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Parameter Estimation of Lehmann Type I Exponential Mixture Distribution

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Abstract

A new class of mixture distribution is derived by mixing Normal and Lehmann Type I Exponential distribution. Properties of the distribution are given and the unknown parameters are estimated by the method of maximum likelihood estimation. The distribution is shown to be a better fit than the scale mixture of normal distribution of Andrews and Mallows (1974).

Introduction

Suppose that Y has a standard normal distribution and that σ has some distribution on $(0,\infty)$ with a continuous or discrete density $f(\sigma)$ $[\sigma > 0]$. Then the distribution of $X = Y\sigma$ is referred to as a scale mixture of normals and with a scale mixing density $f(\sigma)$. A wide class of continuous unimodal and symmetric distributions on the real line may be constructed as scale mixtures of normals. Andrews and Mallows (1974) and West (1984) studied the well known mixture distribution like discrete mixtures or contaminated normals, the student t family, logistic, Laplace or double-exponential and stable family. Their properties have been useful in many practical applications. Mike West (1987), showed that the exponential power family of distributions of Box & Tiao(1973) is a subset of the class of scale mixtures of normals..

Recently, the authors(Seethalakshmi and Saavithri, 2014) have introduced a new three parameter distribution by assuming the parameter σ of normal distribution to have a Lehmann Type I Exponential distribution on $(0,\infty)$. This distribution is called as Lehmann Type I Exponential Mixture Distribution (LEMD). The scale mixtures of normals obtained by Andrews and Mallows is a particular case of this new scale mixture of normals. Earlier , the parameters were estimated using the method of moments. In this paper, the parameters are estimated using the method of maximum likelihood estimation. The Akaike Information Criterion(AIC) is used to test the goodness of fit and it is shown that LEMD is better fit than scale mixture of normal distribution of Andrews and Mallows . LEMD was used to model the stock return data in financial market. The volatility parameter σ was treated as a function of economic factor(Seethalakshmi and Saavithri, 2014), the ratio of expected profit to expected revenue. Economic ratio was assumed to follow exponential distribution.

Scale mixtures of Normal Distribution

Preliminaries

Andrews and Mallows showed that a standard classical Laplace random variable Y has the distribution $Y \overset{d}{=} \sqrt{Z}X$ -- (1.1)

where the random variables Y and Z have the exponential and standard normal distributions respectively. The variable Z is allowed to take only positive values. A random variable Y, which can be expressed as in (1.1) is referred to as a normal variance mixture model or a scale mixture of Gaussians. A scalar μ is the actual mean value so that the mean of Y should be non zero. Then the marginal probability density function(pdf) of Y is obtained by averaging over Z, as in $P_Y(y) = \int_0^\infty \frac{1}{\sqrt{2\pi z}} \exp\left[-\frac{(y-\mu)^2}{2z} \right] P_Z(z)\, dz$ where $P_Z(z)$ is the pdf of Z. If Z is an exponential random variable with pdf $P_Z(z) = \frac{1}{\lambda}\exp\left(-\frac{z}{\lambda} \right)$ and X is a standard normal variable, then Y generated as $Y = \mu + \sqrt{Z}\ X$ will have pdf $P_Y(y) = \frac{1}{2}\sqrt{\frac{2}{\lambda}}\exp\left(-\sqrt{\frac{2}{\lambda}}|y-\mu| \right)$ --- (1.2)

Equation (1.2) is the pdf of a Laplace distribution centered at μ. Gupta and Kundu(1999) proved that Lehmann Type I Exponential Distribution is more suitable for modeling life time data. Exponential distribution is a particular case of Lehmann Type I Exponential Distribution by putting α = 1. In this paper σ is assumed to follow a Lehmann Type I Exponential Distribution with probability density function $P_Z(z) = \frac{\alpha}{\lambda}\left(1 - e^{-\frac{z}{\lambda}} \right)^{\alpha-1} e^{-\frac{z}{\lambda}}$. Then the pdf of the Lehmann Type I Exponential Mixture Distribution (LEMD) is

$$P_X(x) = \frac{\alpha}{2}\sum_{n=0}^{\infty}\sqrt{\frac{2}{\lambda(n+1)}}(-1)^n \binom{\alpha-1}{n} \exp\left(-\sqrt{\frac{2(n+1)}{\lambda}}|x-\mu| \right) \text{------------} \quad (1.3)$$

$$\alpha, \lambda > 0 \quad ; \quad -\infty < \mu < \infty$$

The series (1.3) converges whenever α is an integer. Here α is the shape parameter, μ is the location parameter and λ is the scale parameter. When α = 1, LEMD coincides with the scale mixture distribution of Andrews and Mallows.

