above equation is written in a more compact manner as

$$F_n(\tau) = \int_0^{\infty} \rho\, f_n(\rho)\, J_n(\rho\tau)\, d\rho \quad \ldots \tag{20}$$

Now, replacing $\tau$ by $s$ and $\rho$ by $x$ in the above equation gives,

$$F_n(s) = \int_0^{\infty} x\, f_n(x)\, J_n(sx)\, dx = \mathcal{H}_n[f(x)] \quad \ldots \tag{21}$$

This is the generalized relationship between the Infinite Fourier Transform and the Infinite Hankel Transform [12].

Similarly, it can be shown that

$$f_n(x) = \mathcal{F}_n[F_n(s)] \quad \ldots \tag{22}$$

In the next section, we present the statement and the proof of the Duality Theorem corresponding to Hankel Transform.

## Duality Theorem

*Statement:* - If $f(x)$ and $F(s)$ form an Infinite Hankel Transform pair, i.e., if $F(s) = H[f(x)]$, then, $f(s) = H[F(x)]$, i.e., $F(x)$ and $f(s)$ form a Hankel Transform pair.

*Proof:* - By definition, it follows that the Infinite Hankel Transform of a function $f(x)$ is given by

$F(s) = \int_0^\infty f(x).xJ_n(sx)\,dx\ \ldots$ (23)

$f(x) = \int_0^\infty F(s).sJ_n(sx)\,ds\ \ldots$ (24)

Replacing $x$ by $s$ and vice versa in Eq. (24), we get,

$f(s) = \int_0^\infty F(x).xJ_n(xs)dx = \int_0^\infty F(x).xJ_n(sx)dx\ \ldots$ (25)

Comparing Eq. (25) with Eqs. (23) and (24), we see that the RHS of it is nothing but the Infinite Hankel Transform of $F(x)$, with the time – and frequency – variables, i.e., $x$ and $s$ variables being interchanged respectively.

Thus, we can formally write the Duality Theorem of the Infinite Hankel Transform in a compact form.

$f(s) = H[F(x)]\ \ldots$ (26)

Eq. (26) is the final step in the derivation of the Duality Theorem for Infinite Hankel Transform.

We shall verify the above theorem with three illustrative examples in the next section.

## Illustration of Duality Theorem

**Example A:**

Consider the computation of the zeroth order Infinite Hankel Transform of the function,

$f(x) = \frac{e^{-ax}}{x}, n = 0\ \ldots$ (27)

The Infinite Hankel Transform of $f(x)$ with $n = 0$ is given by

$$F(s) = \int_0^\infty \frac{e^{-ax}}{x}.xJ_0(sx)\,dx$$

$\Rightarrow F(s) = \int_0^\infty e^{-ax}.J_0(sx)\,dx\ \ldots$ (28)

Consider the evaluation of the following integral

$I = \int_0^\infty e^{-\alpha x}J_0(\beta x)dx, \alpha, \beta > 0\ \ldots$ (29)

Substituting $n = 0$ in Eq. (7) and noting that $\cos(-\theta) = \cos\theta$, we get,

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\,sin\,\theta)\,d\theta$$

$\Rightarrow J_0(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x\,sin\,\theta)\,d\theta\ \ldots$ (30)

Replacing $x$ by $\beta x$ in Eq. (30), we get,

$J_0(\beta x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(\beta x \sin\theta)\, d\theta$ … (31)

Multiplying both sides of Eq. (31) by $e^{-\alpha x}$ and integrating from 0 to ∞, we get,

$$I = \int_0^{\infty} e^{-\alpha x} J_0(\beta x) dx = \frac{2}{\pi}\int_0^{\infty}\int_0^{\pi/2} e^{-\alpha x}\cos(\beta x \sin\theta)\, d\theta dx$$

after changing the order of integration in the above double integral.

