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A New Class of Extension Exponential Distribution: Properties and Application

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Abstract

A functional composition of the cumulative distribution function of one probability distribution with the inverse cumulative distribution function of another is called the transmutation map. In this article, we generalize the extension exponential distribution using the quadratic rank transmutation map studied by Shaw and Buckley (2007) to develop a transmuted extension exponential distribution. The new distribution exhibits decreasing, increasing and bathtub hazard rate depending on its parameters. A comprehensive mathematical treatment of this distribution is provided. Some expressions for the moment generating function, moments, and orderstatistics of the new distribution are derived. The model parameters are estimated by the maximum likelihood method. Finally, an application to real data sets is illustrated.

Keywords: Extension Exponential distribution, Moments, Order Statistics, Transmutation Map, Maximum Likelihood Estimation, Reliability Function.

Introduction and Motivation

In lifetime data analysis, monotone hazard rate occurs commonly in practice. Such situations are currently modeled using the families of the Weibull or the Gamma distributions. Among them, the Weibull distribution is more popular. Because the survival function of the Gamma distribution cannot be expressed in a closed form and one needs to obtain the survival function or the failure rate by numerical integration, while the Weibull distribution has a nice survival and hazard function. One can refer to Murthy et al. (2004) for details about Weibull models. Gupta and Kundu (2001) presented the Exponentiated Exponential distribution. This family has lots of properties which are quite similar to those of a Gamma distribution but it has an explicit expression of the survival function like a Weibull distribution. Gupta and

Kundu (2007) provided a detailed review and some developments on the Exponentiated Exponential distribution.

Nadarajaha and Haghighi (2011) introduced a new extension of the exponential distribution as an alternative to the gamma, Weibull and the Exponentiated Exponential distributions. the cumulative distribution function of NH distribution is given by

$$G(x) = 1 - e^{1-(1+\theta x)^{\alpha}}, x > 0, \tag{1.1}$$

where $\theta > 0$ is the scale parameter, and $\alpha > 0$ is the shape parameter. The corresponding probability density and failure rate functions are given by

$$g(x) = \alpha\theta(1+\theta x)^{\alpha-1} e^{1-(1+\theta x)^{\alpha}}, \tag{1.2}$$

and

$$h(x) = \frac{g(x)}{\overline{G}(x)} = \alpha\theta(1+\theta x)^{\alpha-1}. \tag{1.3}$$

Note that Equation (1.2) has two parameters just like the gamma, Weibull and the EE distributions. Note also that Equation (1.2) has closed form survival function and hazard rate functions just like the Weibull and the EE distributions. For $\alpha = 1$, Equation (1.2) reduces to the exponential distribution. As we shall see later, Equation (1.2) has the attractive feature of always having the zero mode and yet allowing for increasing, decreasing and constant hazard rate function.(see, Nadarajaha and Haghighi (2011)). Also Nadarajah and Haghighi (2011) presented some motivations for introducing their new distribution. The first motivation is based on the relationship between the probability density function in (1.2) and its failure rate function. The NH density function can be monotonically decreasing and yet its failure rate function can be increasing. The Gamma, Weibull and Exponentiated Exponential distributions do not allow for an increasing failure function when their respective densities are monotonically decreasing. The second motivation is related with the ability (or the inability) of the NH distribution to model data that have their mode fixed at zero. The Gamma, Weibull and Exponentiated Exponential distributions are not suitable for situations of this kind. The third motivation is based on the following mathematical relationship: if Y is a Weibull random variable with shape parameter α and scale parameter θ , then the density in Eq. (1.2) is the same as that of the random variable $Z = Y - \frac{1}{\theta}$ truncated at zero; that is, the NH distribution can be interpreted as a truncated Weibull distribution. For further details about this new model as well as general properties, the reader is referred to Nadarajah and Haghighi (2011).

In this paper, we introduce a new lifetime distribution by transmuted and compounding extension of the exponential distribution named transmuted extension exponential (TEE) distribution. The concept of transmuted explained in the following subsection

Transmutation Map

In this subsection we demonstrate transmuted probability distribution. Let F_1 and F_2 be the cumulative distribution functions, of two distributions with a common sample space. The general rank transmutation as given in Shaw and Buckley (2007) is defined as

$$G_{R12}(u) = F_2(F_1^{-1}(u)) \text{ and } G_{R21}(u) = F_1(F_2^{-1}(u)).$$

Note that the inverse cumulative distribution function also known as quantile function is defined as

$$F^{-1}(y) = \inf_{x \in R}\{F(x) \geq y\} \text{ for } y \in [0,1].$$

The functions $G_{R12}(u)$ and $G_{R21}(u)$ both map the unit interval $I = [0,1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $G_{Rij}(0) = 0$ and $G_{Rij}(0) = 1$. A quadratic Rank Transmutation Map (QRTM) is defined as

$$G_{R12}(u) = u + \lambda u(1-u), \; |\lambda| \leq 1, \tag{1.4}$$

from which it follows that the cdf's satisfy the relationship

$$F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2 \tag{1.5}$$

which on differentiation yields,

$$f_2(x) = f_1(x)[(1+\lambda) - 2\lambda F_1(x)] \tag{1.6}$$

where $f_1(x)$ and $f_2(x)$ are the corresponding pdfs associated with cdf $F_1(x)$ and $F_2(x)$ respectively. An extensive information about the quadratic rank transmutation map is given in Shaw and Buckley (2007). Observe that at $\lambda = 0$ we have the distribution of the base random variable. The following Lemma proved that the function $f_2(x)$ in given (1.6) satisfies the property of probability density function.