The main aim of this paper is to estimate the unknown parameter of LEMD using maximum likelihood estimation for different sample sizes and to test the goodness of fit. The paper is organized as follows: In section 2, LEMD is introduced and some of its properties are discussed. Section 3 gives the estimation of parameters. In section 4, data sets are analyzed to test the goodness of fit of the proposed distribution.

Definition

Density and Distribution functions

A LEMD is a probability distribution on $(-\infty,\infty)$ given by the density function

$$P_X(x)=\frac{\alpha}{2}\sum_{n=0}^{\infty}\sqrt{\frac{2}{\lambda(n+1)}}(-1)^n\binom{\alpha-1}{n}\exp\left(-\sqrt{\frac{2(n+1)}{\lambda}}|x-\mu|\right) \text{ ---------- } (2.1.1)$$

where $\binom{\alpha-1}{n}=\frac{(\alpha-1)(\alpha-2)...(\alpha-(n-1))}{1.2.3....n}$ and α is an integer. Here α is a shape parameter > 0 , λ is the scale parameter > 0 and the location parameter is μ where $-\infty<\mu<\infty$.

The cumulative distribution function corresponding to density (2.1.1) is

$$F_X(x)=\begin{cases}\alpha\sum_{n=0}^{\infty}(-1)^n\frac{\binom{\alpha-1}{n}}{n+1}\left\{\frac{1}{2}e^{-\sqrt{\frac{2(n+1)}{\lambda}}(\mu-x)}\right\} & if \quad x\le\mu\\ \alpha\sum_{n=0}^{\infty}(-1)^n\frac{\binom{\alpha-1}{n}}{n+1}\left\{1-\frac{1}{2}e^{-\sqrt{\frac{2(n+1)}{\lambda}}(x-\mu)}\right\} & if \quad x\ge\mu\end{cases} \text{ ----- } (2.1.2)$$

A mixture normal generalized exponential distribution with parameters α, μ and λ is denoted by X ~ LEMD(α, μ, λ). The shape parameter controls the skewness of the distribution. This can be observed from the following figures.

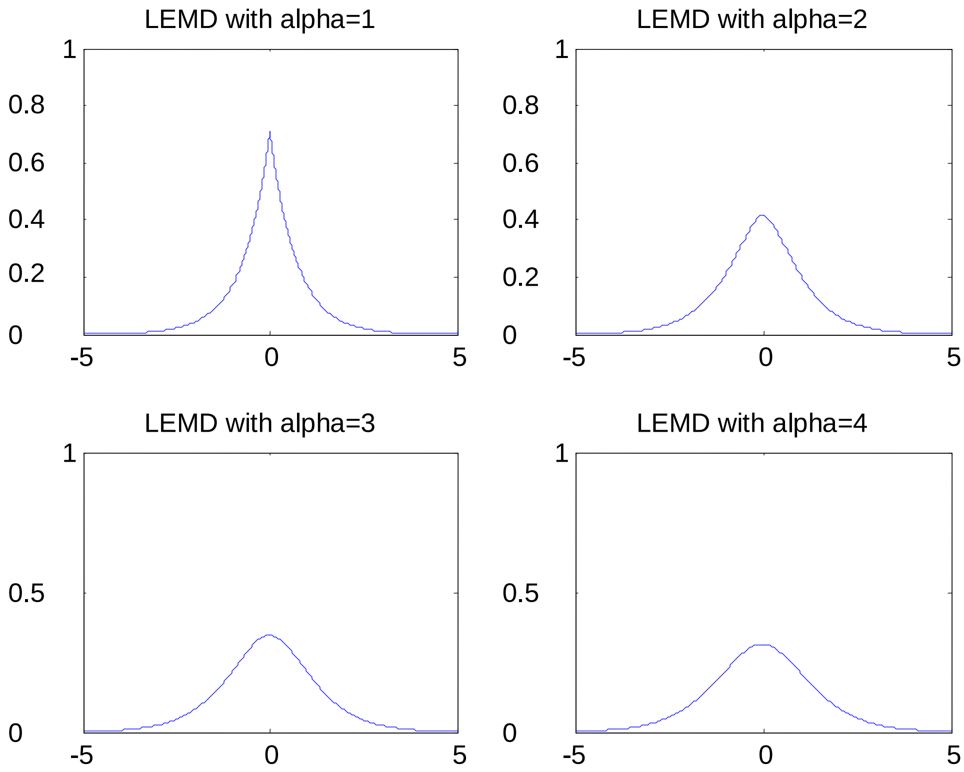

Figure 2.1:

