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Linear Fractional Differential Equations and Sumudu Transform

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Abstract:

The aims of this paper are applying the Sumudu transform to find exact solution of linear fractional differential equations. Sumudu transform is a powerful tool in differential equations. It will allow us to transform fractional differential equations into algebraic equations and then by solving these algebraic equations, and used to get exact solutions of fractional differential equations.

Therefore, we can obtain the exact solution by using the Inverse Sumudu transform. Some examples are included to demonstrate the validity and applicability of the presented technique.

Keywords: Fractional differential equations; Sumudu Transform; Inverse Sumudu Transform.

1. Introduction

Although fractional derivatives have a long mathematical history, for many years ago they were not used in physics and mathematics. One possible explanation of such unpopularity could be that there are multiple nonequivalent definitions of fractional derivatives [1]. Another difficulty is that fractional derivatives have no evidence geometrical interpretation because of their nonlocal character [2]. However, during the last ten years fractional calculus starts to attract much more attention of physicists and mathematicians.

Linear partial differential equations (PDE) with integer or fractional order have played a very important role in various fields of science and engineering.

It is important to obtain exact or approximate solutions of linear fractional partial differential equations (FPDE). But in general, there exists no method that gives an exact solution for FPDEs and most of the obtained solutions are only approximations. Searching of exact solutions of (FPDEs) in mathematical and other scientific applications is still quite challenging and needs new methods. Computing the exact solution of these equations is of considerable importance, because the exact solutions can help to understand the mechanism and complexity of phenomena that have been modeled by PDEs with integer or fractional order.

Moreover, solutions to the fractional differential equations have been investigated by many authors [1, 8]. During the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, and fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian's decomposition method [9–13], He's variational iteration method [14–16], homotopy perturbation method [17–19], homotopy analysis method [20].

The aim of this paper is to find the exact solution of linear fractional differential equations by using Sumudu transform.

2. Preliminaries and notations

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this article.

Definition 1:

A real function $f(x), x > 0$ is said to be in space C_μ, $\mu \in \Re$ if there exists a real number $p > \mu$, such that $f(x) = t^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $f^n \in C_\mu, n \in N$.

Definition 2:

The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u)\,du \quad , \quad \alpha > 0 \tag{1}$$

$$J^0 f(t) = f(t).$$

Some properties of the operator J^α, which are needed here, are as follows:

For $f^n \in C_\mu, n \in N$, $\alpha, \beta \geq 0$ and $\gamma \geq -1$:

(1) $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$

(2) $J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}$ (2)

Definition 3:

The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t) \tag{3}$$

For $m-1<\alpha\leq m, m\in N, t>0$ and $f\in C_{-1}^{m}$

Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative.

Similar to the integer-order integration, the Riemann-Liouville fractional integral operator is a linear operation:

$$J^{\alpha}\left(\sum_{i=1}^{n} c_i\, f_i(x)\right)=\sum_{i=1}^{n} c_i\, J^{\alpha} f_i(x) \tag{4}$$

Where $\{c_i\}_{i=1}^{n}$ are constants?

In the present work, the fractional derivatives are considered in the Caputo sense. The reason for adopting the Caputo definition, as pointed by [10], is as follows: to therefore familiar to us. In contrast, for the Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives solve differential equations (both classical and fractional), we need to specify additional conditions in order to produce a unique solution. For the case of the Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical differential equations, and are (and/or integrals) of the unknown solution at the initial point $x=0$, which are functions of *x*. These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details see [2].

3. Sumudu operations:

The Sumudu transform is a powerful tool in applied mathematics and engineering. Virtually every beginning course in differential equations at the undergraduate level introduces this technique for solving linear differential equations. The Sumudu transform is indispensable in certain areas of control theory.

3.1 Sumudu transforms:

Given a function $f(x)$ defined for $0<x<\infty$, the Sumudu transform $F(u)$ is defined as

$$F(u)=\int_{0}^{\infty} f(ux)\, e^{-x}\, dx \tag{5}$$

at least for those *s* for which the integral converges.

