International Journal of Applied Engineering Research
ISSN 0973-4562 Volume 10, Number 6 (2015) pp. 15777-15797

http://www.ripublication.com

On the Solution of System of Interval Linear Equations

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ABSTRACT

By using a new set of arithmetic operations on interval numbers, we propose a new algorithm based on interval versions of Gaussian Elimination method for the determinants of interval matrices and the Cramer's rule for the solution of interval linear systems

Keywords: Interval arithmetic, Interval matrix, System of interval linear equations, Gaussian algorithm.

1. Introduction

Systems of linear equations play an important role in many theoretical as well as practical real world problems in structural mechanics, solid mechanics, heat transfer analysis, Electrical Engineering, Control Theory, Remote Sensing and GISs, Quality control in manufacturing processes, Hurwitz stability in control theory applications etc. The physical problem can be formulated as a mathematical model and then transformed into system of linear equations and obtained the solution in various ways

It is inevitable to deal with uncertainties in real world problems. These uncertainties may arise due to measurement inaccuracy, round-off errors and various other kinds of inexact data. These uncertain parameters can be mathematically modeled by bounded intervals and problem may be expressed as an interval linear system. Significant research in this field is directed towards the use of intervals to represent the uncertain quantities in such systems. In the existing literature, several methods available for computing the smallest box $\tilde{x}$ containing the exact solution of the system. Several authors such as Alperbazaran, Kolev, Popova, Walter Kraemer, Iwona Skalna etc have studied the system of linear equations with interval parameters.

Kolev [14] developed a method for outer interval solution of linear parametric systems. Popova [22] also introduced a method on the solution of parameterised linear systems. Rohn [26]proved that solvability of a general rectangular system of linear interval equations can be characterized in terms of nonnegative solvability of a finite number of systems of linear equations. Dymova and Sevastjanov [4] have proposed a new approach for the solution of interval and fuzzy equations based on the generalized procedure of interval extension called "interval extended zero". Shohreh Abolmasoumi and Majid Alav [27] have proposed a new algorithm based on gradient vector in order to obtain the lower bound and upper bound of the interval solution. Walter Kraemer [26] investigated computing and visualizing solution sets of interval linear systems. The main objective of this paper is to find interval solution of linear interval system that is to determine the smallest interval vector containing all possible solutions.

In this paper, we propose a new algorithm based on interval versions of Gaussian Elimination method for the determinants of interval matrices and the Cramer's rule for the solution of interval linear systems.

The rest of this paper is organized as follows: In Section 2, we extend the Sengupta and Pal's [3] method of comparison of interval numbers to the set of all generalized intervals D. We recall the generalized interval arithmetic on the set of generalized interval numbers D proposed by Nirmala et.al [20][21]. In Section 3, we recall the notion of interval matrices, arithmetic operations on interval matrices, determinant of square matrices and some of its properties. We propose a simple method based on interval versions of Gaussian Elimination algorithm for the determinant and the Cramer's rule for the solution of interval linear systems. Numerical example is also provided to show the efficiency of the proposed algorithm.

2. Preliminary Notes

Let $IR = \{\tilde{a} = [a_1, a_2] : a_1 \le a_2 \text{ and } a_1, a_2 \in R\}$ be the set of all proper intervals and $\overline{IR} = \{\tilde{a} = [a_1, a_2] : a_1 > a_2 \text{ and } a_1, a_2 \in R\}$ be the set of all improper intervals on the real line R. If $a_1 = a_2 = a$, then $\tilde{a} = [a, a] = a$ is a real number (or a degenerate interval). We shall use the terms "interval" and "interval number" interchangeably. The mid-point and width (or half-width) of an interval number $\tilde{a} = [a_1, a_2]$ are defined as $m(\tilde{a}) = \left(\frac{a_1 + a_2}{2}\right)$ and $w(\tilde{a}) = \left(\frac{a_2 - a_1}{2}\right)$. We denote the set of generalized intervals (proper and improper) by $D = IR \cup \overline{IR} = \{[a_1, a_2] : a_1, a_2 \in R\}$. The set of generalized intervals D is a group with respect to addition and multiplication operations of zero free intervals, while maintaining the inclusion monotonicity.

The "dual" is an important monadic operator proposed by Kaucher[13] that reverses the end-points of the intervals expresses an element to element symmetry between proper and improper intervals in D. For $\tilde{a} = [a_1, a_2] \in D$, its dual is defined by dual$(\tilde{a})$ = dual$[a_1, a_2] = [a_2, a_1]$. The opposite of an interval $\tilde{a} = [a_1, a_2]$ is opp $\{[a_1, a_2]\} = [-a_1, -a_2]$

which is the additive inverse of $[a_1,\ a_2]$ and $\left[\frac{1}{a_1},\ \frac{1}{a_2}\right]$ is the multiplicative inverse of $[a_1,\ a_2]$, provided $0 \notin [a_1,\ a_2]$.