Form of the density for various values of the shape parameter α when $\lambda = 1$

Theorem 2.2

If X ~ LEMD(α, μ, λ), then the survival function is given by

S(x, α, μ, λ) = 1-F(, x, α, μ, λ)

$$= \begin{cases} 1-\frac{1}{2}e^{-\sqrt{\frac{2}{\lambda}}(\mu-x)} & ; \quad x\leq\mu \\ \frac{1}{2}e^{-\sqrt{\frac{2}{\lambda}}(x-\mu)} & ; \quad x\geq\mu \end{cases} \quad \text{for } \alpha=1 \text{ ---------------------------------} \quad (2.2.1)$$

This is the survival function of Laplace distribution.

When α =2,

$$S(x, \alpha, \mu, \lambda) = \begin{cases} 1-e^{-\sqrt{\frac{2}{\lambda}}(\mu-x)} + \frac{1}{2}e^{-\sqrt{\frac{4}{\lambda}}(\mu-x)} & ; \quad x\leq\mu \\ e^{-\sqrt{\frac{2}{\lambda}}(x-\mu)} - \frac{1}{2}e^{-\sqrt{\frac{4}{\lambda}}(x-\mu)} & ; \quad x\geq\mu \end{cases} \text{-------------} (2.2.2)$$

Similarly the survival function for different values of α can be obtained.
If X ~ LEMD(α , μ , λ), then the hazard function is given by

$$h(x,\alpha,\mu,\lambda)=\frac{P(x,\alpha,\mu,\lambda)}{S(x,\alpha,\mu,\lambda)} = \begin{cases} \dfrac{\frac{1}{2}\sqrt{\frac{2}{\lambda}}e^{-\sqrt{\frac{2}{\lambda}}(\mu-x)}}{1-\frac{1}{2}e^{-\sqrt{\frac{2}{\lambda}}(\mu-x)}} & ; x \le \mu \\ \sqrt{\dfrac{2}{\lambda}} \quad ; & x \ge \mu \end{cases} \quad for\ \alpha=1 \quad (2.2.3)$$

This is the hazard function of Laplace distribution.
When α =2,

$$h(x,\alpha,\mu,\lambda)=\begin{cases} \dfrac{\sqrt{\frac{2}{\lambda}}e^{-\sqrt{\frac{2}{\lambda}}(\mu-x)} - \sqrt{\frac{1}{\lambda}}e^{-\sqrt{\frac{4}{\lambda}}(\mu-x)}}{1-e^{-\sqrt{\frac{2}{\lambda}}(\mu-x)}+\frac{1}{2}e^{-\sqrt{\frac{4}{\lambda}}(\mu-x)}} & ; \quad x \le \mu \\ \dfrac{\sqrt{\frac{2}{\lambda}}e^{-\sqrt{\frac{2}{\lambda}}(x-\mu)} - \sqrt{\frac{1}{\lambda}}e^{-\sqrt{\frac{4}{\lambda}}(x-\mu)}}{e^{-\sqrt{\frac{2}{\lambda}}(x-\mu)}-\frac{1}{2}e^{-\sqrt{\frac{4}{\lambda}}(x-\mu)}} & ; \quad x \ge \mu \end{cases} \text{-----------------}(2.2.4)$$

Similarly the hazard function for different values of α can be obtained.

Theorem 2.3
If X ~ LEMD(α , μ , λ), then the characteristic function is given by

$$\phi_X(t)=\alpha\sum_{n=0}^{\infty}\frac{2}{\lambda}(-1)^n\binom{\alpha-1}{n}\frac{e^{it\mu}}{\left(\frac{2(n+1)}{\lambda}+t^2\right)} \text{-------------------------------}(2.3.1)$$

and its moment generating function is given by

$$M_X(t)=\alpha\sum_{n=0}^{\infty}\frac{2}{\lambda}(-1)^n\binom{\alpha-1}{n}\frac{e^{t\mu}}{\left(\frac{2(n+1)}{\lambda}-t^2\right)} \text{-----------------------------}(2.3.2)$$

Theorem 2.4

Moments

Let X ~ LEMD(α, μ, λ). Then the moments about the origin are given by

$$\mu_1' = \alpha\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\binom{\alpha-1}{n}\mu \quad \text{------------} \quad (2.4.1)$$

$$\mu_2' = \alpha\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\binom{\alpha-1}{n}\left(\mu^2+\frac{\lambda}{n+1}\right) \quad \text{------------} (2.4.2)$$

$$\mu_3' = \alpha\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\binom{\alpha-1}{n}\left(\mu^3+\frac{3\mu\lambda}{n+1}\right) \quad \text{------------} (2.4.3)$$

$$\mu_4' = \alpha\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\binom{\alpha-1}{n}\left(\mu^4+\frac{6\mu^2\lambda}{n+1}+\frac{6\lambda^2}{(n+1)^2}\right) \quad \text{------------} (2.4.4)$$