Lemma: $f_2(x)$ given in (1.6) is a well-defined probability density function.

Proof.

Rewriting $f_2(x)$ as $f_2(x) = f_1(x)[1 - \lambda(2F_1(x) - 1)]$ we observe that $f_2(x)$ is nonnegative. We need to show that the integration over the support of the random variable is equal one. Consider the case when the support of $f_1(x)$ is $(-\infty, \infty)$. In this case we have

$$\int_{-\infty}^{\infty} f_2(x)dx = \int_{-\infty}^{\infty} f_1(x)\left[(1+\lambda)-2\lambda F_1(x)\right]dx$$

$$= (1+\lambda)\int_{-\infty}^{\infty} f_1(x)dx - \lambda\int_{-\infty}^{\infty} 2f_1(x)F_1(x)dx$$

$$= (1+\lambda)-\lambda$$

$$= 1$$

Similarly, other cases where the support of the random variable is a part of real line follows. Hence $f_2(x)$ is a well-defined probability density function. We call $f_2(x)$ the transmuted probability density of a random variable with base density $f_1(x)$. Also note that when $\lambda = 0$ then $f_2(x) = f_1(x)$ .This proves the required result.

Many authors dealing with the generalization of some well- known distributions. Aryal and Tsokos (2009) defined the transmuted generalized extreme value distribution and they studied some basic mathematical characteristics of transmuted Gumbel probability distribution and it has been observed that the transmuted Gumbel can be used to model climate data. Also Aryal and Tsokos (2011) presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Elbatal and Aryal (2013) presented the transmuted additive Weibull distribution that extends the additive Weibull distribution. Elbatal (2013) studied the transmuted modified inverse Weibull distribution. Elbatal andElgarhy(2013) introduced transmuted quasi Lindley distribution. Elbatal et al. (2014) proposed the transmuted Exponentiated Fr ˆechet Distribution with properties and applications. Merovci and Elbatal (2013) introduce a new lifetime distribution by transmuted and compounding Lindley and geometric distributions named transmuted Lindley geometric distribution.

The rest of the paper is organized as follows. In section 2, we introduce the transmuted extension exponential (TEE) distribution. In section 3, we give some statistical properties of the TEE such as quintiles, moments and moment generating function.The minimum, maximum and median order statistics models are discussed in Section 4. In section 5, MLEs of the TEE are discussed. In section 6, simulation and some applications of the TEE are provided. In the last section, we draw some conclusions about this paper.

Transmuted Extension Exponential Distribution

In this section we studied the transmuted extension exponential (TEE) distribution. Now using (1.1) and (1.2) in (1.5) we have the cdf of transmuted extension exponential (TEE) distribution

$$F_{TEE}(x,\theta,\alpha,\lambda) = G(x)\left[(1+\lambda) - \lambda G(x)\right]$$

$$= \left[1 - e^{1-(1+\theta x)^{\alpha}}\right]\left[1 + \lambda e^{1-(1+\theta x)^{\alpha}}\right] \qquad (2.1)$$

where λ is the transmuted parameter. The corresponding probability density function (pdf) of the transmuted extension exponential (TEE) distribution is given by

$$f_{TEE}(x,\theta,\alpha,\lambda) = \alpha\theta(1+\theta x)^{\alpha-1} e^{1-(1+\theta x)^{\alpha}} \left\{(1-\lambda) + 2\lambda e^{1-(1+\theta x)^{\alpha}}\right\}$$

$$= (1-\lambda)\, e\,\alpha\,\theta\,(1+\theta x)^{\alpha-1} e^{-(1+\theta x)^{\alpha}} \qquad (2.2)$$

$$+2\lambda\, e^{2}\alpha\,\theta(1+\theta x)^{\alpha-1}\, e^{-2(1+\theta x)^{\alpha}},$$

Respectively. Figures 1 and 2 illustrate plot the pdf and cdf of TEE distribution for selected values of the parameters