That is, $\tilde{a}+(-\text{dual}\ \tilde{a}) = \tilde{a}-\text{dual}(\tilde{a}) = [a_1,\ a_2]-\text{dual}([a_1,\ a_2])$ $=[a_1,\ a_2]-[a_2,\ a_1]=[a_1-a_1,\ a_2-a_2]=[0,0]$ and $\tilde{a}\times\left(\frac{1}{\text{dual}(\tilde{a})}\right)=[a_1,\ a_2]\times\left(\frac{1}{\text{dual}([a_1,\ a_2])}\right)$ $=[a_1,\ a_2]\times\frac{1}{[a_2,\ a_1]}=[a_1,a_2]\times\left[\frac{1}{a_1},\frac{1}{a_2}\right]=\left[\frac{a_1}{a_1},\frac{a_2}{a_2}\right]=[1,1]$

2.1. Comparing Interval Numbers

Let $\preceq$ be an extended order relation between the interval numbers $\tilde{a}=[a_1,\ a_2]$, $\tilde{b}=[b_1,\ b_2]$ in D, then for $m(\tilde{a})<m(\tilde{b})$, we construct a premise $(\tilde{a}\circ\ \tilde{b})$ which implies that $\tilde{a}$ is inferior to $\tilde{b}$ (or $\tilde{b}$ is superior to $\tilde{a}$).

An acceptability function $A_{\preceq}: D\times D\to[0,\infty)$ is defined as:

$A_{\circ}\ (\tilde{a},\tilde{b})=A(\tilde{a}\circ\ \tilde{b})=\frac{(m(\tilde{b})-m(\tilde{a}))}{(w(\tilde{b})+w(\tilde{a}))}$, where $w(\tilde{b})+w(\tilde{a})\neq 0$. $A_{\preceq}$ may be interpreted as the grade of acceptability of the first interval number $\tilde{a}$ to be inferior to the second interval number $\tilde{b}$.

2.2. A New Interval Arithmetic

Ganesan and Veeramani [6] proposed new interval arithmetic on IR. We extend these arithmetic operations to the set of generalized interval numbers D and incorporating the concept of dual.

For $\tilde{a}=[a_1,a_2],\ \tilde{b}=[b_1,b_2]\in D$ and for $*\in\{+,\ -,\ \cdot,\ \div\}$, we define $\tilde{a}*\tilde{b}=[m(\tilde{a})*m(\tilde{b})-k,\ \ m(\tilde{a})*m(\tilde{b})+k]$, $k=\min\left\{\left((m(\tilde{a})*m(\tilde{b})\right)-\alpha,\ \ \beta-\left((m(\tilde{a})*m(\tilde{b})\right)\right\}$, α and β are the end points of the interval $\tilde{a}\square\tilde{b}$ under the existing interval arithmetic. In particular for any two $\tilde{a}=[a_1,a_2],\ \tilde{b}=[b_1,b_2]\in D$,

(i) Addition:

$$\tilde{a}+\tilde{b}=[a_1,a_2]+[b_1,b_2],=[\{m(\tilde{a})+m(\tilde{b})\}-k,\ \{m(\tilde{a})+m(\tilde{b})\}+k],\ \text{where}\ \ k=\left(\frac{(b_2+a_2)-(b_1+a_1)}{2}\right).$$

(ii) Subtraction:

$$\tilde{a}-\tilde{b}=[a_1,a_2]-[b_1,b_2]\ =[\{m(\tilde{a})-m(\tilde{b})\}-k,\ \{m(\tilde{a})-m(\tilde{b})\}+k],\text{where}\ k=\left(\frac{(b_2+a_2)-(b_1+a_1)}{2}\right).$$

Also if $\tilde{a}=\tilde{b}$ i.e. $[a_1,\ a_2]=[b_1,\ b_2]$, then

$$\tilde{a}-\tilde{b}=\tilde{a}-\text{dual}(\tilde{a})=[a_1,\ a_2]-\text{dual}([a_1,\ a_2])=[a_1,\ a_2]-[a_2,\ a_1]=[a_1-a_1,\ a_2-a_2]=[0,0]$$

(iii). Multiplication:

$\tilde{a}.\tilde{b} = [a_1, a_2][b_1, b_2] = \left[\{(m(\tilde{a})\, m(\tilde{b})\} - k,\ \{(m(\tilde{a})m(\tilde{b})\} + k\right]$, where $k = \min\{(m(\tilde{a})m(\tilde{b})) - \alpha,\ \beta - (m(\tilde{a})m(\tilde{b}))\}$

$\alpha = \min\left(a_1b_1, a_1b_2, a_2b_1, a_2b_2\right)$ and $\beta = \max\left(a_1b_1, a_1b_2, a_2b_1, a_2b_2\right)$

(iv). Division:

$1 \div \tilde{a} = \frac{1}{\tilde{a}} = \frac{1}{[a_1, a_2]} = \left[\frac{1}{m(\tilde{a})} - k,\ \frac{1}{m(\tilde{a})} + k\right]$, where $k = \min\left\{\frac{1}{a_2}\left(\frac{a_2 - a_1}{a_1 + a_2}\right), \frac{1}{a_1}\left(\frac{a_2 - a_1}{a_1 + a_2}\right)\right\}$ and $0 \notin [a_1, a_2]$.

