International Journal of Applied Engineering Research.
ISSN 0973-4562 Volume 10, Number 4 (2015), pp. 11437–11453

http://www.ripublication.com/ijaer.htm

M/G/1 Feedback Queue with Three Stage Heterogeneous Service and Deterministic Server Vacations

S. Vanitha
SSN College of Engineering,
Old Mahabalipuram Road,
Kalavakkam - 603110, Chennai, India.
E-mail: vanithasudhakar05@gmail.com

Abstract

We study a single server queue with Poisson input, three stages of general heterogeneous services and deterministic vacations of constant duration. The server provides service to customers one by one on a first come first served basis. Just after completion of third stage of service, a customer may leave the system or may opt to repeat the service in which case this customer rejoins the queue. Further, just after completion of a customer's third stage of service, the server may take a vacation of random length or may opt to continue staying in the system to serve the next customer. We find the time probability generating function in terms of Laplace transforms and derive explicitly the corresponding steady state results.

AMS subject classification: 60K25, 60K30.
Keywords: Poisson arrivals, general heterogeneous service, probability generating function, idle state, steady state, deterministic vacations, supplementary variable technique.

1. Introduction

The queueing system with server vacations can be used to model a system wherein the server stops working during a vacation. Such system has wide applicability in analyzing the performance of various real life traffic situations of day-to-day as well as industrial queues. There have been extensive studies in queues with vacations by prominent researchers. Levy and Yechailai, Takagi, Doshi, Keilson and Servi. Gaver, Fuhrman, Shantikumar, Cramer and Madan are a few among many authors who have studied queues with server vacations with varying vacation policies

An $M/G/1$ queue with vacation model is often referred as a tool of understanding congestion phenomena in local networks. Since the past two or three decades,it has emerged as an important area of study in real life problems such as telecommunication engineering, manufacturing and production industries, computer and communication networks etc. Several contributions have been made by dealing queueing systems of $M/G/1$ type which include Bertsimas, Madan, Choudhury, Thangaraj and Vanitha etc.

The single server queue with phases of service with vacations has been paid attention recently by several researchers. Presently such type of models have been the subject matter of current research mainly due to its applications in computer and communications systems. Madan (2000) have introduced the concept of two phase queueing system by considering a single server queue under Bernoulli schedule server vacations. Madan (2001) has studied a single server queue with two stage heterogeneous service and deterministic server vacations. Choi and Kim (2003) have considered a two phase queueing systems with server vacations and Bernoulli feedback. Choudhury and Madan (2004) have studied two phase batch arrival queueing system with Bernoulli schedule vacation.

Choudhury and Paul (2005) have dealt with an $M/G/1$ queue with two phases of heterogeneous services and Bernoulli feedback. Here the queue size distribution at random and at service completion epoch are derived. Li and Wang (2006) have studied an $M/G/1$ retrial queue with two phase service and feedback where the server is subject to starting failures and breakdowns during service. Badamchi and Shankar (2008) have considered a single server queue with two phases of heterogeneous service with Bernoulli feedback and Bernoulli vacation.

Maragatha sundari and Srinivasan (2012) have analyzed $M/G/1$ feedback queue with three stage service times with multiple server vacation. Ayyappan and Sathiya (2013) have considered three stage batch arrival feedback queue with restricted admissibility policy. In this paper, we consider a single server feedback queue with three stage heterogenous service under Bernoulli schedule server vacations. This paper is organized as follows. The mathematical description of our model is given in Section 2. Definitions and equations governing the system are given in Section 3. The time dependent solution have been obtained in Section 4 and the corresponding steady state results have been explicitly in Section 5. Mean queue size and mean system size are computed in Section 6. Mean waiting time in the queue and in the system are given in Section 7.

2. Assumptions Underlying the Model

The following assumptions describe the mathematical model

- Customers arrive at the system one by one in according to a Poisson stream with arrival rate $\lambda(> 0)$.

- Each customer undergoes three stages of heterogeneous service provided by a single server on a first come first served basis. The service time of the three stages follow different general (arbitrary) distributions with distribution function $B_j(v)$ and the density function $b_j(v)$, $j = 1, 2, 3$.

- After completion of third stage of service if the customer is dissatisfied with its service for certain reason or if it received unsuccessful service, the customer may immediately join the tail of the original queue with probability p $(0 \leq p < 1)$. Otherwise the customer may depart forever from the system with probability $q = (1 - p)$.
- Let $\mu_i(x)\mathrm{d}x$ be the conditional probability of completion of the i^{th} stage of service during the interval $(x, x + \mathrm{d}x]$ given that elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \; i = 1, 2, 3 \tag{2.1}$$

and therefore,

$$b_i(v) = \mu_i(v)e^{-\int_0^v \mu_i(x)\mathrm{d}x}, \; i = 1, 2, 3. \tag{2.2}$$

- As soon as the third stage of a customer is complete, then with probability θ the server decides to take a vacation and with probability $1 - \theta$, server continues to be available for the next service.
- We assume that whenever the server takes a vacation, it is of constant duration $d(> 0)$.
- Various Stochastic Processes involved in the system are independent of each other.

3. Definitions, Notations and the Time - Dependent Equations Governing the System

We define

$W_n^{(1)}(x, t)$ **:** Probability that at time t, the server is providing first stage of service and there are $n \geq 0$ customers in the queue excluding the one being served and the elapsed served time of this customer is x. Consequently $W_n^{(1)}(t) = \int_0^\infty W_n^{(1)}(x, t)\mathrm{d}x$ denotes the probability that at time t, there are n customers in the queue excluding the one customer in the first stage of service irrespective of the value of x.