Parameter Estimation

In this section, the parameters are estimated by maximizing log likelihood function based on the observations (x1,y1),(x2,y2),…,(xn,yn) from the LEMD population. The log likelihood function can be written as

$$\log L = \begin{cases} \log\left[\prod_{i=1}^{n_1}\left(\frac{\alpha}{2}\sum_{n=0}^{\infty}\sqrt{\frac{2}{\lambda(n+1)}}(-1)^n\binom{\alpha-1}{n}e^{-\sqrt{\frac{2(n+1)}{\lambda}}(\mu-x)_i}\right)\right] & \text{for } x_i \le \mu \\ \log\left[\prod_{j=1}^{n_2}\left(\frac{\alpha}{2}\sum_{n=0}^{\infty}\sqrt{\frac{2}{\lambda(n+1)}}(-1)^n\binom{\alpha-1}{n}e^{-\sqrt{\frac{2(n+1)}{\lambda}}(x_j-\mu)}\right)\right] & \text{for } x_j \ge \mu \end{cases} \quad \text{-(3.1)}$$

For different integer values of α, the values of λ for which log L is maximum are obtained. As λ increases the value of log L decreases , therefore the large possible value of λ is chosen for the corresponding α and maximum log L value.

Data Analysis and Goodness of fit

In this section, different data sets are analyzed and compared with normal variance mixture model in the context of goodness of fit.

Data set I consists of the return data for 200 days (02/01/’12 to 16/10/’12) of the TCS Index.(obtained from www.yahoofinance.com) . In this data, σ , the volatility, follows Lehmann Type I Exponential distribution. The corresponding results are given in the following table.

Table 4.1:

Scale Mixture of Normal	LEMD	Skew	Kurtosis	Mean (GED)	Variance (σ^2)	Std(σ)
$\hat{\phi} = 0.0121$ $\lambda = 2\hat{\phi}^2 = 0.000292$ Log = 544.3811 AIC = - 1087.8	$\mu = 0.0025$ $\lambda = 0.00014$ $\alpha = 3$ Log = 545.9076 AIC= - 1087.2	- 1.2156e-005	4.2148	0.0025 667	2.6678e -008	5.4792 e-004

Skew - Skewness Std - Standard deviatio

From the above table, it is obvious that LEMD has the maximum log likelihood value and minimum AIC value, which proves that LEMD is a better fit for this financial data. 1/λ gives the mean economic ratio.

Data set II

A new turbine steam generator requires a 0.98 reliability over a 1000-cycle period. A cycle occurs whenever the super heated steam reaches a temperature in excess of 700°C. The first 40 generators had failiures occurring at the following times

347	396	433	513	624	673	1008	1035	1055	1066
1162	1266	1298	1361	1367	1549	1561	1576	1708	1834
1840	2497	2554	2656	2666	2686	3261	3278	3281	3338
3421	4238	4242	4481	4845	5022	5744	6013	6238	13446

Table 4.2 shows that LEMD has the maximum log likelihood value and minimum AIC value, which proves that LEMD is a better fit for this data

Table 4.2:

Scale Mixture of Normal	LEMD	Skew	Kurtosis	Mean (GED)	Variance (σ^2)	Std(σ)
$\hat{\phi} = 1.6598$ $\lambda = 2\hat{\phi}^2 = 5.5097$ Log = - 87.9930 AIC = 178.9860	$\mu = 2.6895$ $\lambda = 2.5170$ $\alpha = 3$ Log = - 87.182 AIC= 178.25	1.3375e-005	4.2148	4.6145	8.6230	2.9365

Skew - Skewness Std - Standard deviatio

Data set III consists of 35 failure times were observed from among 50 units placed on test. The test was terminated at the 35th failiure. The failures are

1.3	7.3	7.8	13.3	13.9	19.4	19.7	22.3	22.8	26.7
29.7	30.2	30.9	32.2	33	36.8	37	41.7	46.7	50.4
51.4	60	61.3	61.4	65.6	65.8	72.6	78.4	100.4	100.6
111.4	118.2	119.4	132.1	139.7					

Table 4.3:

Scale Mixture of Normal	LEMD	Skew	Kurtosis	Mean (GED)	Variance (σ^2)	Std (σ)
$\widehat{\phi} = 0.3131$ $\lambda = 2\widehat{\phi}^2 = 0.1961$ Log = - 18.6180 AIC = 39.2360	$\mu = 0.5350$ λ=0.0882 α=3 Log = - 16.703 AIC= 37.4062	0.0012	4.2102	0.1617	0.0106	0.1030

Skew - Skewness Std - Standard deviatio

From the above table, it is obvious that LEMD has the maximum log likelihood value and minimum AIC value, which proves that LEMD fits better than the normal variance mixture model.