Also if $\tilde{a} = \tilde{b}$ i.e. $[a_1, a_2] = [b_1, b_2]$, then

$$\frac{\tilde{a}}{\tilde{b}} = \frac{\tilde{a}}{\text{dual}(\tilde{a})} = [a_1, a_2] \times \frac{1}{\text{dual}([a_1, a_2])} = [a_1, a_2] \times \frac{1}{[a_2, a_1]} = [a_1, a_2] \times \left[\frac{1}{a_1}, \frac{1}{a_2}\right] = \left[\frac{a_1}{a_1}, \frac{a_2}{a_2}\right] = [1,1]$$

From (iii), it is clear that $\lambda\tilde{a} = \begin{cases} [\lambda a_1, \lambda a_2], & \text{for } \lambda \geq 0 \\ [\lambda a_2, \lambda a_1], & \text{for } \lambda < 0 \end{cases}$

3. Main Results

An interval matrix $\tilde{A}$ is a matrix whose elements are interval numbers. An interval matrix $\tilde{A}$ will be written as $\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & .... & \tilde{a}_{1n} \\ ... & ... & ... \\ \tilde{a}_{m1} & ... & \tilde{a}_{mn} \end{pmatrix} = (\tilde{a}_{ij})_{1 \leq i \leq m,\ 1 \leq j \leq n}$, where each $\tilde{a}_{ij} = [\underline{a_{ij}}, \overline{a_{ij}}]$ (or) $\tilde{A} = [\underline{A}, \overline{A}]$ for some $\underline{A}, \overline{A}$ satisfying $\underline{A} \leq \overline{A}$. We use $D^{m \times n}$ to denote the set of all (m × n) interval matrices. The midpoint of an interval matrix $\tilde{A}$ is the matrix of midpoints of its interval elements defined as $m(\tilde{A}) = \begin{pmatrix} m(\tilde{a}_{11}) & .... & m(\tilde{a}_{1n}) \\ ... & ... & ... \\ m(\tilde{a}_{m1}) & ... & m(\tilde{a}_{mn}) \end{pmatrix}$.

The width of an interval matrix $\tilde{A}$ is the matrix of widths of its interval elements defined as $w(\tilde{A}) = \begin{pmatrix} w(\tilde{a}_{11}) & .... & w(\tilde{a}_{1n}) \\ ... & ... & ... \\ w(\tilde{a}_{m1}) & ... & w(\tilde{a}_{mn}) \end{pmatrix}$ which is always nonnegative. We use O to denote the null matrix $\begin{pmatrix} 0 & .... & 0 \\ ... & ... & ... \\ 0 & ... & 0 \end{pmatrix}$ and $\tilde{O}$ to denote the null interval matrix $\begin{pmatrix} \tilde{0} & .... & \tilde{0} \\ ... & ... & ... \\ \tilde{0} & ... & \tilde{0} \end{pmatrix}$. Also we use I to denote the identity matrix $\begin{pmatrix} 1 & .... & 0 \\ ... & 1 & ... \\ 0 & ... & 1 \end{pmatrix}$ and $\tilde{I}$ to denote the identity interval matrix $\begin{pmatrix} \tilde{1} & .... & \tilde{0} \\ ... & \tilde{1} & ... \\ \tilde{0} & ... & \tilde{1} \end{pmatrix}$. If $m(\tilde{A}) = m(\tilde{B})$, then the interval matrices $\tilde{A}$ and $\tilde{B}$ are said to be equivalent and is denoted by $\tilde{A} \approx \tilde{B}$. In particular if $m(\tilde{A}) = m(\tilde{B})$ and $w(\tilde{A}) = w(\tilde{B})$, then $\tilde{A} = \tilde{B}$. If $m(\tilde{A}) = O$, then we say that $\tilde{A}$ is a zero interval matrix and is denoted by $\tilde{O}$. In particular if $m(\tilde{A}) = O$ and $w(\tilde{A}) = O$, then $\tilde{A} = \begin{pmatrix} [0,0] & .... & [0,0] \\ ... & ... & ... \\ [0,0] & ... & [0,0] \end{pmatrix}$. Also if $m(\tilde{A}) = O$ and $w(\tilde{A}) \neq O$, then $\tilde{A} = \begin{pmatrix} \tilde{0} & .... & \tilde{0} \\ ... & ... & ... \\ \tilde{0} & ... & \tilde{0} \end{pmatrix} \approx \tilde{O}$. If $\tilde{A} \not\approx \tilde{O}$ (i.e. $\tilde{A}$ is not equivalent to $\tilde{O}$), then $\tilde{A}$ is said to be a non-zero interval matrix. If $m(\tilde{A}) = I$ then we say that $\tilde{A}$ is an identity interval matrix and is denoted by $\tilde{I}$. In particular if $m(\tilde{A}) = I$ and $w(\tilde{A}) = O$, then $\tilde{A} = \begin{pmatrix} [1,1] & .... & [0,0] \\ ... & [1,1] & ... \\ [0,0] & ... & [1,1] \end{pmatrix}$.