Thus, we can write the above integral as

$$I = \int_0^{\infty} e^{-\alpha x} J_0(\beta x) dx = \frac{2}{\pi}\int_0^{\infty} \{\int_0^{\pi/2} e^{-\alpha x}\cos(\beta x \sin\theta)\, d\theta\} dx$$

Evaluation of the inner integral on the R.H.S. yields

$I = \frac{2}{\pi}\int_0^{\infty} \frac{\alpha\, d\theta}{\alpha^2+\beta^2 sin^2\theta} = \frac{2\alpha}{\pi}\int_0^{\infty} \frac{cosec^2\theta\, d\theta}{\alpha^2 cosec^2\theta+\beta^2}$, after multiplying the Numerator and Denominator by $cosec^2\theta$.

$$I = \frac{-2\alpha}{\pi}\int_0^{\infty} \frac{-cosec^2\theta\, d\theta}{(\alpha^2+\beta^2) + \alpha^2\, cot^2\theta}$$

Substituting $\alpha \cot\theta = t$ and corresponding simplification reduces the above integral to

$$I = \frac{2}{\pi}\int_0^{\infty} \frac{dt}{\{\sqrt{\alpha^2+\beta^2}\}^2 + t^2} = \frac{2}{\pi\sqrt{\alpha^2+\beta^2}}\left(\frac{\pi}{2}\right)$$

This leads to

$I = \int_0^{\infty} e^{-\alpha x} J_0(\beta x) dx = \frac{1}{\sqrt{\alpha^2+\beta^2}}$ … (32)

Now, returning to Eq. (28) and using the result of Eq. (32) by substituting $\alpha = a$ and $\beta = s$ yields

$F(s) = (a^2 + s^2)^{-1/2}$ … (33)

Next, we shall find the zeroth order Inverse Hankel Transform of

$F(s) = \frac{e^{-as}}{s}$ … (34)

Invoking Eq. (3) and substituting $F(s)$ into it gives

$$f(x) = H^{-1}[F(s)] = \int_0^{\infty} \frac{e^{-as}}{s} . sJ_n(sx) ds$$

$\Rightarrow f(x) = \int_0^{\infty} e^{-as} . J_n(xs) ds$ … (35)

Eq. (35) is very much similar to Eq. (32) and substituting $\alpha = a$ and $\beta = x$ yields

$\Rightarrow f(x) = \frac{1}{\sqrt{a^2+x^2}} = (a^2 + x^2)^{-1/2}$ … (36)

Next, we apply the Duality Property just derived for finding the zeroth order Inverse Hankel Transform of Eq. (34). It has been established from Eq. (32) that the Hankel Transform of $f(x) = \frac{e^{-ax}}{x}, n = 0$, is

$$F(s) = \frac{1}{\sqrt{a^2+s^2}} = (a^2 + s^2)^{-1/2} \quad \dots \tag{37}$$

By the principle of Duality Theorem proved in the previous section, replacing $s$ by $x$ and vice versa in the above equation gives,

$$F(x) = \frac{1}{\sqrt{a^2+x^2}} = (a^2 + x^2)^{-1/2} \quad \dots \tag{38}$$

Eq. (38) is exactly equal to Eq. (36).

**Example B:**

Consider the computation of the first order Infinite Hankel Transform of the function,

$$f(x) = x^{-2}e^{-ax}, n = 1 \quad \dots \tag{39}$$

The Infinite Hankel Transform of $f(x)$ with $n = 1$ is given by

$$F(s) = \int_0^\infty x^{-2}e^{-ax}.\, xJ_1(sx)\, dx$$

$$\Rightarrow F(s) = \int_0^\infty e^{-ax}x^{-1}.J_1(sx)\, dx \quad \dots \tag{40}$$

From the Theory of Bessel Functions, it is well known that

$$\int_0^\infty e^{-\alpha x}x^{-1}.J_1(\beta x)\, dx = \frac{\sqrt{\alpha^2+\beta^2}-\alpha}{\beta} \quad \dots \tag{41}$$

Substituting $\alpha = a$ and $\beta = s$ in Eq. (41), we get,

$$F(s) = \int_0^\infty e^{-ax}x^{-1}.J_1(sx)\, dx$$

$$F(s) = H[x^{-2}e^{-ax}, n = 1] = \frac{\sqrt{a^2+s^2}-a}{s} \quad \dots \tag{42}$$