Data set IV consists of CSIAC software Real time command data (72 data , obtained from the web source)

Table 4.4:

Scale Mixture of Normal	LEMD	Skew	Kurtosis	Mean (GED)	Variance (σ^2)	Std(σ)
$\widehat{\phi} = 2.0107$ $\lambda = 2\widehat{\phi}^2 = 8.0856$ Log = - 93.6646 AIC = 189.3291	$\mu = 1.7832$ λ=3.5870 α=3 Log = - 89.692 AIC= 183.384	- 8.9475e-006	4.2148	6.5762	17.5128	4.1848

Skew - Skewness Std - Standard deviatio

From the above table, it is obvious that LEMD has the maximum log likelihood value and minimum AIC value, which proves that LEMD fits better than the normal variance mixture model. The following figures are the pdf curves of the above four data .

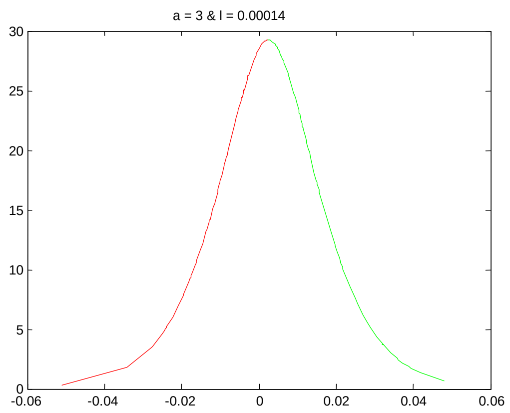

Figure 4.1:

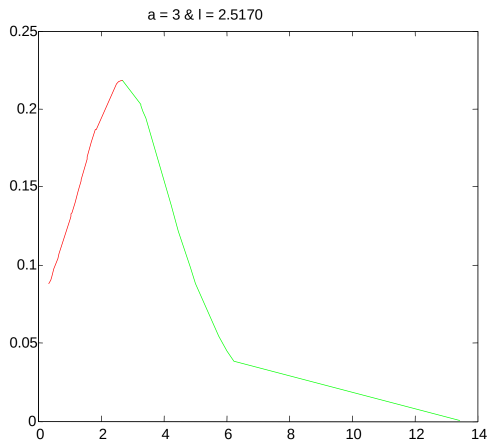

Figure 4.2:

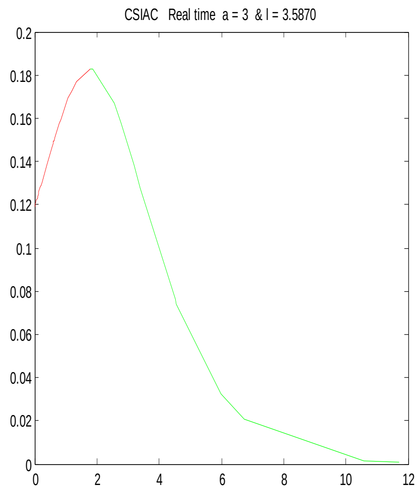

Figure 4.3:

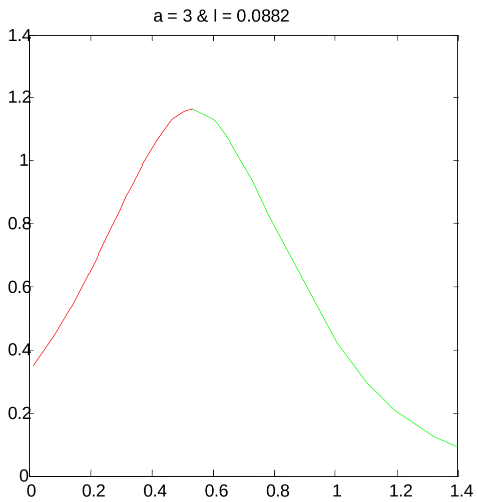

Figure 4.4:

Summary

This paper deals with the problem of estimating the parameters of the distribution LEMD. The parameters are estimated by the method of maximum likelihood estimation. Also it is shown that the distribution is better fit than the exponential scale mixture distribution. In conclusion, LEMD has a promising future in modeling financial and other reliability data.

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