Also, if $m(\tilde{A}) = I$ and $w(\tilde{A}) \neq O$, then $\begin{pmatrix} \tilde{1} & .... & \tilde{0} \\ ... & \tilde{1} & ... \\ \tilde{0} & ... & \tilde{1} \end{pmatrix} \approx \tilde{I}$.

3.1. Arithmetic Operations on Interval Matrices

We define arithmetic operations on interval matrices as follows: If $\tilde{A}, \tilde{B} \in D^{m\times n}$, $\tilde{\mathbf{x}} \in D^n$ and $\tilde{\alpha} \in D$, then

(i). $\tilde{\alpha}\tilde{A} \approx (\tilde{\alpha}\, \tilde{a}_{ij})_{1 \le i \le m,\ 1 \le j \le n}$

(ii). $(\tilde{A} + \tilde{B}) \approx (\tilde{a}_{ij} + \tilde{b}_{ij})_{1 \le i \le m,\ 1 \le j \le n}$

(iii). $(\tilde{A} - \tilde{B}) \approx \begin{cases} (\tilde{a}_{ij} - \tilde{b}_{ij})_{1 \le i \le m,\ 1 \le j \le n}, & \text{if } \tilde{A} \not\approx \tilde{B} \\ \tilde{A} - \text{dual}(\tilde{A}) \approx \tilde{0} = 0, & \text{if } \tilde{A} \approx \tilde{B} \end{cases}$

(iv). $\tilde{A}\tilde{B} \approx \left(\sum_{k=1}^{n} \tilde{a}_{ik} \tilde{b}_{kj} \right)_{1 \le i \le m, 1 \le j \le n}$

(v). $\tilde{A}\tilde{\mathbf{x}} \approx \left(\sum_{j=1}^{n} \tilde{a}_{ij}\, \tilde{\mathbf{x}}_j \right)_{1 \le i \le m}$

3.2. Determinant of an Interval Matrix

We define the determinant of a square interval matrix as in the case of real square matrix except that the determinant of an interval matrix is an interval number. That is $\det \tilde{A} = |\tilde{A}| = \sum \tilde{a}_{ij} \tilde{A}_{ij}$ where $\tilde{A}_{ij}$ is the cofactor of $\tilde{a}_{ij}$ in the usual sense. It is easy to see that most of the properties of determinants of classical matrices are hold good (up to equivalent) for the determinants of interval matrices under the modified interval arithmetic.

3.3. Properties of Determinants

Let $\tilde{A}$ be an interval matrix of order (n × n). Then

(i). det $\tilde{A} \approx$ det $\tilde{A}^T$, for all (n × n) interval matrices $\tilde{A}$.

(ii). If one row of $\tilde{A}$ consists entirely of zeros then det $\tilde{A} \approx \tilde{0}$.

(iii). If $\tilde{B}$ is an interval matrix obtained from $\tilde{A}$ by multiplying one row of $\tilde{A}$ by the scalar λ, then det $\tilde{B} \approx \lambda$ det $\tilde{A}$.

(iv). If $\tilde{B}$ is an interval matrix obtained from $\tilde{A}$ by interchanging two rows of $\tilde{A}$ then det $\tilde{B} \approx$ - det $\tilde{A}$.

(v). If two rows of $\tilde{A}$ are identical, then det $\tilde{A} \approx \tilde{0}$.

Theorem 3.1. If $\tilde{A}$ is an (n × n) upper triangular or lower triangular interval matrix, then $|\tilde{A}|$ is equivalent to the product of the elements in its main diagonal.

Theorem 3.2. If $\tilde{B}$ is an interval matrix obtained from $\tilde{A}$ by adding a multiple of one row of $\tilde{A}$ to another row of $\tilde{A}$, then det $\tilde{A} \approx$ det $\tilde{B}$.

Theorem 3.3. Let $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ be a system of linear equations involving interval numbers. If the (n × n) interval matrix $\tilde{A}$ is invertible, then it is possible to find a smallest box $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n)$ which containing the exact solution of the system $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$, where each $\tilde{x}_i = \frac{|\tilde{A}^{(i)}|}{|\tilde{A}|}$, $\tilde{A}^{(i)}$ is the interval matrix obtained when the ith column of $\tilde{A}$ is replaced by the vector $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, ..., \tilde{b}_n)$.

3.4 Algorithm for the Solution of System of Interval Linear Equations

We use the generalized interval arithmetic proposed by Nirmala et.al by extending Kaucher's[13][13] interval arithmetic. This arithmetic operation satisfying group properties with respect to addition and multiplication operations and satisfying the distributive relations between intervals, while maintaining the inclusion monotonicity. We extent the Interval Gaussian Elimination Algorithm proposed by Nirmala et.al [21] by incorporating Cramers rule suitably to find the solution of system of interval linear equations.