Next, we shall find the first order Inverse Hankel Transform of

$$F(s) = s^{-2}e^{-as}, n = 1 \quad \dots \tag{43}$$

Invoking Eq. (3) and substituting $F(s)$ into it gives

$$f(x) = H^{-1}[F(s)] = \int_0^\infty s^{-2}e^{-as}.\, sJ_1(sx)ds$$

$$f(x) = \int_0^\infty s^{-1}e^{-as}.J_1(sx)ds \qquad \dots \tag{44}$$

Eq. (44) is similar to the integral mentioned in Eq. (41). Substituting $\alpha = a$ and $\beta = x$ into it gives

$$f(x) = H^{-1}[F(s)] = \frac{\sqrt{a^2+x^2}-a}{x} \ \ldots \tag{45}$$

Next, we apply the Duality Property just derived for finding the first order Inverse Hankel Transform of Eq. (43). It has been established from Eq. (42) that the Hankel Transform of $f(x) = x^{-2}e^{-ax}, n = 1$, is

$$F(s) = \frac{\sqrt{a^2+s^2}-a}{s} \ldots \tag{46}$$

By the principle of Duality Theorem proved in the previous section, replacing $s$ by $x$ and vice versa in the above equation gives,

$$F(x) = \frac{\sqrt{a^2+x^2}-a}{x} \ \ldots \tag{47}$$

Eq. (47) is exactly equal to Eq. (45).

**Example C:**

Consider the computation of the zeroth order Infinite Hankel Transform of the function,

$$f(x) = e^{-ax}, n = 0 \ \ldots \tag{48}$$

The Infinite Hankel Transform of $f(x)$ with $n = 0$ is given by

$$F(s) = \int_0^\infty e^{-ax}.xJ_0(sx)\,dx \ \ldots \tag{49}$$

From the Theory of Bessel Functions, it is well known that

$$\int_0^\infty e^{-\alpha x}x.J_0(\beta x)\,dx = \alpha(\alpha^2+\beta^2)^{-3/2} \ \ldots \tag{50}$$

Substituting $\alpha = a$ and $\beta = s$ in Eq. (50), we get,

$$F(s) = H[e^{-ax}, n = 0] = a(a^2+s^2)^{-3/2} \ \ldots \tag{51}$$

Next, we shall find the first order Inverse Hankel Transform of

$$F(s) = e^{-as}, n = 0 \ \ldots \tag{52}$$

Invoking Eq. (3) and substituting $F(s)$ into it gives

$$f(x) = H^{-1}[F(s)] = \int_0^\infty e^{-as}.sJ_0(sx)ds \ \ldots \tag{53}$$

Eq. (53) is similar to the integral mentioned in Eq. (50). Substituting $\alpha = a$ and $\beta = x$ into it gives

$$f(x) = H^{-1}[F(s)] = a(a^2+x^2)^{-3/2} \ \ldots \tag{54}$$

Next, we apply the Duality Property just derived for finding the first order Inverse Hankel Transform of Eq. (52). It has been established from Eq. (51) that the Hankel Transform of $f(x) = e^{-ax}, n = 0$, is

$$F(s) = a(a^2+s^2)^{-3/2} \ \ldots \tag{55}$$

By the principle of Duality Theorem proved in the previous section, replacing $s$ by $x$ and vice versa in the above equation gives,

$$F(x) = a(a^2 + x^2)^{-3/2} \quad \ldots \tag{56}$$

Eq. (56) is exactly equal to Eq. (54).

Thus, the above three examples proves the veracity of the Duality Theorem corresponding to the Infinite Hankel Transform. If a Hankel Transform pair is known, then, it is possible to compute the Inverse Hankel Transform of the unknown function which has a similar form to the function whose Hankel Transform is known by merely interchanging the time – and frequency – domain variables. Hence, in such instances, the use of the Duality Theorem for Hankel Transform totally saves computation time by eradicating the need of integration, thereby proving to be a versatile and indispensable tool in Signal Analysis, Image Processing, and Electromagnetic Field Theory. It is also used in solving partial differential equations in mathematics. It thus saves computation time and computation cost to a maximum extent.

## Conclusions

This paper gives a new property of the Hankel Transform which has been not been mentioned and derived until now in the literature. Given a function whose Hankel Transform is known, by using this simple but beautiful Duality Property, we can find the Inverse Hankel Transform of a function whose shape resembles the time – domain function (whose Hankel Transform exists). Thus, by doing so, the actual labour of integration involved in finding the Inverse Hankel Transform is eradicated thereby saving computation cost and time, which proves the usage of this duality property. Hankel Transform can be used in applications such as Signal Processing, Communications, Image Processing, and Mechanics.

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