$W_n^{(2)}(x, t)$ **:** Probability that at time t, the server is providing second stage of service and there are $n \geq 0$ customers in the queue excluding the one being served and the elapsed served time of this customer is x. Consequently $W_n^{(2)}(t) =$

$\int_0^\infty W_n^{(2)}(x,t)\mathrm{d}x$ denotes the probability that at time t, there are n customers in the queue excluding the one customer in the second stage of service irrespective of the value of x.

$W_n^{(3)}(x,t)$ **:** Probability that at time t, the server is providing third stage of service and there are $n \geq 0$ customers in the queue excluding the one being served and the elapsed served time of this customer is x. Consequently $W_n^{(3)}(t) = \int_0^\infty W_n^{(3)}(x,t)\mathrm{d}x$ denotes the probability that at time t, there are n customers in the queue excluding the one customer in the third stage of service irrespective of the value of x.

$V_n(t)$ **:** Probability that at time t, there are $n \geq 0$ customers in the queue and the server is away on vacation.

$Q_n(t)$ **:** Probability that at time t, there is no customer in the system and the server is idle but available in the system. Finally, we assume that k_r is the probability of r arrivals during a vacation period of duration d so that,

$$K_r = \frac{e^{-\lambda d}(\lambda d)^r}{r!}, r = 0, 1, 2, \ldots. \tag{3.1}$$

The model is then governed by the following time dependent forward system equations

$$\frac{\partial}{\partial x}W_n^{(1)}(x,t) + \frac{\partial}{\partial t}W_n^{(1)}(x,t) + (\lambda + \mu_1(x))W_n^{(1)}(x,t) = \lambda W_{n-1}^{(1)}(x,t), \quad n = 1, 2, \ldots \tag{3.2}$$

$$\frac{\partial}{\partial x}W_0^{(1)}(x,t) + \frac{\partial}{\partial t}W_0^{(1)}(x,t) + (\lambda + \mu_1(x))W_0^{(1)}(x,t) = 0, \tag{3.3}$$

$$\frac{\partial}{\partial x}W_n^{(2)}(x,t) + \frac{\partial}{\partial t}W_n^{(2)}(x,t) + (\lambda + \mu_2(x))W_n^{(2)}(x,t) = \lambda W_{n-1}^{(2)}(x,t), \quad n = 1, 2, \ldots \tag{3.4}$$

$$\frac{\partial}{\partial x}W_0^{(2)}(x,t) + \frac{\partial}{\partial t}W_0^{(2)}(x,t) + (\lambda + \mu_2(x))W_0^{(2)}(x,t) = 0, \tag{3.5}$$

$$\frac{\partial}{\partial x}W_n^{(3)}(x,t) + \frac{\partial}{\partial t}W_n^{(3)}(x,t) + (\lambda + \mu_3(x))W_n^{(3)}(x,t) = \lambda W_{n-1}^{(3)}(x,t), \quad n = 1, 2, \ldots \tag{3.6}$$

$$\frac{\partial}{\partial x}W_0^{(3)}(x,t) + \frac{\partial}{\partial t}W_0^{(3)}(x,t) + (\lambda + \mu_3(x))W_0^{(3)}(x,t) = 0, \tag{3.7}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}V_0(t) = \theta q \int_0^\infty W_0^{(3)}(x,t)\mu_3(x)\mathrm{d}x, \tag{3.8}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}V_n(t) = \theta p \int_0^\infty W_{n-1}^{(3)}(x,t)\mu_3(x)\mathrm{d}x + \theta q \int_0^\infty W_n^{(3)}(x,t)\mu_3(x)\mathrm{d}x, \quad n = 1,2,\ldots, \tag{3.9}$$

$$\frac{d}{dt}Q(t) = -\lambda Q(t) + V_0(t)K_0 + q(1-\theta)\int_0^\infty W_0^{(3)}(x,t)\mu_3(x)\mathrm{d}x\ . \tag{3.10}$$

Equations (3.2) to (3.10) are to be solved subject to the following boundary conditions:

$$W_0^{(1)}(0,t) = Q(t)\lambda + V_0(t)K_1 + V_1(t)K_0 + (1-\theta)q\int_0^\infty W_1^{(3)}(x,t)\mu_3(x)\mathrm{d}x, + (1-\theta)p\int_0^\infty W_0^{(3)}(x,t)\mu_3(x)\mathrm{d}x\ , \tag{3.11}$$

$$W_n^{(1)}(0,t) = V_0(t)K_{n+1} + V_1(t)K_n + \cdots + V_n(t)K_1 + V_{n+1}(t)K_0 + + (1-\theta)q\int_0^\infty W_{n+1}^{(3)}(x,t)\mu_3(x)\mathrm{d}x + (1-\theta)p\int_0^\infty W_n^{(3)}(x,t)\mu_3(x)\mathrm{d}x, \quad n = 1,2,\ldots, \tag{3.12}$$

$$W_n^{(2)}(0,t) = \int_0^\infty W_n^{(1)}(x,t)\mu_1(x)\mathrm{d}x,\ n = 0,1,\ldots. \tag{3.13}$$

$$W_n^{(3)}(0,t) = \int_0^\infty W_n^{(2)}(x,t)\mu_3(x)\mathrm{d}x,\ n = 0,1,\ldots. \tag{3.14}$$

Next, we assume that initially the system starts when there is no customer in the system and the server is idle but available in the system so that the initial conditions are given by

$$Q(0) = 1,\ V_0(0) = 0 = V_n(0), n \geq 0. \tag{3.15}$$

4. Generating functions of the queue length: The time-dependent solution

In this section we obtain the transient solution for the above set of differential-difference equations. We define the probability generating functions,

$$\left.\begin{aligned} W^{(1)}(x,z,t) &= \sum_{n=0}^{\infty} z^n W^{(1)}(x,t), \\ W^{(1)}(z,t) &= \sum_{n=0}^{\infty} z^n W^{(1)}(t), \\ W^{(2)}(x,z,t) &= \sum_{n=0}^{\infty} z^n W^{(2)}(x,t), \\ W^{(2)}(z,t) &= \sum_{n=0}^{\infty} z^n W^{(2)}(t), \\ W^{(3)}(x,z,t) &= \sum_{n=0}^{\infty} z^n W^{(3)}(x,t), \\ W^{(3)}(z,t) &= \sum_{n=0}^{\infty} z^n W^{(3)}(t), \\ V(z,t) &= \sum_{n=0}^{\infty} z^n V_n(t). \end{aligned}\right\} \tag{4.1}$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\overline{f}(s) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t, \ \Re(s) > 0. \tag{4.2}$$