Step 1: Given a system of interval linear equations $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$.

Step 2: Consider the augmented interval matrix $[\tilde{A} | \tilde{\mathbf{b}}]$.

Step 3: Using Interval Gaussian Elimination Algorithm, reduce the augmented interval matrix in the form of an upper triangular interval matrix.

Step 4: By Theorems (3.1) and (3.2), find the determinant of $\tilde{A}$ from the reduced form of the augmented interval matrix $[\tilde{A} | \tilde{\mathbf{b}}]$.

Step 5: $\tilde{A}^{(i)}$ is the interval matrix obtained when the i th column of $\tilde{A}$ is replaced by the last column $\tilde{\mathbf{b}}$ of the reduced augmented interval matrix.

Step 6: As in Step 4, find the determinant of $|\tilde{A}^{(i)}|, i = 1, 2, 3 \cdots$

Step 7: Find the smallest box $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n)$ (Interval vector) which contains the exact solution of the system $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$, where each $\tilde{x}_i = \frac{|\tilde{A}^{(i)}|}{|\tilde{A}|}$.

3.5. Algorithm for the solution of system of linear interval equations using Int Lab

```
A=input('Enter the interval augmented matrix: ')
[n,nn]=size(A);
A1=A;
for i=1:n-1
for k=i+1:n
% Division and Multiplication of two interval numbers
a=inf(A1(i,i));
b=sup(A1(i,i));
c=infsup(a,b);
m1=(a+b)/2 ;
if(a<=0 && b>=0)
if(m1>0)
m=m1/2;
f=m1-m;
g=m1+m;
h=infsup(f,g);
m2=(f+g)/2;
k1=(1/g)*((g-f)/(g+f));
k2=(1/f)*((g-f)/(g+f));
k=min(k1,k2);
n1=(1/m2)-k ;
n2=(1/m2)+k;
n3=infsup(n1,n2);
a=inf(n3)
b=sup(n3)
c=inf(A1(k,i));
d=sup(A1(k,i));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k3=min(k1,k2);
n1=(m1*m2)-k3;
n2=(m1*m2)+k3;
```

```
m(k,i)=-infsup(n1,n2);
a=inf(m(k,i));
b=sup(m(k,i));
c=inf(A1(i,:));
d=sup(A1(i,:));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k4=min(k1,k2);
n1=(m1*m2)-k4;
n2=(m1*m2)+k4;
if(A1(k,i)==A1(k,i))
s1=infsup(0,0);
if(A1(i,i)/A1(i,i)==inf)
s2=infsup(1,1);
else
end
s2=infsup(1,1);
A1(k,i)=s1*s2;
A1(k,:)=A1(k,:)+infsup(n1,n2);
A1(k,i)=infsup(0,0);
full(A1);
end
else
m3=-m1/2;
f1=m1-m3;
g1=m1+m3;
h1=infsup(f1,g1);
m4=(f1+g1)/2;
k3=(1/g1)*((g1-f1)/(g1+f1));
k4=(1/f1)*((g1-f1)/(g1+f1));
k1=min(k3,k4);
n4=(1/m4)-k1;
n5=(1/m4)+k1;
n6=infsup(n4,n5);
a=inf(n6);
b=sup(n6);
c=inf(A1(k,i));
d=sup(A1(k,i));
m1=(a+b)/2;
```

```
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k3=min(k1,k2);
n1=(m1*m2)-k3;
n2=(m1*m2)+k3;
h=infsup(n1,n2);
m(k,i)=-h;
a=inf(m(k,i));
b=sup(m(k,i));
c=inf(A1(i,:));
d=sup(A1(i,:));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k4=min(k1,k2);
n1=(m1*m2)-k4;
n2=(m1*m2)+k4;
f=infsup(n1,n2);
if(A1(k,i)==A1(k,i))
s1=infsup(0,0);
if((A1(i,i)/A1(i,i))==inf)
s2=infsup(1,1) ;
else
end
s2=infsup(1,1) ;
A1(k,i)=s1*s2;
A1(k,:)=A1(k,:)+infsup(n1,n2);
A1(k,i)=infsup(0,0);
full(A1);
end
end
else
k5=(1/b)*((b-a)/(a+b)) ;
k6=(1/a)*((b-a)/(a+b));
```

```
k7=min(k5,k6);
n7=(1/m1)-k7 ;
n8= (1/m1)+k7;
n9=infsup(n7,n8);
a=inf(n9);
b=sup(n9);
c=inf(A1(k,i));
d=sup(A1(k,i));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2 ;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k3=min(k1,k2);
n1=(m1*m2)-k3;
n2=(m1*m2)+k3;
h=infsup(n1,n2);
m(k,i)=-h;
a=inf(m(k,i));
b=sup(m(k,i));
c=inf(A1(i,:));
d=sup(A1(i,:));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k4=min(k1,k2);
n1=(m1*m2)-k4;
n2=(m1*m2)+k4;
f=infsup(n1,n2);
if(A1(k,i)==A1(k,i))
s1=infsup(0,0);
if(A1(i,i)/A1(i,i)==inf)
s2=infsup(1,1) ;
else
end
s2=infsup(1,1);
```

```
A1(k,:)=A1(k,:)+f;
A1(k,i)=infsup(0,0);
full(A1);
end
end
end
end
%Product of the diagonal matrix
diag(A);
pro_dia =infsup(1,1);
for m=1:n
a=inf(pro_dia);
b=sup(pro_dia);
c=inf(diag(A1(m,m)));
d=sup(diag(A1(m,m))) ;
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k4=min(k1,k2);
n1=(m1*m2)-k4;
n2=(m1*m2)+k4;
n3=infsup(n1,n2);
pro_dia=n3;
end
p1=n3;
fprintf('The solution of delta is \t')