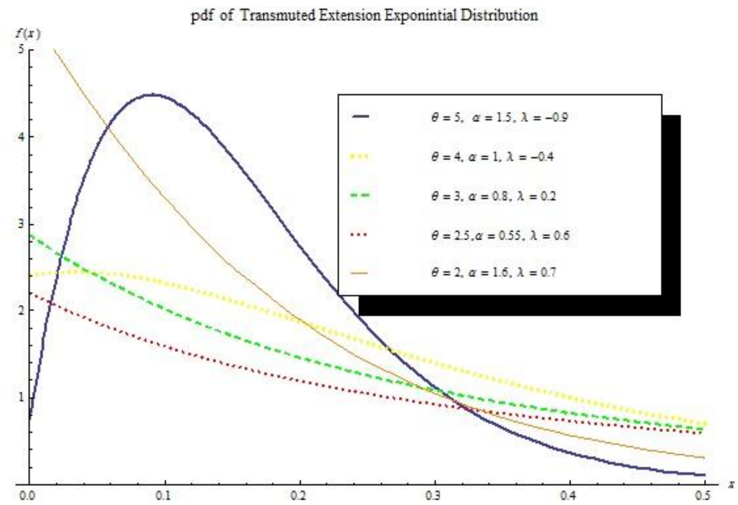

Figure 1: Pdf of Thetee Distribution

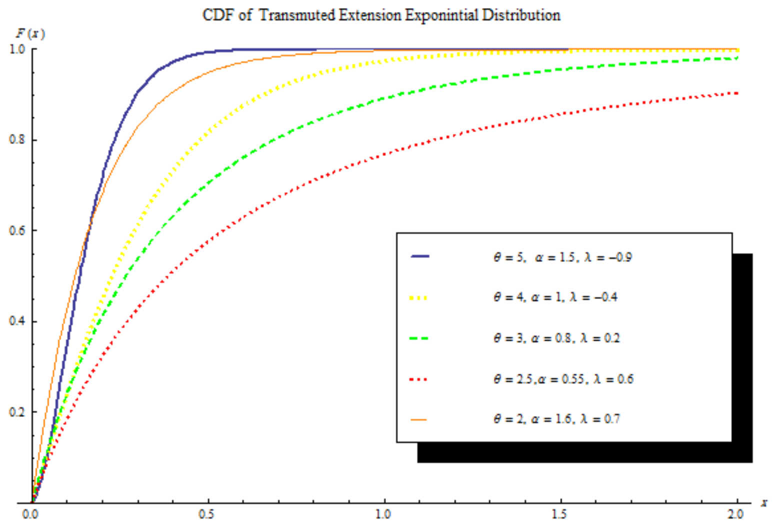

Figure 2: CDF of The TEE Distribution

It is observed that the transmuted extension exponential distribution is an extended model to analyze data from complex situations and it generalizes some of the widely used distributions. For instance when $\alpha = 1$ it reduces to transmuted Exponential distribution . The extension exponential distribution is clearly a special case for $\lambda = 0$. (See, Nadarajah et al. (2011)). When $\lambda = 0$ and $\alpha = 1$ then the resulting distribution is an Exponential distribution.

The reliability function (RF) of the transmuted extension exponential distribution is denoted by $R_{TEE}(x)$ also known as the survivor function and is defined as

$$R_{TEE}(x) = 1 - F_{TEE}(x) = 1 - \left[1 - e^{1-(1+\theta x)^{\alpha}}\right]\left[1 + \lambda e^{1-(1+\theta x)^{\alpha}}\right] \qquad (2.3)$$

It is important to note that $R_{TEE}(x) + F_{TEE}(x) = 1$. One of the characteristic in reliability analysis is the hazard rate function (HF) defined by

$$h_{TEE}(x) = \frac{f_{TEE}(x)}{R_{TEE}(x)}$$

$$= \frac{\alpha\theta(1+\theta x)^{\alpha-1} e^{1-(1+\theta x)^{\alpha}}\left[(1-\lambda) + 2\lambda e^{1-(1+\theta x)^{\alpha}}\right]}{1 - \left[1 - e^{1-(1+\theta x)^{\alpha}}\right]\left[1 + \lambda e^{1-(1+\theta x)^{\alpha}}\right]} \qquad (2.4)$$

It is important to note that the units for $h_{TEE}(x)$ is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters.The cumulative hazard function of the transmuted extension exponential distribution is denoted by $H_{TEE}(x)$ and is defined as

$$H_{TEE}(x) = -\ln\left|\left[1 - e^{1-(1+\theta x)^{\alpha}}\right]\left[1 + \lambda e^{1-(1+\theta x)^{\alpha}}\right]\right| \qquad (2.5)$$

It is important to note that the units for $H_{TEE}(x)$ is the cumulative probability of failure per unit of time, distance or cycles, we can show that for all choice of parameters the distribution has the decreasing patterns of cumulative instantaneous failure rates.

Figures 3 and 4 illustrate plot survival and hazard (failure) rate functions of TEE distribution for selected values of the parameters.