Let $f(x)$ be a continuous function on the interval $[0,\infty)$ which is of exponential order, that is, for some $c\in\Re$ and $x>0$

$$\sup\frac{|f(x)|}{e^{cx}}<\infty .$$

In this case the Sumudu transform (5) exists for all $\frac{1}{u} > c$ [27].

Some of the useful Sumudu transforms which are applied in this paper, are as follows:

For $S[f(x)] = F(u)$ and $S[g(x)] = G(u)$

$$S[f(x) + g(x)] = F(u) + G(u),$$

$$S[x^{\beta}] = u^{\beta}\,\Gamma(\beta+1) \quad , \quad \beta > -1,$$

$$S[f^{(n)}(x)] = \frac{F(u)}{u^n} - \frac{f(0)}{u^n} - \frac{f'(0)}{u^{n-1}} - \cdots - \frac{f^{(n-1)}(0)}{u} \tag{6}$$

$$S\left[\int_0^x f(t)\,dt\right] = u\,F(u)$$

$$S\left[\int_0^x f(x-t)\,g(t)\,dt\right] = u\,F(u)\,G(u). \tag{7}$$

Lemma 1:

The Sumudu transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ can be obtained in the form of:

$$S[I^{\alpha} f(x)] = u^{\alpha} F(u).$$

Proof:

The Sumudu transform of Riemann-Liouville fractional integral operator of order $\alpha > 0$ is:

$$S[I^{\alpha} f(x)] = S\left[\frac{1}{\Gamma(\alpha)}\int_0^x (x-\alpha)^{\alpha-1} f(t)\,dt\right] = \frac{1}{\Gamma(\alpha)} u\,F(u)\,G(u),$$

Where

$$G(u) = S[x^{\alpha-1}] = u^{\alpha-1}\Gamma(\alpha)$$

Lemma 2:

The Sumudu transform of Caputo fractional derivative for $m-1 < \alpha \le m$, $m \in N$, can be obtained in the form of:

$$S[D^{\alpha} f(x)] = u^{m-\alpha}\left[\frac{F(u)}{u^m} - \frac{f(0)}{u^m} - \frac{f'(0)}{u^{m-1}} - \cdots - \frac{f^{(m-1)}(0)}{u}\right]$$

Proof:

The Sumudu transform of Caputo fractional derivative of order $\alpha > 0$ is :

$$S\left[D^{\alpha} f(x)\right] = S\left[J^{m-\alpha} f^{(m)}(x)\right] = u^{m-\alpha} S\left[f^{(m)}(x)\right].$$

Using equation 7. Now, we can transform fractional differential equations into algebraic equations and then by solving this algebraic equation, we can obtain the unknown Sumudu function $F(u)$.

3.2. Inverse Sumudu transform:

The function $f(x)$ in (5) is called the inverse Sumudu transform of $F(u)$ and will be denoted by $f(x) = S^{-1}\left[F(u)\right]$ in the Paper. In practice when one uses the Sumudu transform to, for example, solve a differential equation, one has to at some

Point invert the Sumudu transform by finding the function $f(x)$ which corresponds to some specified $F(u)$.

The Inverse Sumudu Transform of $F(u)$ is defined as:

$$f(x) = S^{-1}\left[F(u)\right] = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} u\, F(u)\, e^{\frac{x}{u}}\, du\ ,$$

Where σ large enough that is $F(u)$ is defined for the real part of $\frac{1}{u} \geq \sigma$ surprisingly, this formula isn't really useful. Therefore, in this section some useful function $f(x)$ is obtained from their Sumudu transform. In the first we define the most important special functions used in fractional calculus the Mittag-Leffler functions and the generalized Mittag-Leffler functions

For $\alpha\ ,\ \beta > 0$ and $z \in C$

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(n\alpha + 1)},$$

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(n\alpha + \beta)}.$$

Now, we prove some Lemmas which are useful for finding the function $f(x)$ from its Sumudu transform.