display(infsup(n3))
for j=1:n
V=A1(1:n,1:n);
V(:,(j))=A1(:,(n+1));
for i=1:n-1
for k=i+1:n
% Division and Multiplication of two interval numbers
a=inf(V(i,i));
b=sup(V(i,i));
c=infsup(a,b);
m1=(a+b)/2 ;
if(a<=0 && b>=0)
if(m1>0)
m=m1/2;
```

```
f=m1-m;
g=m1+m;
h=infsup(f,g);
m2=(f+g)/2;
k1=(1/g)*((g-f)/(g+f));
k2=(1/f)*((g-f)/(g+f));
k=min(k1,k2);
n1=(1/m2)-k ;
n2=(1/m2)+k;
n3=infsup(n1,n2);
a=inf(n3);
b=sup(n3);
c=inf(V(k,i));
d=sup(V(k,i));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k3=min(k1,k2);
n1=(m1*m2)-k3;
n2=(m1*m2)+k3;
m(k,i)=-infsup(n1,n2);
a=inf(m(k,i));
b=sup(m(k,i));
c=inf(V(i,:));
d=sup(V(i,:));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k4=min(k1,k2);
n1=(m1*m2)-k4;
n2=(m1*m2)+k4;
if(V(k,i)==V(k,i))
s1=infsup(0,0);
if(V(i,i)/V(i,i)==inf)
```

```
s2=infsup(1,1) ;
else
end
s2=infsup(1,1);
V(k,i)=s1*s2;
V(k,:)=V(k,:)+infsup(n1,n2)
V(k,i)=infsup(0,0);
full(V);
end
else
m3=-m1/2;
f1=m1-m3;
g1=m1+m3;
h1=infsup(f1,g1);
m4=(f1+g1)/2;
k3=(1/g1)*((g1-f1)/(g1+f1));
k4=(1/f1)*((g1-f1)/(g1+f1));
k1=min(k3,k4);
n4=(1/m4)-k1;
n5=(1/m4)+k1;
n6=infsup(n4,n5);
a=inf(n6);
b=sup(n6);
c=inf(V(k,i));
d=sup(V(k,i));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k3=min(k1,k2);
n1=(m1*m2)-k3;
n2=(m1*m2)+k3;
h=infsup(n1,n2);
m(k,i)=-h;
a=inf(m(k,i));
b=sup(m(k,i));
c=inf(V(i,:));
d=sup(V(i,:));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
```

```
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k4=min(k1,k2);
n1=(m1*m2)-k4;
n2=(m1*m2)+k4;
f=infsup(n1,n2);
if(V(k,i))==(V(k,i))
s1=infsup(0,0);
if(V(i,i)/V(i,i)==inf)
s2=infsup(1,1) ;
else
end
s2=infsup(1,1);
V(k,i)=s1*s2;
V(k,:)=V(k,:)+f;
V(k,i)=infsup(0,0);
full(V);
end
end
else
k5=(1/b)*((b-a)/(a+b));
k6=(1/a)*((b-a)/(a+b));
k7=min(k5,k6);
n7=(1/m1)-k7 ;
n8= (1/m1)+k7;
n9=infsup(n7,n8);
a=inf(n9);
b=sup(n9);
c=inf(V(k,i));
d=sup(V(k,i));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2 ;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k3=min(k1,k2);
n1=(m1*m2)-k3;
n2=(m1*m2)+k3;
```

```
h=infsup(n1,n2);
m(k,i)=-h;
a=inf(m(k,i));
b=sup(m(k,i));
c=inf(V(i,:));
d=sup(V(i,:));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k4=min(k1,k2);
n1=(m1*m2)-k4;
n2=(m1*m2)+k4;
f=infsup(n1,n2);
if(V(k,i))==(V(k,i))
s1=infsup(0,0);
if(V(i,i))/(V(i,i)==inf)
s2=infsup(1,1) ;
else
end
s2=infsup(1,1) ;
V(k,:)=V(k,:)+f;
V(k,i)=infsup(0,0);
full(V);
end
end
end
end
%Product of the diagonal matrix
diag(V(:,j));
pro_dia =infsup(1,1);
for m=1:n
a=inf(pro_dia);
b=sup(pro_dia);
c=inf(diag(V(m,m)));
d=sup(diag(V(m,m)));
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
```

```
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k4=min(k1,k2);
n1=(m1*m2)-k4;
n2=(m1*m2)+k4;
n3=infsup(n1,n2);
pro_dia=n3;
p2=n3;
end
a=inf(p1);
b=sup(p1);
c=infsup(a,b);
m1=(a+b)/2 ;
if(a<=0 && b>=0)
if(m1>0)
m=m1/2;
f=m1-m;
g=m1+m;
h=infsup(f,g);
m2=(f+g)/2;
k1=(1/g)*((g-f)/(g+f));
k2=(1/f)*((g-f)/(g+f));
k=min(k1,k2);
n1=(1/m2)-k ;
n2=(1/m2)+k;
n3=infsup(n1,n2) ;
a=inf(p2);
b=sup(p2);
c=inf(n3);
d=sup(n3);
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k=min(k1,k2);
n1=(m1*m2)-k;
n2=(m1*m2)+k;
hhh=infsup(n1,n2)
else
```

```
m3=-m1/2;
f1=m1-m3;
g1=m1+m3;
h1=infsup(f1,g1);
m4=(f1+g1)/2;
k3=(1/g1)*((g1-f1)/(g1+f1));
k4=(1/f1)*((g1-f1)/(g1+f1));
k1=min(k3,k4);
n4=(1/m4)-k1;
n5=(1/m4)+k1;
n6=infsup(n4,n5);
a=inf(p2);
b=sup(p2);
c=inf(n6);
d=sup(n6);
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k=min(k1,k2);
n1=(m1*m2)-k;
n2=(m1*m2)+k;
hhh=infsup(n1,n2)
end
else
k5=(1/b)*((b-a)/(a+b)) ;
k6=(1/a)*((b-a)/(a+b));
k7=min(k5,k6);
n7=(1/m1)-k7 ;
n8= (1/m1)+k7;
n9=infsup(n7,n8);
a=inf(p2);
b=sup(p2);
c=inf(n9);
d=sup(n9);
m1=(a+b)/2;
m2=(c+d)/2;
I1=infsup(a,b);
I2=infsup(c,d);
m=I1*I2;
alpha=inf(m);
```

```
beta=sup(m);
k1=(m1*m2)-alpha;
k2=beta-(m1*m2);
k=min(k1,k2);
n1=(m1*m2)-k;
n2=(m1*m2)+k;
hhh=infsup(n1,n2)
end
end
```