Taking the Laplace transforms of equations (3.2) to (3.14) and using (3.15), we obtain

$$\frac{\partial}{\partial x}\overline{W}_n^{(1)}(x,s) + (s+\lambda+\mu_1(x))\overline{W}_n^{(1)}(x,s) = \lambda\overline{W}_{n-1}^{(1)}(x,s), \quad n = 1,2,\ldots \tag{4.3}$$

$$\frac{\partial}{\partial x}\overline{W}_0^{(1)}(x,s) + (s+\lambda+\mu_1(x))\overline{W}_0^{(1)}(x,s) = 0, \tag{4.4}$$

$$\frac{\partial}{\partial x}\overline{W}_n^{(2)}(x,s) + (s+\lambda+\mu_2(x))\overline{W}_n^{(2)}(x,s) = \lambda\overline{W}_{n-1}^{(2)}(x,s), \quad n = 1,2,\ldots \tag{4.5}$$

$$\frac{\partial}{\partial x}\overline{W}_0^{(2)}(x,s) + (s+\lambda+\mu_2(x))\overline{W}_0^{(2)}(x,s) = 0, \tag{4.6}$$

$$\frac{\partial}{\partial x}\overline{W}_n^{(3)}(x,s) + (s+\lambda+\mu_3(x))\overline{W}_n^{(3)}(x,s) = \lambda\overline{W}_{n-1}^{(3)}(x,s), \quad n = 1,2,\ldots \tag{4.7}$$

$$\frac{\partial}{\partial x}\overline{W}_0^{(3)}(x,s) + (s+\lambda+\mu_3(x))\overline{W}_0^{(3)}(x,s) = 0, \tag{4.8}$$

$$s\overline{V}_0(s) = \theta q\int_0^\infty \overline{W}_0^{(3)}(x,s)\mu_3(x)\mathrm{d}x, \tag{4.9}$$

$$s\overline{V}_n(s) = \theta p\int_0^\infty \overline{W}_{n-1}^{(3)}(x,s)\mu_3(x)\mathrm{d}x + \theta q\int_0^\infty \overline{W}_n^{(3)}(x,s)\mu_3(x)\mathrm{d}x, \quad n = 1,2,\ldots, \tag{4.10}$$

$$s\overline{Q}(s) = -\lambda\overline{Q}(s) + 1 + \overline{V}_0(s)K_0 + (1-\theta)q\int_0^\infty \overline{W}_0^{(3)}(x,s)\mu_3(x)\mathrm{d}x, \tag{4.11}$$

$$\overline{W}_0^{(1)}(0,s) = \overline{Q}(s)\lambda + \overline{V}_0(s)K_1 + \overline{V}_1(s)K_0 + q(1-\theta)\int_0^\infty \overline{W}_1^{(3)}(x,s)\mu_3(x)\mathrm{d}x, + (1-\theta)p\int_0^\infty \overline{W}_0^{(3)}(x,s)\mu_3(x)\mathrm{d}x, \tag{4.12}$$

$$\overline{W}_n^{(1)}(0,s) = \overline{V}_0(s)K_{n+1} + \overline{V}_1(s)K_n + \cdots + \overline{V}_n(s)K_1 + \overline{V}_{n+1}(s)K_0 + +q(1-\theta)\int_0^\infty \overline{W}_{n+1}^{(3)}(x,s)\mu_3(x)\mathrm{d}x + (1-\theta)p\int_0^\infty \overline{W}_n^{(3)}(x,s)\mu_2(x)\mathrm{d}x, \quad n = 1,2,\ldots, \tag{4.13}$$

$$\overline{W}_n^{(2)}(0,s) = \int_0^\infty \overline{W}_n^{(1)}(x,s)\mu_1(x)\mathrm{d}x, \ n = 0,1,\ldots, \tag{4.14}$$

$$\overline{W}_n^{(3)}(0,s) = \int_0^\infty \overline{W}_n^{(2)}(x,s)\mu_2(x)\mathrm{d}x, \ n = 0,1,\ldots. \tag{4.15}$$

Now multiplying equation (4.3) by z^n and summing over n from 1 to ∞, adding to equation (4.4) and using the generating functions defined in (4.1), we get

$$\frac{\partial}{\partial x}\overline{W}^{(1)}(x,z,s) + (s+\lambda-\lambda z+\mu_1(x))\overline{W}^{(1)}(x,z,s) = 0, \tag{4.16}$$

Performing similar operations on equations (4.5) to (4.10) we obtain

$$\frac{\partial}{\partial x}\overline{W}^{(2)}(x,z,s) + (s+\lambda-\lambda z+\mu_2(x))\overline{W}^{(2)}(x,z,s) = 0, \tag{4.17}$$

$$\frac{\partial}{\partial x}\overline{W}^{(3)}(x,z,s) + (s+\lambda-\lambda z+\mu_3(x))\overline{W}^{(3)}(x,z,s) = 0, \tag{4.18}$$

$$s\overline{V}(z,s) = \theta(q+pz)\int_0^\infty \overline{W}^{(3)}(x,z,s)\mu_3(x)\mathrm{d}x. \tag{4.19}$$