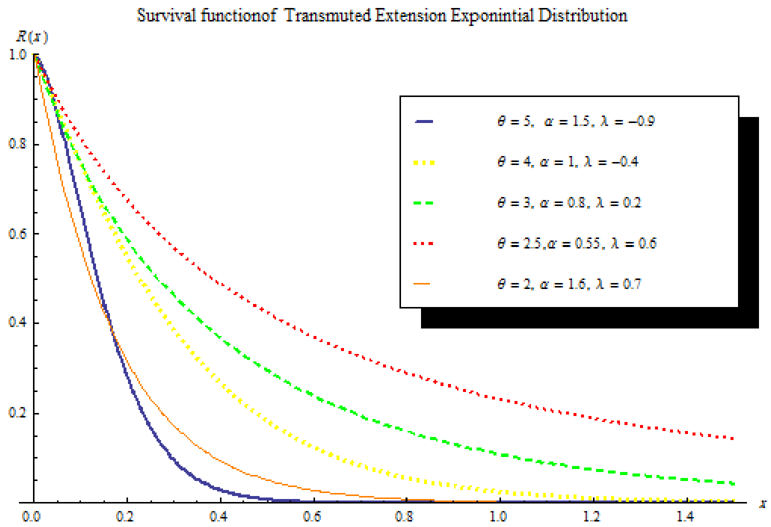

Figure 3: Survival Function of theTEE Distribution

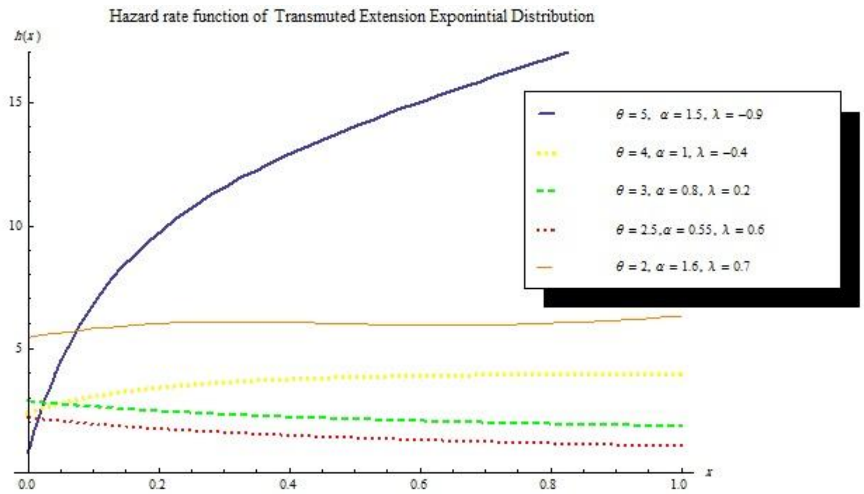

Figure 4: Hazard Function of the TEE Distribution

Statistical Properties

This section is devoted to studying statistical properties of the (TEE) distribution.

Quantiles and Random Generating

The quantile x_q of the transmuted extension exponential $TEE(\theta,\alpha,\lambda,x)$ is obtained from the following equation

$$x_q = \frac{1}{\theta}\left\{\left[1-\log\left(\frac{(\lambda-1)+\sqrt{(1-\lambda)^2-4\lambda(1-q)}}{2\lambda}\right)\right]^{\frac{1}{\alpha}}\right\}$$

This equation has no closed form solution in x_q , so we have to use a numerical technique to get the quantiles. In particular, put $q = 0.5$ in the above equation one gets the median of $TEE(\theta,\alpha,\lambda,x)$.

Random Number Generation

The random number generation as x of the $TEE(\theta,\alpha,\lambda,x)$ is defined by the following relation

$$Q(u) = \frac{1}{\theta}\left\{\left[1-\log\left(\frac{(\lambda-1)+\sqrt{(1-\lambda)^2-4\lambda(1-u)}}{2\lambda}\right)\right]^{\frac{1}{\alpha}}\right\} \text{ where } u \sim U(0,1)$$

The above equation doesn't have a closed form solution so we generate u as uniform random variables from $U(0,1)$ and solve for x in order to generate random numbers from TEE distribution.

Moments

In this subsection we discuss the r_{th} moment for (TEE) distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem (3.1).

If X has $TEE(\theta,\alpha,\lambda,x)$ then the r_{th} moment of X is given by

$$\mu_r^{'}(x) = \frac{e}{\theta^r}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\left[(1-\lambda)\Gamma\left(\frac{i}{\alpha}+1,1\right)+\frac{\lambda e}{2^{\frac{i}{\alpha}}}\Gamma\left(\frac{i}{\alpha}+1,2\right)\right].$$

Proof

Let X be a random variable with density function (2.2). The r_{th} ordinary moment of the (TEE) distribution is given by

$$\mu_r'(x) = E(X^{r)} = \int_0^\infty x^r f(x)dx$$

$$= (1-\lambda)e\alpha\theta \int_0^\infty x^r (1+\theta x)^{\alpha-1} e^{-(1+\theta x)^\alpha} dx \tag{3.2}$$

$$+2\lambda e^2 \alpha\theta \int_0^\infty x^r (1+\theta x)^{\alpha-1} e^{-2(1+\theta x)^\alpha} dx.$$

For $r > 0$ integer, it follows that

$$\int_0^\infty x^r (1+\theta x)^{\alpha-1} e^{-j(1+\theta x)^\alpha} = \frac{1}{\alpha\theta^{r+1}} \sum_{i=0}^{r} \frac{(-1)^{r-i}}{j^{\frac{i}{\alpha}+1}} \binom{r}{i} \Gamma(\frac{i}{\alpha}+1, j), \tag{3.3}$$

where $\Gamma(a,z) = \int_z^\infty x^{a-1} e^{-x} dx$ denotes the complementary incomplete gamma function, which can be evaluated in MATHEMATICA, R, etc. Then, the r_{th} moment of Xcan be expressed as

$$\mu_r'(x) = \frac{(1-\lambda)e}{\theta^r} \sum_{j=0}^{\infty} (-1)^{r-i} \binom{r}{i} \Gamma(\frac{i}{\alpha}+1, 1)$$

$$+\frac{2\lambda e^2}{\theta^r} \sum_{i=0}^{r} \frac{(-1)^{r-i}}{2^{\frac{i}{\alpha}+1}} \binom{r}{i} \Gamma(\frac{i}{\alpha}+1, 2) \tag{3.4}$$

$$= \frac{e}{\theta^r} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \left[(1-\lambda)\Gamma(\frac{i}{\alpha}+1, 1) + \frac{\lambda e}{2^{\frac{i}{\alpha}}} \Gamma(\frac{i}{\alpha}+1, 2) \right]$$

This completes the proof. Additionally, we notice that if we put $\lambda = 0,$ in (3.4) we get the r_{th} moment of extension exponential distribution

$$\mu_r'(x) = \frac{e}{\theta^r} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \Gamma(\frac{i}{\alpha}+1, 1) \tag{3.5}$$

(See, Nadarajah and Haghighi (2011)).