3.6. Numerical Example

Consider an example given in Karkar Nora, Benmohamed Khier etal.[12]. The system of interval equations $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ be given with

$$\tilde{A} = \begin{pmatrix} [3.7,\ 4.3] & [-1.5,\ -0.5] & [0,\ 0] \\ [-1.5,\ -0.5] & [3.7,\ 4.3] & [-1.5,\ -0.5] \\ [0,\ 0] & [-1.5,\ -0.5] & [3.7,\ 4.3] \end{pmatrix} \text{and } \tilde{\mathbf{b}} = \begin{pmatrix} [-14, 0] \\ [-9, 0] \\ [-3, 0] \end{pmatrix}.$$

Let us solve this problem using the proposed algorithm. The augmented interval matrix of the given system is

$$\left[\tilde{A}|\tilde{\mathbf{b}}\right] = \begin{pmatrix} [3.7,\ 4.3] & [-1.5,\ -0.5] & [0,\ 0] & [-14, 0] \\ [-1.5,\ -0.5] & [3.7,\ 4.3] & [-1.5,\ -0.5] & [-9,\ 0] \\ [0,\ 0] & [-1.5,\ -0.5] & [3.7,\ 4.3] & [-3,\ 0] \end{pmatrix}$$

By Theorems (3.1) and (3.2), the reduced form of the augmented interval matrix $[\tilde{A}\,|\,\tilde{\mathbf{b}}]$ is given by

$$[\tilde{A}\,|\,\tilde{\mathbf{b}}] = \begin{pmatrix} [3.7,\ 4.3] & [-1.5,\ -0.5] & [0,\ 0] & [-14, 0] \\ [0,\ 0] & [3.2581, 4.2419] & [-1.5, -0.5] & [-12.5,\ 0] \\ [0,\ 0] & [0, 0] & [3.2256, 4.2411] & [-6.3334, 0] \end{pmatrix}$$

The determinant of coefficient interval matrix $\tilde{A}$ of the system $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ is given by

$$|\tilde{A}| = \begin{vmatrix} [3.7,\ 4.3] & [-1.5,\ -0.5] & [0,\ 0] \\ [0,\ 0] & [3.2581,\ 4.2419] & [-1.5,\ -0.5] \\ [0,\ 0] & [0, 0] & [3.2256,\ 4.2411] \end{vmatrix} = [38.8846,\ 73.1159] \text{ and } |\tilde{A}| \not\approx \tilde{0}.$$