For the boundary conditions, we multiply both sides of equation (4.11) by z, multiply both sides of equation (4.12) by z^{n+1}, sum over n from 1 to ∞, add the two results and use equation (4.1) to get

$$\begin{aligned} z\overline{W}^{(1)}(0,z,s) &= \lambda z\overline{Q}(s) + (1-\theta)(q+pz)\int_0^\infty \overline{W}^{(3)}(x,z,s)\mu_3(x)\mathrm{d}x \\ &\quad -(1-\theta)q\int_0^\infty \overline{W}_0^{(3)}(x,s)\mu_3(x)\mathrm{d}x + \overline{V}(z,s)e^{-\lambda d[1-z]} - \overline{V}_0(s)K_0. \end{aligned} \tag{4.20}$$

Performing similar operation on equation (4.13) and (4.14),we have

$$\overline{W}^{(2)}(0,z,s) = \int_0^\infty \overline{W}^{(1)}(x,z,s)\mu_1(x)\mathrm{d}x, \tag{4.21}$$

$$\overline{W}^{(3)}(0,z,s) = \int_0^\infty \overline{W}^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x. \tag{4.22}$$

Using equation (4.11), equation (4.20) become

$$\begin{aligned} z\overline{W}^{(1)}(0,z,s) &= (1-\theta)(q+pz)\int_0^\infty \overline{W}^{(3)}(x,z,s)\mu_3(x)\mathrm{d}x + \overline{V}(z,s)e^{-\lambda d[1-z]} \\ &\quad +[1-s\overline{Q}(s)] + \lambda\overline{Q}(s)[z-1]. \end{aligned} \tag{4.23}$$

Integrating equation (4.16) from 0 to x yields

$$\overline{W}^{1}(x,z,s) = \overline{W}^{(1)}(0,z,s)\, e^{-(s+\lambda-\lambda z)x-\int_0^x \mu_1(t)dt}, \tag{4.24}$$

where $\overline{W}^{(1)}(0, z, s)$ is given by equation (4.23) Again integrating equation (4.24) by parts with respect to x yields

$$\overline{W}^{(1)}(z, s) = \overline{W}^{(1)}(0, z, s)\left[\frac{1 - \overline{b}_1(s + \lambda - \lambda z)}{s + \lambda - \lambda z}\right], \tag{4.25}$$

where

$$\overline{b}_1(s + \lambda - \lambda z) = \int_0^\infty e^{-(s+\lambda-\lambda z)x} db_1(x) \tag{4.26}$$

is the Laplace-Stieltjes transform of the first stage service time $b_1(x)$. Now multiplying both sides of equation (4.24) by $\mu_1(x)$ and integrating over x, we obtain

$$\int_0^\infty \overline{W}^{(1)}(x, z, s)\mu_1(x)\mathrm{d}x = \overline{W}^{(1)}(0, z, s)\overline{b}_1(s + \lambda - \lambda z). \tag{4.27}$$

Similarly, on integrating equations (4.17)and (4.18) from 0 to x, we get

$$\overline{W}^2(x, z, s) = \overline{W}^{(2)}(0, z, s)\, e^{-(s+\lambda-\lambda z)x - \int_0^x \mu_2(t)dt}, \tag{4.28}$$

$$\overline{W}^3(x, z, s) = \overline{W}^{(3)}(0, z, s)\, e^{-(s+\lambda-\lambda z)x - \int_0^x \mu_3(t)dt}, \tag{4.29}$$

where $\overline{W}^{(2)}(0, z, s)$ and $\overline{W}^{(3)}(0, z, s)$ are given by equations (4.21) and (4.22). Again integrating equations (4.28) and (4.29) by parts with respect to x yield

$$\overline{W}^{(2)}(z, s) = \overline{W}^{(2)}(0, z, s)\left[\frac{1 - \overline{b}_2(s + \lambda - \lambda z)}{s + \lambda - \lambda z}\right], \tag{4.30}$$

where

$$\overline{b}_2(s + \lambda - \lambda z) = \int_0^\infty e^{-(s+\lambda-\lambda z)x} db_2(x) \tag{4.31}$$

is the Laplace-Stieltjes transform of the second stage service time $b_2(x)$ and

$$\overline{W}^{(3)}(z, s) = \overline{W}^{(3)}(0, z, s)\left[\frac{1 - \overline{b}_3(s + \lambda - \lambda z)}{s + \lambda - \lambda z}\right], \tag{4.32}$$

where

$$\overline{b}_3(s + \lambda - \lambda z) = \int_0^\infty e^{-(s+\lambda-\lambda z)x} db_3(x) \tag{4.33}$$

is the Laplace-Stieltjes transform of the third stage service time $b_3(x)$. We see that by virtue of equations (4.28) and (4.29)

$$\int_0^\infty \overline{W}^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x = \overline{W}^{(2)}(0,z,s)\overline{b}_2(s+\lambda-\lambda z), \tag{4.34}$$

$$\int_0^\infty \overline{W}^{(3)}(x,z,s)\mu_3(x)\mathrm{d}x = \overline{W}^{(3)}(0,z,s)\overline{b}_3(s+\lambda-\lambda z). \tag{4.35}$$

By using equations (4.27) and (4.34), equations (4.21) and (4.22) reduce to

$$\overline{W}^{2}(0,z,s) = \overline{W}^{(1)}(0,z,s)\overline{b}_1(s+\lambda-\lambda z), \tag{4.36}$$

$$\overline{W}^{3}(0,z,s) = \overline{W}^{(2)}(0,z,s)\overline{b}_2(s+\lambda-\lambda z). \tag{4.37}$$