Lemma 3:

For $\alpha\ ,\ \beta > 0,\ a \in C$ and $\dfrac{1}{u^{\alpha}} > |a|$ we have the following inverse Sumudu transform formula

$$S^{-1}\left[\frac{u^{\beta-1}}{1+a u^{\alpha}}\right] = x^{\beta-1} E_{\alpha,\beta}\left(-a x^{\alpha}\right).$$

Proof:

$\dfrac{u^{\beta}}{1+a u^{\alpha}}$ by using the series expansion can be rewritten as

$$\frac{u^{\beta}}{1+a u^{\alpha}} = u^{\beta}\ \frac{1}{1+a u^{\alpha}} = u^{\beta}\sum_{n=0}^{\infty}\left(-a u^{\alpha}\right) = \sum_{n=0}^{\infty}\left(-a\right)^{n} u^{n\alpha+\beta-1}\ .$$

The inverse Sumudu transform of above function is

$$\sum_{n=0}^{\infty}\frac{\left(-a\right)^{n} u^{n\alpha+\beta-1}}{\Gamma\left(n\alpha+\beta\right)} = x^{\beta-1}\sum_{n=0}^{\infty}\frac{\left(-a x^{\alpha}\right)^{n}}{\Gamma\left(n\alpha+\beta\right)} = x^{\beta-1}E_{\alpha,\beta}\left(-a\ x^{\alpha}\right).$$

Lemma 4:

For $\alpha \geq \beta > 0\ ,\ a \in \Re$ and $\alpha\ ,\ \beta > 0$ we have

$$S^{-1}\left[\frac{u^{\alpha(n+1)-1}}{(1+a u^{\alpha-\beta})^{n+1}}\right] = x^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{\left(-a\right)^{k}\binom{n+k}{k}}{\Gamma\left((\alpha-\beta)k+(n+1)\alpha\right)}x^{k(\alpha-\beta)}\ .$$

Proof:

Using the series expansion of $\left(1+x\right)^{-n-1}$ of the from

$$\frac{1}{\left(1+x\right)^{n+1}} = \sum_{k=0}^{\infty}\binom{n+k}{k}\left(-x\right)^{k}$$

We have:

$$\frac{u^{\alpha(n+1)-1}}{(1+a u^{\alpha-\beta})^{n+1}} = u^{\alpha(n+1)-1}\frac{1}{(1+a u^{\alpha-\beta})^{n+1}} = u^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\binom{n+k}{k}\left(-a\ u^{\alpha-\beta}\right)^{k}$$

Giving the inverse Sumudu transform of above function can prove the Lemma.

Lemma 5:

For $\alpha \geq \beta\ ,\ \alpha > \gamma\ ,\ a \in \Re$ and $\left|\dfrac{u^{\alpha-1}}{1+a u^{\alpha-\beta}}\right| > |b|$ we have

$$S^{-1}\left[\frac{u^{\alpha+\beta-\gamma-1}}{u^{\beta}+a u^{\alpha}+b u^{\alpha+\beta}}\right]= x^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^{n}(-a)^{k}\binom{n+k}{k}}{\Gamma\left(k(\alpha-\beta)+(n+1)\alpha-\gamma\right)}x^{k(\alpha-\beta)+n\alpha} .$$

Proof:

$u^{\alpha+\beta-\gamma-1}/u^{\beta}+a u^{\alpha}+b u^{\alpha+\beta}$ by using the series expansion can be rewritten as

$$\frac{u^{\alpha+\beta-\gamma-1}}{u^{\beta}+au^{\alpha}+bu^{\alpha+\beta}}=\frac{u^{\alpha+\beta-\gamma-1}}{u^{\beta}+au^{\alpha}}\ \frac{1}{1+\dfrac{bu^{\alpha+\beta}}{u^{\beta}+au^{\alpha}}}=\sum_{n=0}^{\infty}\frac{(-b)^{n}\,u^{\alpha+\beta-\gamma-1}}{u^{\beta}+au^{\alpha}+bu^{\alpha+\beta}}$$

Now by using Lemma 4 the Lemma can be proved.