From the reduced $[\tilde{A}\,|\,\tilde{\mathbf{b}}]$, form an interval matrix $\tilde{A}^{(i)}$, $i = 1, 2, 3$ by replacing the ith column of $\tilde{A}$ by $\tilde{\mathbf{b}}$, we have

$$|\tilde{A}^{(1)}| = \begin{vmatrix} [-14, 0] & [-1.5,\ -0.5] & [0,\ 0] \\ [-12.5,\ 0] & [3.2581,\ 4.2419] & [-1.5,\ -0.5] \\ [-6.3334,\ 0] & [0, 0] & [3.2256,\ 4.2411] \end{vmatrix}$$ Similarly, we have

$$|\tilde{A}^{(2)}| = \begin{vmatrix} [3.7,\ 4.3] & [-14, 0] & [0,\ 0] \\ [0,\ 0] & [-12.5,\ 0] & [-1.5,\ -0.5] \\ [0,\ 0] & [-6.3334,\ 0] & [3.2256,\ 4.2411] \end{vmatrix} \text{ and } |\tilde{A}^{(3)}| = \begin{vmatrix} [3.7,\ 4.3] & [-1.5,\ -0.5] & [-14, 0] \\ [0,\ 0] & [3.2581,\ 4.2419] & [-12.5,\ 0] \\ [0,\ 0] & [0, 0] & [-6.3334,\ 0] \end{vmatrix}$$

The determinants of the above reduced interval matrices are

$$|\tilde{A}^{(1)}| = \begin{vmatrix} [-14, 0] & [-1.5, -0.5] & [0, 0] \\ [0, 0] & [3.2581, 6.0275] & [-1.5, -0.5] \\ [0, 0] & [0, 0] & [3.2256, 4.4360] \end{vmatrix} = [-248.9990, 0],$$

$$|\tilde{A}^{(2)}| = \begin{vmatrix} [3.7, 4.3] & [-14, 0] & [0, 0] \\ [0, 0] & [-12.5, 0] & [-1.5, -0.5] \\ [0, 0] & [0, 0] & [3.2256, 5.2544] \end{vmatrix} = [-212, 0] \text{ and } |\tilde{A}^{(3)}| = \begin{vmatrix} [3.7, 4.3] & [-1.5, -0.5] & [-14, 0] \\ [0, 0] & [3.2581, 4.2419] & [-12.5, 0] \\ [0, 0] & [0, 0] & [-6.3334, 0] \end{vmatrix} = [-95.0010, 0]$$

Applying the proposed algorithm, the smallest box containing the exact solution of system of interval linear system $\tilde{A}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$ is

$$\tilde{x}_1 = \frac{|\tilde{A}^{(1)}|}{|\tilde{A}|} = \frac{[-248.9990, 0]}{[38.8846, 73.1159]} = [-4.4446, 0], \quad \tilde{x}_2 = \frac{|\tilde{A}^{(2)}|}{|\tilde{A}|} = \frac{[-212, 0]}{[38.8846, 73.1159]} = [-3.7842, 0],$$

$$\tilde{x}_3 = \frac{|\tilde{A}^{(3)}|}{|\tilde{A}|} = \frac{[-95.0010, 0]}{[38.8846, 73.1159]} = [-1.6957, 0].$$

Hence the solution set (box or interval matrix) obtained by the proposed algorithm is

$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} [-4.4446, 0] \\ [-3.7842, 0] \\ [-1.6957, 0] \end{pmatrix}.$$

For the same problem, Ganesan [7] obtained the solution set (box)

$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} [-4.482, 0] \\ [-3.816, 0] \\ [-1.776, 0.006] \end{pmatrix}.$$

Using interval Gaussian elimination with existing interval arithmetic,

Ning et al[19] obtained the solution set (box) $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} [-6.38, 0] \\ [-6.40, 0] \\ [-3.40, 0] \end{pmatrix}$. Using Hansen's technique of [10] or Rohn's reformulation of [25], Ning et al[19] obtained the solution set (wider box) $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} [-6.38, 1.12] \\ [-6.40, 1.54] \\ [-3.40, 1.40] \end{pmatrix}$.

Ning et al [19] obtained the solution set (much wider box) $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} [-6.38, 1.67] \\ [-6.40, 2.77] \\ [-3.40, 2.40] \end{pmatrix}$.

Karkar Nora, Benmohamed Khier etal. [12] obtained the solution set (box or interval matrix) as

$$\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} [-4.482, 0] \\ [-3.816, 0] \\ [-1.776, 0.006] \end{pmatrix}.$$

It is to be noted that the solution set (box) obtained by our method is sharper than the solution sets obtained by other techniques.

4 Conclusion

We proposed a new algorithm based on interval version of Gaussian elimination algorithm for the determinant and Cramers rule for the solution of interval linear system. By applying proposed method, we obtained a better solution comparing with the solutions obtained other methods.

Acknowledgement

The authors gratefully acknowledge (BRNS-DAE) Board of Research in Nuclear Sciences, Department of Atomic Energy, Government of India for its support through the funded research project grant No.2008/36/35/BRNS/1999 for the investigation presented here.

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