Using equations (4.36) and (4.37), equations (4.34) and (4.35) become

$$\int_0^\infty \overline{W}^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x = \overline{W}^{(1)}(0,z,s)\overline{b}_1(s+\lambda-\lambda z)\overline{b}_2(s+\lambda-\lambda z), \tag{4.38}$$

$$\int_0^\infty \overline{W}^{(3)}(x,z,s)\mu_3(x)\mathrm{d}x = \overline{W}^{(1)}(0,z,s)\overline{b}_1(s+\lambda-\lambda z)\overline{b}_2(s+\lambda-\lambda z) \overline{b}_3(s+\lambda-\lambda z). \tag{4.39}$$

By using above equation, (4.23) reduces to

$$\overline{W}^{(1)}(0,z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]} + [1-s\overline{Q}(s)] + \lambda\overline{Q}(s)[z-1]}{z-(1-\theta)(q+pz)\overline{b}_1(s+\lambda-\lambda z)\overline{b}_2(s+\lambda-\lambda z)\overline{b}_3(s+\lambda-\lambda z)}. \tag{4.40}$$

Substituting the value of $\overline{W}^{(1)}(0,z,s)$ into equation (4.25), we get

$$\overline{W}^{(1)}(z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]} + [1-s\overline{Q}(s)] + \lambda\overline{Q}(s)[z-1]}{z-(1-\theta)(q+pz)\overline{b}_1(s+\lambda-\lambda z)\overline{b}_2(s+\lambda-\lambda z)\overline{b}_3(s+\lambda-\lambda z)} \left[\frac{1-\overline{b}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right]. \tag{4.41}$$

Now using equations (4.36) and (4.40), equation (4.30) become

$$\overline{W}^{(2)}(z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]} + [1-s\overline{Q}(s)] + \lambda\overline{Q}(s)[z-1]}{z-(1-\theta)(q+pz)\overline{b}_1(s+\lambda-\lambda z)\overline{b}_2(s+\lambda-\lambda z)\overline{b}_3(s+\lambda-\lambda z)} \overline{b}_1(s+\lambda-\lambda z)\left[\frac{1-\overline{b}_2(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right]. \tag{4.42}$$

Making use of equations (4.36), (4.37) and (4.40), equation (4.32) become

$$\overline{W}^{(3)}(z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]} + [1 - s\overline{Q}(s)] + \lambda\overline{Q}(s)[z-1]}{z - (1-\theta)(q+pz)\overline{b}_1(s+\lambda-\lambda z)\overline{b}_2(s+\lambda-\lambda z)\overline{b}_3(s+\lambda-\lambda z)} \overline{b}_1(s+\lambda-\lambda z)\overline{b}_2(s+\lambda-\lambda z)\left[\frac{1-\overline{b}_3(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right]. \quad (4.43)$$

By virtue of equation (4.39), (4.19) can be re-written as

$$s\overline{V}(z,s) = \theta(q+pz)\overline{W}^{(1)}(0,z,s)\overline{b}_1(s+\lambda-\lambda z)\overline{b}_2(s+\lambda-\lambda z) \overline{b}_3(s+\lambda-\lambda z). \quad (4.44)$$

Thus $\overline{V}(z,s)$, $\overline{W}^{(1)}(z,s)$, $\overline{W}^{(2)}(z,s)$ and $\overline{W}^{(3)}(z,s)$ can be determined from equations (4.44), (4.41), (4.42) and (4.43) respectively.

5. Steady State Solution

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we supress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t). \quad (5.1)$$

Now multiplying both sides of equation (4.40), (4.41), (4.42), (4.43) and (4.45) by s, taking limit as $s \to 0$, applying property (5.1), we get

$$\begin{aligned}
\lim_{s\to 0} s\overline{W}^{(1)}(0,z,s) &= W^{(1)}(0,z) \\
&= \frac{s\overline{V}(z,s)e^{-\lambda d[1-z]} + s[1 - s\overline{Q}(s)] + \lambda s\overline{Q}(s)[z-1]}{z - (q+pz)\overline{b}_1(s+\lambda-\lambda z)\overline{b}_2(s+\lambda-\lambda z)\overline{b}_3(s+\lambda-\lambda z)} \\
&= \frac{V(z)e^{-\lambda d[1-z]} + \lambda Q[z-1]}{z - (1-\theta)(q+pz)\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z)}, \quad (5.2)
\end{aligned}$$

$$W^{(1)}(z) = \frac{V(z)e^{-\lambda d[1-z]} + \lambda Q[z-1]}{z - (1-\theta)(q+pz)\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z)} \left[\frac{1-\overline{b}_1(\lambda-\lambda z)}{\lambda-\lambda z}\right], \quad (5.3)$$

$$W^{(2)}(z) = \frac{V(z)e^{-\lambda d[1-z]} + \lambda Q[z-1]}{z-(1-\theta)(q+pz)\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z)} \overline{b}_1(\lambda-\lambda z)\left[\frac{1-\overline{b}_2(\lambda-\lambda z)}{\lambda-\lambda z}\right], \quad (5.4)$$

$$W^{(3)}(z) = \frac{V(z)e^{-\lambda d[1-z]} + \lambda Q[z-1]}{z-(1-\theta)(q+pz)\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z)} \overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\left[\frac{1-\overline{b}_3(\lambda-\lambda z)}{\lambda-\lambda z}\right], \quad (5.5)$$

$$V(z) = \theta(q+pz)W^{(1)}(0,z)\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z) \quad (5.6)$$

which on using (5.2) becomes

$$V(z) = \theta(q+pz)\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z) \left[\frac{V(z)e^{-\lambda d[1-z]} + \lambda Q[z-1]}{z-(1-\theta)(q+pz)\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z)}\right]. \quad (5.7)$$

Equation (5.7) can be further simplified to

$$V(z) = \frac{\lambda\theta\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z)(q+pz)Q[z-1]}{DR}, \quad (5.8)$$

where

$$DR = z-(1-\theta)(q+pz)\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z) + \theta(q+pz)\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z)[1-e^{-\lambda d[1-z]}]. \quad (5.9)$$

Then substituting for $V(z)$ from equation (5.8) into equations (5.3), (5.4) and (5.5)

$$W^{(1)}(z) = \frac{\left[\overline{b}_1(\lambda-\lambda z)-1\right]Q}{DR}, \quad (5.10)$$

$$W^{(2)}(z) = \frac{\overline{b}_1(\lambda-\lambda z)\left[\overline{b}_2(\lambda-\lambda z)-1\right]Q}{DR}, \quad (5.11)$$

$$W^{(3)}(z) = \frac{\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\left[\overline{b}_3(\lambda-\lambda z)-1\right]Q}{DR}. \quad (5.12)$$

where *DR* is given by equation (5.9).