Based on the first four moments of the (TEE) distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the (TEE) distribution can obtained as

$$A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^{\frac{3}{2}}}, \tag{3.6}$$

and

$$k(\Phi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^{2}}. \tag{3.7}$$

Moment Generating Function
In this subsection we derived the moment generating function of (TEE) distribution.

Theorem (3.2):
If X has (TEE) distribution, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = \left\{ \frac{\alpha\theta^2}{\theta+1} \sum_{j=0}^{\infty} \sum_{i=0}^{j} (-1)^j \binom{\alpha-1}{j} \binom{j}{i} \left(\frac{\theta}{\theta+1} \right)^i \right. \tag{3.8}$$

$$\left. \left[(1+\lambda)\binom{\alpha-1}{j} - 2\lambda \binom{2\alpha-1}{j} \right] \left[\frac{\Gamma(i+1)}{(\theta(j+1)-t)^{i+1}} \left(1 + \frac{i+1}{(\theta(j+1)-t)} \right) \right] \right\}$$

Proof
We start with the well-known definition of the moment generating function given by

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx$$

$$= \frac{(1+\lambda)\alpha\theta^2}{\theta+1} \int_0^{\infty} (1+x) e^{-(\theta-t)x} \left[1 - (1 + \frac{\theta x}{\theta+1}) e^{-\theta x} \right]^{\alpha-1} dx \tag{3.9}$$

$$- \frac{2\lambda\alpha\theta^2}{\theta+1} \int_0^{\infty} (1+x) e^{-(\theta-t)x} \left[1 - (1 + \frac{\theta x}{\theta+1}) e^{-\theta x} \right]^{2\alpha-1} dx$$

using (3.5) and (3.7) into (3.9) we get

$$M_X(t) = \left\{ \frac{\alpha\theta^2}{\theta+1} \sum_{j=0}^{\infty} \sum_{i=0}^{j} (-1)^j \binom{\alpha-1}{j} \binom{j}{i} \left(\frac{\theta}{\theta+1} \right)^i \right. \tag{3.10}$$

$$\left. \left[(1+\lambda)\binom{\alpha-1}{j} - 2\lambda \binom{2\alpha-1}{j} \right] \left[\frac{\Gamma(i+1)}{(\theta(j+1)-t)^{i+1}} \left(1 + \frac{i+1}{(\theta(j+1)-t)} \right) \right] \right\}$$

Which completes the proof.

Distribution of The Order Statistics
In fact, the order statistics have many applications in reliability and life testing. The order statistics arise in the study of reliability of a system. Let $X_1, X_2, ..., X_n$ be a simple random sample from $TEE(\theta, \alpha, \lambda, x)$ with cumulative distribution function and probability density function as in (2.1) and (2.2), respectively. Let $X_{(1\,:\,n)} \leq$

$X_{(2\ :\ n)} \leq \ldots \leq X_{(n\ :\ n)}$ denote the order statistics obtained from this sample. In reliability literature, $X_{(i\ :\ n)}$ denote the lifetime of an $(n-i+1)-$ out $-$ of $-n$ system which consists of n independent and identically components. Then the pdf of $X_{(i\ :\ n)}, 1 \leq i \leq n$ is given by

$$f_{i\ :\ n}(x) = \frac{1}{\beta(i, n-i+1)} \left[F(x_{(i)})\right]^{i-1} \left[1 - F(x_{(i)})\right]^{n-i} f(x_{(i)}) \tag{4.1}$$

also, the joint pdf of $X_{(i\ :\ n)}$, $X_{(j\ :\ n)}$ and $1 \leq i \leq j \leq n$ is

$$f_{i\ :\ j\ :\ n}(x_i, x_j) = C \left[F(x_i)\right]^{i-1} \left[F(x_j) - F(x_i)\right]^{j-i-1} \left[1 - F(x_j)\right]^{n-j} f(x_i) f(x_j) \tag{4.2}$$

Where

$$C = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

We defined the first order statistics $X_{(1)} = Min(X_1, X_2, \ldots, X_n)$, the last order statistics as $X_{(n)} = Max(X_1, X_2, \ldots, X_n)$ and median order X_{m+1}.

Distribution of Minimum, Maximum and Median

Let $X_{(1\ :\ n)} \leq X_{(2\ :\ n)} \leq \ldots \leq X_{(n\ :\ n)}$ be independently identically distributed order random variables from the transmuted extension exponential distribution having first , last and median order probability density function are given by the following

$$f_{1\ :\ n}(x) = n \left[1 - F(x_{(1)})\right]^{n-1} f(x_{(1)})$$

$$= n \left\{1 - \left[1 - h_{(1)}\right]\left[1 + \lambda h_{(1)}\right]\right\}^{n-1} \tag{4.3}$$

$$\times \alpha\,\theta(1 + \theta\, x_{(1)})^{\alpha-1} h_{(1)} \left[(1-\lambda) + 2\lambda h_{(1)}\right]$$

where

$$h_{(i)} = e^{1-(1+\theta x_{(i)})^{\alpha}}$$

$$f_{n\ :\ n}(x) = n \left[F(x_{(n)}, \Phi)\right]^{n-1} f(x_{(n)}, \Phi)$$

$$= n \left\{\left[1 - h_{(n)}\right]\left[1 + \lambda h_{(n)}\right]\right\}^{n-1} \tag{4.4}$$

