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Selection Rejection Methodology For Two Dimensional Continuous Random Variables and Its Application To Two Dimensional Normal Distribution

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Abstract

In this paper we have generalized the Selection-Rejection Methodology for one dimensional continuous random variables to two dimensional continuous random variables and applied it to the two dimensional normal distribution.

Keywords: random variable, iterations, target probability distribution and proposal probability distribution.

Introduction

The Selection-Rejection Methodology for one dimensional continuous random variables was developed based on the idea of Acceptance-Rejection Method by the renowned mathematician, Von Neumann, from the University of Berlin. Von Neumann[1] came forward with his method during 1950's but later on Karl Sigman[2] from Columbia University gave the similar methodology in 2007.Again in 1989,Bernard D.Flury[3] from Indiana University came forward with the theory "Acceptance-Rejection Sampling Made Easy".D.P.Kroese[4] from University of Queensland put forward his theory of Acceptance-Rejection in 2011.Selection-Rejection Methodology can be applied to almost all statistical distributions and hence it has got immense physical significance.

Selection-Rejection Methodology For Two Dimensional Continuous Random Variables

Let X,Y be a two dimensional continuous random variable with probability distribution function $f\ x,y\ \ \forall x,y\in R$,where R=set of all real numbers. Let $g\ x,y$ $\forall x,y\in R$ where R=set of all real numbers be another probability density function such that $\frac{f\ x,y}{g\ x,y}\le k\ \ \forall\ x,y\in R$,where $k\ge 1$ is a real number. By successively selecting different values of (X, Y) we will try to make the ratio $\frac{f(x,y)}{kg(x,y)}$ as close to 1 as possible. The probability density function $f(x,y)$ is called target distribution and he probability density function $g(x,y)$ is called proposal distribution.

The step by step procedure for the Selection-Rejection Methodology is as follows.

Step (1):- Let (X, Y) be a two dimensional continuous random variable with probability distribution function $f(x,y)$ $\forall x, y \in R$, where R =set of all real numbers.

Step (2):- Let (X', Y') be another two dimensional continuous random variable (which is independent of (X, Y)) with probability distribution function $g(x,y)$ $\forall x, y \in R$, where R =set of all real numbers.

Step (3):- Let $\frac{f(X',Y')}{g(X',Y')} \le k$ $\forall (X',Y') \in R$, where $k \ge 1$ a real number.

Step (4):- Let $0 < R_1 < 1$ and $0 < R_2 < 1$ be two random numbers.

Step (5):- Set X' in terms of R_1 and set Y' in terms of R_2 depending on the expression obtained for the ratio $\frac{f(X',Y')}{kg(X',Y')}$.

Step (6):- If $R_1 R_2 \le \frac{f(X',Y')}{kg(X',Y')}$, then set $(X, Y) = (X', Y')$ and select the continuous random variable (X', Y'); otherwise reject the variable (X', Y') and repeat the process from step (1).

The probability that the continuous random variable (X', Y') is selected is $\frac{1}{k}$.

The number of iterations required to select (X', Y') is k.

It may be noted that $0 \le \frac{f(X',Y')}{kg(X',Y')} \le 1$

To prove that the probability for the selection of (X, Y) is $\frac{1}{k}$,

Proof: - $P(Select|(X',Y')) = P\left(R_1 R_2 \le \frac{f(X',Y')}{kg(X',Y')}\right) = \frac{f(X',Y')}{kg(X',Y')}$..

$$P((X',Y')\ is\ selected) = \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{f(w,v)}{kg(w,v)} g(w,v)\, dwdv$$

$$= \frac{1}{k}\int_{-\infty}^{x}\int_{-\infty}^{y} f(w,v)\, dwdv = \frac{1}{k} \left[\because \int_{-\infty}^{x}\int_{-\infty}^{y} f(w,v)\, dwdv = 1\right]$$

Hence the proof.

Since the probability of selection (i.e. success) is $\frac{1}{k}$,the number of iterations needed will follow a geometric distribution with $p=\frac{1}{k}$.So, on average it will take k iterations to generate a number.

Application To Two Dimensional Normal Distribution.

Two dimensional normal distribution is given by

$$f\left(x,y\right)=\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{x^2}{2}+\frac{x^2}{2}\right)},x\geq 0,y\geq 0,x,y\in R \tag{1}$$

Here $f\left(x,y\right)$ is the target function.

Let $g(x,y)=e^{-\left(x+y\right)},x\geq 0,y\geq 0$ be the proposal distribution. (2)

Let $h(x,y)=\frac{f(x,y)}{g(x,y)}=\frac{1}{\sqrt{2\pi}}\exp\left(-\left(\frac{x^2+y^2-2x-2y}{2}\right)\right)$ (3)

With the help of differential calculus we can show that $h(x,y)$ attains maximum at $\left(1,1\right)$ and the maximum value of $h(x,y)$ is $\frac{e}{\sqrt{2\pi}}\approx 1.0845(approximately)$.

Choosing $k=\frac{e}{\sqrt{2\pi}}$, we get

$$\frac{f\left(x,y\right)}{kg\left(x,y\right)}=\exp\left(-\frac{\left(x-1\right)^2}{2}\right)\times\exp\left(-\frac{\left(y-1\right)^2}{2}\right) \tag{4}$$

Selection-Rejection Methodology for the two dimensional distribution is as follows

Step (1):- Let $\left(X,Y\right)$ be a two dimensional continuous random variable with probability distribution function $f\left(x,y\right)\ \forall x,y\in R$, where R=set of all real numbers.

Step (2):- Let $\left(X',Y'\right)$ be a two dimensional continuous random variable with probability distribution function $g\left(x,y\right)\ \forall x,y\in R$, where R=set of all real numbers.

Step (3):-Let $0<R_1<1$ and $0<R_2<1$ be two random numbers.

Step (4):- Set $X' = 1 + \sqrt{-2\ln(R_1)}$ and $Y' = 1 + \sqrt{-2\ln(R_2)}$

Step (5):-If $R_1 R_2 \le \exp\left(-\frac{X'-1}{2}^2\right) \times \exp\left(-\frac{Y'-1}{2}^2\right)$,then set $X, Y = X', Y'$ and select X', Y' ;otherwise reject X', Y' and repeat the process from Step(1).

Conclusion

Selection-Rejection Methodology is valid for any dimension of continuous random variable. In this method we approximate the target function to proposal function so that after a number of successive iterations the proposal function becomes almost equal to target function and proposal function is selected.

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