Let $W(z) = W^{(1)}(z) + W^{(2)}(z) + W^{(3)}(z)$. Now from equations (5.10), (5.11) and (5.12),

$$W(z) = \left[\frac{\overline{b}_1(\lambda-\lambda z)\overline{b}_2(\lambda-\lambda z)\overline{b}_3(\lambda-\lambda z)-1}{DR}\right]Q \quad (5.13)$$

We see that at $z = 1$, the right hand side of equations (5.8), (5.10), (5.11) and (5.12) are of the form $\frac{0}{0}$. Therefore, applying L'Hopital's rule, we obtain

$$W^{(1)}(1) = \frac{\lambda Q \mu_2 \mu_3}{\mu_1\mu_2\mu_3 q - \lambda\mu_2\mu_3 - \lambda\mu_1\mu_3 - \lambda\mu_1\mu 2 - \theta\lambda d\mu_1\mu_2\mu_3}, \quad (5.14)$$

$$W^{(2)}(1) = \frac{\lambda Q \mu_1 \mu_3}{\mu_1\mu_2\mu_3 q - \lambda\mu_2\mu_3 - \lambda\mu_1\mu_3 - \lambda\mu_1\mu 2 - \theta\lambda d\mu_1\mu_2\mu_3}, \quad (5.15)$$

$$W^{(3)}(1) = \frac{\lambda Q \mu_1 \mu_2}{\mu_1\mu_2\mu_3 q - \lambda\mu_2\mu_3 - \lambda\mu_1\mu_3 - \lambda\mu_1\mu 2 - \theta\lambda d\mu_1\mu_2\mu_3}, \quad (5.16)$$

$$V(1) = \frac{\lambda\theta Q \mu_1\mu_2\mu_3}{\mu_1\mu_2\mu_3 q - \lambda\mu_2\mu_3 - \lambda\mu_1\mu_3 - \lambda\mu_1\mu_2 - \theta\lambda d\mu_1\mu_2\mu_3}. \quad (5.17)$$

Now to determine the only unknown constant Q, we use (5.13) to (5.16) in the normalizing condition

$$W^{(1)}(1) + W^{(2)}(1) + +W^{(3)}(1) + V(1) + Q = 1. \quad (5.18)$$

and have

$$\begin{aligned} Q &= \frac{\mu_1\mu_2\mu_3[q - \theta\lambda d] - \lambda[\mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3]}{\mu_1\mu_2\mu_3[q - \theta\lambda d] + \theta\lambda\mu_1\mu_2\mu_3}, \\ &= 1 - \lambda\left[\frac{\theta\mu_1\mu_2\mu_3 + \mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3}{\mu_1\mu_2\mu_3[q - \theta\lambda d] + \theta\lambda\mu_1\mu_2\mu_3}\right] \end{aligned} \quad (5.19)$$

where $\lambda < \mu_1\mu_2\mu_3[q - \theta\lambda d]$.

Equation (5.19) gives the steady state probability that there is no customer in the system and the server is idle. It is easy to verify that when there are no server vacations then with $\theta = 0$, equation (5.19) reduces to $Q = 1 - \lambda\left[\frac{1}{\mu_1 q} + \frac{1}{\mu_2 q} + \frac{1}{\mu_3 q}\right]$. Also from equation (5.19), we obtain ρ, the utilisation factor of the system as

$$\rho = 1 - Q = \lambda\left[\frac{\theta\mu_1\mu_2\mu_3 + \mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3}{\mu_1\mu_2\mu_3[q - \theta\lambda d] + \theta\lambda\mu_1\mu_2\mu_3}\right] < 1 \quad (5.20)$$

Substituting for Q found in equation (5.19), we finally have from equations (5.8) and (5.13)

$$W(z) = \frac{\left[\overline{b}_1(\lambda - \lambda z)\overline{b}_2(\lambda - \lambda z)\overline{b}_3(\lambda - \lambda z) - 1\right]\left\{1 - \lambda\left[\frac{\theta\mu_1\mu_2\mu_3 + \mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3}{\mu_1\mu_2\mu_3[q - \theta\lambda d] + \theta\lambda\mu_1\mu_2\mu_3}\right]\right\}}{DR}, \quad (5.21)$$

$$V(z) = \frac{\lambda\theta\overline{b}_1(\lambda - \lambda z)\overline{b}_2(\lambda - \lambda z)\overline{b}_3(\lambda - \lambda z)[z - 1](q + pz)Q}{DR}. \quad (5.22)$$