$$\times \alpha\theta(1 + \theta x_{(n)})^{\alpha-1} h_{(n)} \left[(1-\lambda) + 2\lambda h_{(n)}\right]$$

and

$$f_{m+1:n}(\tilde{x}) = \frac{(2m+1)!}{m!m!}(F(\tilde{x}))^m (1-F(\tilde{x}))^m f(\tilde{x})$$

$$= \frac{(2m+1)!}{m!m!}\{(1-h_{(m+1)}))\ (1+\lambda\, h_{(m+1)})^m \qquad (4.5)$$

$$.[1-(1-h_{(m+1)})]\,(1+\lambda\, h_{(m+1)})^m$$

$$.\alpha\theta\,(1+\theta\, x_{(m+1)})^{\alpha-1} h_{(m+1)}\{(1-\lambda)+2\lambda\, h_{(m+1)}\}.$$

We notice that the minimum, maximum and median order statistics of three parameters transmuted extension exponential distribution have different lifetime distributions when its parameters are changed.

Joint Distribution of The I $_{th}$ And J $_{th}$ Order Statistics

The joint distribution of the the i_{th} and j_{th} order Statistics from transmuted extension exponential is

$$f_{i:j:n}(x_i, x_j) = C\,[F(x_i)]^{i-1}[F(x_j)-F(x_i)]^{j-i-1}[1-F(x_j)]^{n-j} f(x_i) f(x_j)$$

$$= C\,\{[1-h_{(i)}][1+\lambda h_{(i)}]\}^{i-1}$$

$$\times\{[1-h_{(j)}][1+\lambda h_{(j)}]-[1-h_{(i)}][1+\lambda h_{(i)}]\}^{j-i-1} \qquad (4.6)$$

$$\times\{1-[1-h_{(j)}][1+\lambda h_{(j)}]\}^{n-j}$$

$$\times\alpha\theta(1+\theta x_{(i)})^{\alpha-1} h_{(i)}\{(1-\lambda)+2\lambda h_{(i)}\}$$

$$\times\alpha\theta(1+\theta x_{(j)})^{\alpha-1} h_{(j)}\{(1-\lambda)+2\lambda h_{(j)}\}$$

Special case if $i=1$ and $j=n$ we get the joint distribution of the minimum and maximum of order statistics

$$f_{1::n:n}(x_i, x_j) = n(n-1)\,[F(x_{(n)})-F(x_{(1)})]^{n-2} f(x_{(1)}) f(x_{(n)})$$

$$= n(n-1)\,\{h^{\alpha}_{(n)}[(1+\lambda)-\lambda h^{\alpha}_{(n)}]-$$

$$h^{\alpha}_{(1)}[(1+\lambda)-\lambda h^{\alpha}_{(1)}]\}^{n-2} \qquad (4.7)$$

$$\times\frac{\alpha\theta^2}{\theta+1}(1+x_{(1)})e^{-\theta x_{(1)}}\, h^{\alpha-1}_{(1)}\{(1+\lambda)-2\lambda h^{\alpha}_{(1)}\}$$

$$\times\frac{\alpha\theta^2}{\theta+1}(1+x_{(n)})e^{-\theta x_{(n)}}\, h^{\alpha-1}_{(n)}\{(1+\lambda)-2\lambda h^{\alpha}_{(n)}\}.$$

Also we can find the joint of minimum and maximum order statistics of three parameters transmuted generalized Lindley distribution when its parameters are changed.

Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the (TEE) distribution from complete samples only. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $TEE(\theta, \alpha, \lambda, x)$. The log-likelihood function for the vector of parameters $\Phi = (\alpha, \theta, \lambda)$ given by

$$\log L = n + n\log\theta + n\log\alpha + (\alpha-1)\sum_{i=1}^{n}\log(1+\theta x_i) - \sum_{i=1}^{n}(1+\theta x_i)^{\alpha} + \sum_{i=1}^{n}\log\left\{(1-\lambda) + 2\lambda e^{1-(1+\theta x_i)^{\alpha}}\right\}. \tag{5.1}$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (5.1). The components of the score vector are given by

$$\frac{\partial \log L}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log(1+\theta x_i) - \sum_{i=1}^{n}(1+\theta x_i)^{\alpha}\log(1+\theta x_i) - 2\lambda\sum_{i=1}^{n}\frac{e^{1-(1+\theta x_i)^{\alpha}}(1+\theta x_i)^{\alpha}\log(1+\theta x_i)}{\left\{(1-\lambda) + 2\lambda e^{1-(1+\theta x_i)^{\alpha}}\right\}}, \tag{5.2}$$

`

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i}{(1+\theta x_i)} - \alpha\sum_{i=1}^{n}x_i(1+\theta x_i)^{\alpha-1} - 2\lambda\alpha\sum_{i=1}^{n}\frac{x_i e^{1-(1+\theta x_i)^{\alpha}}(1+\theta x_i)^{\alpha-1}}{\left\{(1-\lambda) + 2\lambda e^{1-(1+\theta x_i)^{\alpha}}\right\}}, \tag{5.3}$$

and

$$\frac{\partial \log L}{\partial\lambda} = \sum_{i=1}^{n}\frac{2e^{1-(1+\theta x_i)^{\alpha}} - 1}{\left\{(1-\lambda) + 2\lambda e^{1-(1+\theta x_i)^{\alpha}}\right\}}. \tag{5.4}$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (5.2),(5.3) and (5.4)to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters.