4. Illustrative examples:

This section is applied the method presented in the paper and give an exact solution of some linear fractional differential equations.

Example 1:

As the first example, we consider the following initial value problem in the case of the inhomogeneous Bagley-Torvik equation,

$$D^{2}y(x)+D^{\frac{3}{2}}y(x)+y(x)=1+x, \tag{8}$$

$y(0)=y'(0)=1.$

This equation by using Sumudu transform is converted to

$$\frac{1}{u^{2}}\left[F(u)-y(0)-u\,y'(0)\right]+u^{2-\alpha}\left[\frac{F(u)}{u^{2}}-\frac{y(0)}{u^{2}}-\frac{y'(0)}{u}\right]+F(u)=1+u$$

$$\frac{1}{u^{2}}\left[F(u)-1-u\right]+u^{2-\alpha}\left[\frac{F(u)}{u^{2}}-\frac{1}{u^{2}}-\frac{1}{u}\right]+F(u)=1+u$$

$$F(u)\left[\frac{u^{\alpha}+u^{2}+u^{\alpha+2}}{u^{\alpha+2}}\right]=\left[\frac{u^{\alpha}+u^{2}+u^{\alpha+2}}{u^{\alpha+2}}\right]+u\left[\frac{u^{\alpha}+u^{2}+u^{\alpha+2}}{u^{\alpha+2}}\right]$$

$$F(u)=1+u.$$

Using the inverse Sumudu transform the exact solution of this problem $y(x)=1+x$ can be obtained.

Example 2:

Our second example covers the inhomogeneous linear equation,

$$D^{\alpha} y(x) + y(x) = \frac{2\, x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x \qquad (9)$$

$$y(0) = 0\,, \quad 0 < \alpha \leq 1.$$

Using the Sumudu transform $F(u)$ is obtained as follows

$$u^{2-\alpha}\left[\frac{F(u)}{u} - \frac{y(0)}{u}\right] + F(u) = 2u^{2-\alpha} - u^{1-\alpha} + 2u^2 - u$$

$$F(u)(u^{-\alpha} + 1) = 2u^2 (u^{-\alpha} + 1) - u\,(u^{-\alpha} + 1)$$

$$F(u) = 2u^2 - u\,.$$

Then $y(x) = x^2 - x$ is obtained by using the inverse Sumudu transform.

Example 3:

Consider the following linear initial value problem,

$$D^{\alpha} y(x) + y(x) = 0 \qquad (10)$$

$$y(0) = 1\,, \quad y'(0) = 0.$$

The second initial condition is for $\alpha > 0$ only.

In two cases of α , $S[D^{\alpha} y(x)]$ is obtained as

1. For $\alpha < 1$

$$S[D^{\alpha} y(x)] = u^{2-\alpha}\left[\frac{F(u)}{u^2} - \frac{1}{u^2}\right] = \frac{F(u) - 1}{u^{\alpha}}\,,$$

2. For $\alpha < 1$

$$S[D^{\alpha} y(x)] = u^{2-\alpha}\left[\frac{F(u)}{u^2} - \frac{1}{u^2}\right] = \frac{F(u) - 1}{u^{\alpha}},$$

Which are the same. Now the Sumudu transform, $F(u)$ is obtained as

$$\frac{F(u) - 1}{u^{\alpha}} + F(u) = 0$$

$$F(u) = \frac{1}{1 + u^{\alpha}}$$

Using the lemma 3, the exact solution of this problem can be obtained as:

$$y(x) = E_{\alpha}(-x^{\alpha})$$

Example 4:

Consider the following linear initial value problem,

$$D^{\alpha} y(x) = y(x) + 1 \qquad , \quad 0 < \alpha \le 1 \tag{11}$$

$$y(0) = 0.$$

Using the Sumudu transform $F(u)$ is obtained as follows

$$\frac{F(u)}{u^{\alpha}} = F(u) + 1$$

$$F(u) = \frac{u^{\alpha}}{1 - u^{\alpha}}$$

Using the Lemma 3 the exact solution of this problem can be obtained as:

$$y(x) = x^{\alpha} E_{\alpha,\alpha+1}(x^{\alpha})$$

Example 5:

Consider the composite fractional oscillation equation

$$y''(x) - a D^{\alpha} y(x) - b y(x) = 8 \qquad , \quad 1 < \alpha \le 2 \tag{12}$$

$$y(0) = y'(0) = 0.$$

Using the Sumudu transform, $F(u)$ is obtained as follows

$$\frac{F(u)}{u^2} - a u^{2-\alpha} \frac{F(u)}{u^2} - b F(u) = 8$$

$$F(u) = \frac{8\, u^{\alpha+2}}{u^{\alpha} - a u^2 - b u^{\alpha+2}}.$$

Using the lemma 5 the exact solution of this problem can be obtained as:

$$y(x) = 8 x^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{b^n \; a^k \binom{n+k}{k}}{\Gamma(k(2-\alpha) + 2(n+1) + 1)} x^{k(2-\alpha)+2n}$$

Example 6:

Consider the following system of fractional algebraic-differential equations,

$$D^{\alpha} x(t) - t\, y'(t) + x(t) - (1+t)\, y(t) = 0 \qquad , \quad 0 < \alpha \le 1 \tag{13}$$

$$y(t) - \sin t = 0,$$

Subject to the initial conditions

$$x(0) = 1 \qquad , \qquad y(0) = 0.$$

Using the Sumudu transform $F(u)=S^{-1}[y(t)]$ and $G(u)=S^{-1}[x(t)]$ is obtained as follows

$$u^{1-\alpha}\left[\frac{G(u)}{u}-\frac{1}{u}\right]-u\frac{d}{du}F(u)+F(u)-G(u)-F(u)-u^{2}\frac{d}{du}F(u)-uF(u)=0$$

$$F(u)=\frac{u}{1+u^{2}} \quad , \quad F'(u)=\frac{1-u^{2}}{(1+u^{2})^{2}}$$

$$G(u)\left(\frac{1+u^{\alpha}}{u^{\alpha}}\right)=\frac{2u\,(1+u)}{(1+u^{2})^{2}}+u^{1-\alpha}$$

$$G(u)=\frac{2u^{\alpha+1}}{1+u^{\alpha}}\cdot\frac{1+u}{(1+u^{2})^{2}}+\frac{u}{1+u^{\alpha}}\,.$$

The exact solution for $\alpha=1$ is $x(t)=t\sin t+e^{-t}$. Using the Lemma 3 and 4 the exact solution for $0<\alpha\leq 1$ can be obtained as:

$$x(t)=2x^{\alpha+1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}(k+1)\,t^{2k}$$

$$\left(\frac{t^{n\alpha+1}}{\Gamma((n+1)\alpha+2k+3)}+\frac{t^{n\alpha}}{\Gamma((n+1)\alpha+2k+2)}\right)+E_{\alpha}(-t^{\alpha})$$

$$=2\,t^{\alpha+1}\sum_{k=0}^{\infty}(-1)^{\left[\frac{k}{2}\right]}\left(\left[\frac{k}{2}\right]+1\right)t^{k}\,E_{\alpha,\alpha+k+2}(-t^{\alpha})+E_{\alpha}(-t^{\alpha})$$

$$y(t)=\sin t$$

5. Conclusions:

Sumudu transform has been introduced by Watugala [37] . This transformation technique is very good to solve FPDEs. In this paper, the application of Sumudu transform is investigated to obtain an exact solution of some linear fractional differential equations. The fractional derivatives are described in the Caputo sense which obtained by Riemann-Liouville fractional integral operator. It is going to solve some problems. It shows that the Sumudu transform is a powerful and efficient technique for obtaining analytic solution of linear fractional differential equations.

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