Let $P_q(z) = V(z)+W(z)$ denote the probability generating function of the queue length irrespective of whether the server is away on vacation or is available in the system. Then adding equations (5.21) and (5.22) and simplifying, we have

$$P_q(z) = \frac{\left\{\overline{b}_1(\lambda - \lambda z)\overline{b}_2(\lambda - \lambda z)\overline{b}_3(\lambda - \lambda z)\left[(\lambda\theta z - \lambda\theta)(q + pz) + 1\right] - 1\right\} Q}{DR}. \tag{5.23}$$

where Q and DR are given by equations (5.19) and (5.9) respectively. We note that in the case when there are no server vacations, we let $\theta = 0$ in equation (5.23) and have

$$P_q(z) = \left[\frac{\overline{b}_1(\lambda - \lambda z)\overline{b}_2(\lambda - \lambda z)\overline{b}_3(\lambda - \lambda z) - 1}{z - (q + pz)\overline{b}_1(\lambda - \lambda z)\overline{b}_2(\lambda - \lambda z)\overline{b}_3(\lambda - \lambda z)}\right]\left[1 - \lambda\left(\frac{1}{\mu_1 q} + \frac{1}{\mu_2 q} + \frac{1}{\mu_3 q}\right)\right]. \tag{5.24}$$

Further let $P(z)$ denote the probability generating function of the number in the system. Then from equations (5.19) and (5.24)

$$\begin{aligned} P(z) &= Q + zP_q(z), \\ &= \frac{N^{'}(z)}{D^{'}(z)}, \end{aligned}$$

where

$$\begin{aligned} N^{'}(z) &= \overline{b}_1(\lambda - \lambda z)\overline{b}_2(\lambda - \lambda z)\overline{b}_3(\lambda - \lambda z)\left\{[z(\lambda\theta z - \lambda\theta)(q + pz) + 1]\right. \\ &\quad \left. + \theta[1 - e^{-\lambda d[1-z]}] - (q + pz)\right\} Q, \\ D^{'}(z) &= z - (1-\theta)(q + pz)\overline{b}_1(\lambda - \lambda z)\overline{b}_2(\lambda - \lambda z)\overline{b}_3(\lambda - \lambda z) \\ &\quad + \theta(q + pz)\overline{b}_1(\lambda - \lambda z)\overline{b}_2(\lambda - \lambda z)\overline{b}_3(\lambda - \lambda z)[1 - e^{-\lambda d[1-z]}]. \end{aligned}$$

In the particular case when there are no server vacations, we let $\theta = 0$ in equation and get

$$P(z) = \frac{\overline{b}_1(\lambda - \lambda z)\overline{b}_2(\lambda - \lambda z)\overline{b}_3(\lambda - \lambda z)[z - (q + pz)]}{z - (q + pz)\overline{b}_1(\lambda - \lambda z)\overline{b}_2(\lambda - \lambda z)\overline{b}_3(\lambda - \lambda z)}. \tag{5.25}$$

6. The Expected Number in the Queue and in the System

Let L_q denote the expected number of customers in the queue. Then we have from equation (), $L_q = \frac{d}{dz}P_q(z)$ at $z = 1$. Since $P_q(z)$ is indeterminate of the $\frac{0}{0}$ form at $z = 1$, we let $P_q(z) = \frac{N(z)}{D(z)}$, where $N(z)$ and $D(z)$ are respectively the numerator and denominator of the right side of equation. Then use the following well-known result in queueing theory [see Madan (1991)].

$$L_q = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}P_q(z) = P_q^{'}(1) = \lim_{z\to 1}\frac{D^{'}(z)N^{''}(z) - N^{'}(z)D^{''}(z)}{2(D^{'}(z))^2},$$

$$= \lim_{z\to 1}\frac{D^{'}(1)N^{''}(1) - N^{'}(1)D^{''}(1)}{2(D^{'}(1))^2}, \tag{6.1}$$

where primes stand for the derivatives with respect to z.We carry out the desired derivatives of the right hand side of equation at $z = 1$,using the fact that $\overline{b}_i(0) = 1$, $-\overline{b}_i^{'}(0) = E(v_i) = \frac{1}{\mu_i}$ and $\overline{b}_i^{''}(0) = E(v_i^2) = \frac{2}{\mu_i^2}$, $i = 1, 2, 3$, the second moment of the service time for the ith type of service. After a lot of algebraic simplifications, we obtain

$$N^{'}(1) = \lambda\left[E(v_1) + E(v_2) + E(v_3) + \theta\right]Q, \tag{6.2}$$

$$N^{''}(1) = \Big\{\lambda^2\left[E(v_1^2) + E(v_2^2) + E(v_3^2)\right] + 2\lambda^2\left[E(v_1)E(v_2) + E(v_2)E(v_3) + E(v_1)E(v_3)\right] + 2\lambda^2\theta\left[E(v_1) + E(v_2) + E(v_3)\right] + 2\lambda\theta p\Big\}\,Q, \tag{6.3}$$

$$D^{'}(1) = q - \lambda E(v_1) - \lambda E(v_2) - \lambda E(v_3) - \theta\lambda d, \tag{6.4}$$

$$D^{''}(1) = -\lambda^2\left[E(v_1^2) + E(v_2^2) + E(v_3^2)\right] - 2\lambda^2\theta d\left[E(v_1) + E(v_2) + E(v_3)\right] - \lambda^2\theta d^2 - 2p\lambda\left[E(v_1) + E(v_2) + E(v_3)\right]. \tag{6.5}$$

Substituting the above values in equation (6.1) and simplifying we finally get L_q where Q is given by (5.19). We note that the expected number in the system is given by $L = L_q + Q$, where ρ has already been found in equation (5.20).

7. The Expected Waiting Time in the Queue and in the System

The expected waiting time in the queue and in the system are given by

$$W_q = \frac{L_q}{\lambda}, \tag{7.1}$$

$$W = \frac{L}{\lambda}. \tag{7.2}$$

Acknowledgement

The author thanks the management of SSN College of Engineering for providing the necessary requirements during the preparation of this paper.

References

[1] Ayyappan, G. and Sathiya K., *An* $M^{[x]}/G/1$ *feedback queue with three stage heterogeneous service and server vacations having restricted admissibility*, Journal of computations and modelling, 3, pp. 203–225, 2013.