The estimators and the corresponding summary statistics are obtained by our proposed model and the Newton iteration method using Math CAD program. For a given samples with different choices of λ, α and θ , we list the average of the maximum likelihood estimators (MLEs), the mean squared error (MSE), relative bias (RAB) and the coverage rate of the 95% confidence interval for λ, α and θ. The numerical results presented in Table1are based on 1000 simulations.

Table 1 summarizes the results of the estimates for λ, αandθ. We observe that $\hat{\lambda}, \hat{\alpha}$ and $\hat{\theta}$estimates the true parameters λ, α and θ quite well with relatively small MSEs and RAB. We also notice that the coverage probabilities of the asymptotic confidence interval are close to the nominal level.

These results indicate that the proposed model and the asymptotic approximation work well under the situation where no censoring occurs.

Table 1: Parameter Estimation for the Complete Simulated Samples

n	Parameters	MLEs	MSE	RAB	95% CI Coverage
10	λ	1.414	0.046	0.178	0.999
	α	0.489	0.007949	0.223	0.031
	θ	1.212	0.064	0.251	0.466
20	λ	1.448	0.002661	0.034	0.979
	α	0.603	0.041	0.509	0.127
	θ	1.452	0.041	0.162	0.8158
30	λ	1.402	0.041	0.168	0.999
	α	0.465	0.004246	0.163	0.001
	θ	1.212	0.043	0.206	1
40	λ	1.449	0.002436	0.035	1
	α	0.613	0.023	0.333	0.119
	θ	1.459	0.065	0.212	1
50	λ	1.442	0.001805	0.03	0.977
	α	0.611	0.023	0.328	0.1
	θ	1.452	0.062	0.207	0.989
60	λ	1.446	0.002114	0.033	0.989
	α	0.612	0.023	0.331	0.058
	θ	1.452	0.062	0.207	0.995
70	λ	1.443	0.001811	0.03	0.978
	α	0.61	0.023	0.327	0.01

	θ	1.452	0.062	0.207	0.999
80	λ	1.442	0.001782	0.03	0.988
	α	0.61	0.022	0.326	0.056
	θ	1.451	0.061	0.206	0.998

Application To Real Data Set

In this Section, we use two real data sets to show that how the TEE distribution can be applied in practice and the TEE can be a better model than one based on the Extension Exponential (EE), Generalized Exponential (GE), Kumaraswamy Exponential (KE) and New Exponential type (NET) distributions. For both data sets, we estimate the unknown parameters of each distribution by the maximum-likelihood method, and with these obtained estimates, we obtain the values of the Kolmogorov-Smirnov (K-S) statistic (the distance between the empirical CDFs and the fitted CDFs), Akaike information criterion (AIC), Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) are used to compare the candidate distributions. The AIC and BIC values are given by -2logL+2k and -2logL + k log n, respectively, where L is the value of the log likelihood function for obtained estimates of the unknown parameters, k is the number of the estimated parameters and n is the sample size. The better distribution corresponds to smaller KS, -2logL, AIC, CAIC and HQIC values.

The first data set represents remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003). The data are as follows

0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Table 2: Maximum-likelihood estimates, AIC , BIC, CAIC and HQIC values, and Kolmogorov-Smirnov statistics for the bladder cancer patients data

The Model	**MLEs**					**Measures**					
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	K-S	-2logL	AIC	BIC	CAIC	HQIC
EE	8.48	0.135				0.768	1330	1334	1339	1334	1336
GE	0.047			0.322		0.337	923.804	927.804	933.805	927.9	930.122
KE	0.162			0.180	0.308	0.263	955.719	961.719	970.257	961.913	965.195
NET	0.837	1.912		1.16		0.946	964.786	970.786	979.342	970.979	974.262
TEE	0.345	0.061	0.184			0.708	838.235	844.235	852.791	844.429	847.712

The asymptotic variance covariance matrix of the MLEs of the TEE model parameters, which is the inverse of the observed Fisher information matrix is given by

0.0000042	0.002639	0.005377
0.002639	0.006372	-0.003321
0.005377	-0.003321	0.001129

and the 95% asymptotic confidence intervals for the model parameters are givenby $\lambda \in (0.31, 0.381), \alpha \in (0.036, 0.064)$ and $\theta \in (0.178, 0.189)$

The second data set represents the waiting times (in minutes) before service of 100 bank customers. The data are as follows

0.8 0.8 1.3 1.5 1.8 1.9 1.9 2.1 2.6 2.7 2.9 3.1 3.2 3.3 3.5 3.6 4.0 4.1 4.2 4.2 4.3 4.3 4.4 4.4 4.6 4.7 4.7 4.8 4.9 4.9 5.0 5.3 5.5 5.7 5.7 6.1 6.2 6.2 6.2 6.3 6.7 6.9 7.1 7.1 7.1 7.1 7.4 7.6 7.7 8.0 8.2 8.6 8.6 8.6 8.8 8.8 8.9 8.9 9.5 9.6 9.7 9.8 10.7 10.9 11.0 11.0 11.1 11.2 11.2 11.5 11.9 12.4 12.5 12.9 13.0 13.1 13.3 13.6 13.7 13.9 14.1 15.4 15.4 17.3 17.3 18.1 18.2 18.4 18.9 19.0 19.9 20.6 21.3 21.4 21.9 23.0 27.0 31.6 33.1 38.5