[2] Badamchi Zadeh A. and Shankar G.H., *A two phase queue system with Bernouuli feedback and Bernoulli schedule server vacation*, Int. J. Inf. Mag. Sci., 19, pp. 329–338, 2008.

[3] Bertsimas, D. and Papaconstantinou, X.,*On the steady state solution of the* $M/C_2(a,b)/s$ *queueing system*, Transport. Sci., 22, pp. 125–138, 1988.

[4] Choi.D and Kim, T.S.,*Analysis of two phase queueing system with vacations and Bernoulli feedback*, Stoc. Anal. Appl., 21, 1009–1019.

[5] Choudhury, G., *An M/G/1 queueing system with two phase service under D policy*, Int. J. Inf. Mag. Sci., 16, pp. 1–17, 2005.

[6] Choudhury, G. and Paul, M., *A two phase queueing system with Bernoulli schedule vacation under multiple vacation policy*, Statistical Methodology, 3, pp. 174–185, 2005.

[7] Choudhury, G. and Paul, M., *On a single server queue with two stage heterogeneous service and deterministic server vacations*, Int. J. Syt. Sci., 32, pp. 837–844, 2006.

[8] Choudhury, G. and Madan K.C., *A two phase batch arrival queueing system witha a vacation time under Bernoulli schedule*, Applied Mathematics and Computation, 149, 337–349.

[9] Cox, D.R., it The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables, Proc. Camb. Phil. Soc., 51, pp. 433–441, 1955.

[10] Cramer, M., *Stationary distributions in a queueing system with vacation times and limited service*, Queueing Systems, 4, pp. 57–68, 1989.

[11] Doshi, B.T., *A note on stochastic decomposition in a GI/G/1 queue with vacation or setup times*, J. Appl. Prob., 22, pp. 419–428, 1985.

[12] Doshi, B.T., *Queueing systems with vacations - A survey*, Queueing Systems, 1, pp. 29–66, 1986.

[13] Fuhrman, S., *A note on the M/G/1 queue with server vacations*, Oper. Res., 31, pp. 13–68, 1981.

[14] Gaver, D.P., *Imbedded Markov chain analysis of a waiting line process in continuous time*, Ann. Math. Stat., 30, pp. 698–720, 1959.

[15] Keilson, J. and Servi, N.D., *Oscillating random walk models for GI/G/1 vacation systems with Bernoulli schedules*, J. Appl. Prob., 23, pp. 790–802, 1986.

[16] Levy, Y. and Yechiali, U., *An M/M/s queue with server vacations*, Infor., 14, pp. 153–163, 1976.

[17] Li J., and Wang J., *An M/G/1 retrial queue with second multi optional service, feedback and reliable server*, Appl. Math. J. Chinese Univ. Ser. B., 21, pp. 252–262.

[18] Madan, K.C., *An M/G/1 queue with optional deterministic server vacations*, Metron LVII, pp. 83–95, 1999.

[19] Madan, K.C., *On a single server queue with two stage general heterogeneous service and Binomial schedule server vacations*, Egyptian Statistical Journal, 44, pp. 837–844, 2000.

[20] Madan, K.C., *On a single server queue with two stage heterogeneous service and deterministic server vacations*, Int. J. Syt. Sci., 32, pp. 837–844, 2001.

[21] Madan, K.C. and Saleh, M.F., *On M/D/1 queue with general server vacations*, Int. J. Inf. Mag. Sci., 12, pp. 25–37, 2001.

[22] Madan, K.C. and Choudhury, G., *An* $M^X/G/1$ *queue with Bernoulli vacation schedule under Restricted Admissibility policy*, Sankhya, 66, pp. 175–193, 2004.

[23] Madan, K.C. and Choudhury, G., *A single server queue with two phases of heterogeneous service under Bernoulli schedule and a general vacation time*, Inf. Mag. Sci., 16, pp. 1–16, 2005.

[24] Madan, K.C. and Choudhury, G., *Steady state analysis of an* $M^x/(G_1, G_2)/1$ *queue with restricted admissibility and random setup time*, Int. J. Inf. Mag. Sci., 17, pp. 33–56, 2006.

[25] Maragatha Sundari S., and Srinivasan S., *Analysis of M/G/1 feedback queue with three stage and multiple server vaction*, Appl. Math. Sci., 6, pp. 6221–6240, 2012.

[26] Maraghi, F.A., Madan, K.C. and Dowman, K.D., *Batch arrival queueing system with random breakdowns and Bernoulli schedule server vacations having general time distribution*, Int. J. Inf. Mag. Sci., 20, pp. 55–70, 2009.

[27] Shantikumar, J.G., *On stochastic decomposition in the M/G/1 type queues with generalised vacations*, Oper. Res., 36, pp. 566–569, 1988.

[28] Takagi, H., *Queueing Analysis: Vacation and Priority Systems*, Vol 1, North Holland, Amsterdam, 1991.

[29] Takine,T., and Sengupta, B., *A single server queue with service interruptions*, Queueing Systems, 26, pp. 285–300, 1997.

[30] Tegham, J., *Control of the service process in a queueing system*, Eur. Jl. Oper. Res., 23, pp. 141–158, 1986.

[31] Thangaraj, V. and Vanitha, S., *A two phase M/G/1 feedback queue with multiple server vacations*, Int. J. Stoc. Anal. Appl., 27(6), pp. 1231–1245, 2009.

[32] Thangaraj, V. and Vanitha, S., *On the analysis of* $M/G/1$ *feedback queue with two stage heterogeneous service and with multiple server vacation*, International Mathematical Forum, 22(5), pp. 829–847, 2009.

[33] Zadeh, A.B. and Shankar, G.H., *A two phase system with Bernoulli feedback and Bernoulli schedule server vacation*, Int. J. Inf. Mag. Sci., 19, pp. 329–338, 2008.