Table 3. Maximum-likelihood estimates, AIC , BIC, CAIC and HQIC values, and Kolmogorov-Smirnov statistics for the waiting times (in minutes) before service of 100 bank customers

The Model	MLEs					Measures					
	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	K-S	-2logL	AIC	BIC	CAIC	HQIC
EE	4.83	0.135				0.796	1092	1096	1102	1096	1098
GE	0.051			0.369		0.375	745.232	749.232	754.442	749.355	751.34
KE	0.146			0.147	0.485	0.503	800.035	806.035	813.85	806.285	809.198
NET	0.966	1.477		0.113		0.968	798.347	804.347	812.162	804.597	807.51
TEE	0.431	0.074	0.107			0.734	671.972	672.972	685.788	678.222	681.135

The asymptotic variance covariance matrix of the MLEs of the TEE model parameters for the waiting times (in minutes) before service of 100 bank customers is given by

0.0000031	0.005607	0.001749
0.005607	0.015	-0.003252
0.001749	-0.003252	0.0006128

and the 95% asymptotic confidence intervals for the model parameters are given by $\lambda \in (0.396, 0.466), \alpha \in (0.026, 0.074)$ and $\theta \in (0.103, 0.112)$

From Table2 and Table3, we observe that the TEE distribution is a competitive distribution compared with other distributions. In fact, based on the values of the AIC and BIC criteria as, we observe that the TEE distribution provides the best fit for the two data sets among all the models considered.

A CDF plot compares the fitted density of the model with the empirical curve of the observed data as in Figure 5 and Figure 6

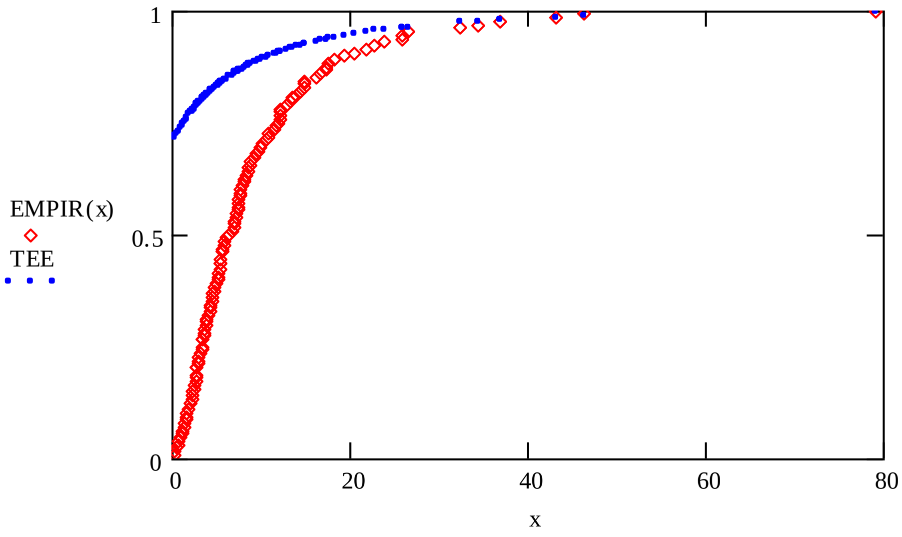

Figure 5: Empirical and fitted TEE cdf of the bladder cancer patients data

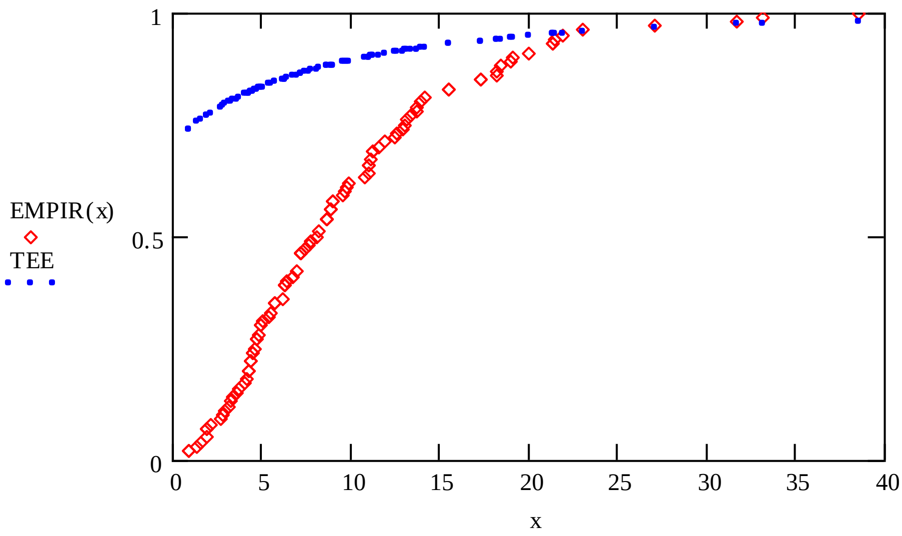

Figure 6: Empirical and fitted TEE CDF of the waiting times (in minutes) before service of 100 bank customers

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