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The Adaptive Determination Of The Relative Importances Of The Objects On The Basis Of The Qualitive Pair Comparisons

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Abstract

The variational approach of the elaboration of the results of the pair qualitative comparisons of the исследуемых objects with the purpose of the determination of their relative importances as positive weights without a priori elaboration of scales of judgments is offered. In the basis of this method the interpretation of elements of matrixes of pair comparisons as functions of the required weights is used.

INTRODUCTION

Decision making in tasks of management science often leads to the necessity of the ordering on a degree of importance of objects (alternatives) from the given finite set A_i ,i=1,...,N. At that in many cases it's expediently to calculate their relative importances (RI) as respective appropriate positive weights

$$w_i > 0, i = 1,..., N, \tag{1}$$

On these weights other restrictions, that determine the set of the allowable vectors, may be imposed

$$\bar{w} = (w_1,..., w_N) \in W , \tag{2}$$

The problem of the determination of the RI is from the class of the semistructured problems [1,2,3,10,11], because qualitative valuations such as "less", "comparable", "more", "most" etc. are often used for the solution of this problem. In many cases the experts from the competent specialists are invited for the forming of these valuations [4,5,9].

Different methods for the solving of the semistructured problems of the comparison of the objects on the basis of the qualitative valuations are developed [1-5,9-11]. Necessity of not only ranking of objects on their importance, but the determination of weights that express degrees of preference (relative importances (RI)), appears sufficiently frequently, especially when considering hierarchical problems of decision making on the choice of the best alternative [1,2,11]. That's why the improvement of methods of the determination of weights of comparing objects on the basis of qualitative valuations is the actual task.

Often in well-known approaches [1,2,11] to the determination of RIs procedures of pair comparisons are used, when for each pair of objects (A_i,A_j) the judgment about degree of supremacy/loss of one of them before other is standed. Results of all comparisons are taken to present as so called matrixes of pair comparisons (MPC), elements C_{ij} of these matrixes must express powers of supremacies/losses of A_i before A_j, i,j=1,...,N in the quantitative form. For giving for elements of MPC concrete numeric values, scales of verbal judgments $\{S_m\}$ with gradations S_m and respective quantitative expressions (numeric gradations of their power) $\{x_m\}$, where x_m are real numbers m=0,..., M, are developed beforehand. Here and hereinafter we expect that amount of using judgments M+1 is finite, moreover it is fixed beforehand at present.

After forming of these scales filling of MPC is realized by the following rule

$$C_{ij} = x_k, \tag{3}$$

if the judgment S_k is voiced by comparing of objects A_i with A_j.

The important moment for scales' forming and for further processing of MPC is the a priori choice of interpretation of elements of MPC in terms of weights of objects. In general case this interpretation is described by means of functional dependencies from sought vector of weights

$$C_{ij} \sim f_{ij}(\vec{w}), \tag{4}$$

where f_{ij} are some functions and the symbol ~ means correspondence, since for real MPC it's unnecessary that exact equality exists.

Sufficiently frequently presentations [1, 2, 11] are used

$$f_{ij}(\vec{w}) = w_i - w_j, \tag{5}$$

$$f_{ij}(\vec{w}) = w_i / w_j, \tag{6}$$

Hereinafter MPCs, elements of which have exactly described by the right part of the (4) upon some possible vector upon given functions f_{ij}, are called ideal MPCs. The special indication is used for them

$$F(\vec{w}) = \{f_{ij}(\vec{w})\}, \quad i, j = 1,..., N. \tag{7}$$

It seems to be natural to call functions f_{ij} structured functions, because these functions determine in a great deal the structure of the MPC. An ideal structure of MPC, chosen a priori and evident symmetry about transpositions of two comparable objects generates general correlations. Elements of real MPC must satisfy to these correlations. The collection of these correlations is taken to entitle as calibration. In particular, askew symmetry and power-mode calibrations exist for presentations (5) and (6) respectively. [1]

$$c_{ij} + c_{ji} = 0, \quad i, j = 1,..., N. \tag{8}$$

$$c_{ij} \cdot c_{ji} = 1, \quad c_{ij} > 0, \quad i, j = 1,..., N. \tag{9}$$

In general case structures of real MPCs that are forming by the rule (3), will differ from the ideal to the extent that at any $\vec{w} \in W$ the whole collection N^2 of equalities of the left and right sides of (4) will not exist.

The basic reasons of such case are the expert mistakes in the choice of the verbal judgments and the a priori fixing of the numeral gradations of their power. At the same time exactly type of the ideal structure of the MPC that is chosen a priori serves a reference for developing computing algorithms of decision of the inverse task of the determination of RIs of comparable objects [1,2,11], that it seems to be wholly logistical and doesn't cause principle objections.

The main difficulty of use of the described procedure of the determination of RIs is stipulated by the difficulty of the logistic motivation of a priori choice of numeric gradations of power of respective verbal expert judgments.

As the examples that demonstrate the essentiality of the influence of the chosen scale $\{x_m\}$ on results of calculations of RIs upon the same verbal judgments of experts we can indicate the data of the tables 3.3a, 3.4a и 3.5a from the work [2].

The purpose of this work is the elaboration of such approach to the elaboration of verbal expert judgments for structure functions in the form of (5) & (6), that using of this approach doesn't require a priori forming of the scale of the numerical gradation of their intensities.

We'll show that these gradations and weights of comparable objects may be computed on the basis of the a priori assignment of the species of ideal structure of the MPC and some additional suggestions. We realized computing experiments with already mentioned data from work [2] to illustrate the efficiency of the offered algorithms.

The approach to the determination of RIs that is elaborated in the work is adaptive in such meaning that numeric gradations of intensities of verbal judgments are computed anew for every concrete expert and every concrete case of the realization of the pair comparisons.

1. BASIC THESISES AND MODELS OF ADAPTATION

The task is to elaborate such procedure for the realization of pair comparisons and elaboration of results of these pair comparisons that permits to reflect adequately qualitative valuations of expert about degrees of the supremacy/loss objects before others in the manner of positive real numbers, which are called RI. At that it's necessary to minimize the influence of used a priori suggestions including the mathematical models that are used in the computing experiments as the basis.

In this connection it's not expedient to form a priori scales of verbal judgments in the form of the evident formulation of their gradations. So, we offer to communicate only numbers 0,1,…, M of the used judgments thus: larger number corresponds to the judgment that expresses the larger intensity of the supremacy.

Here and further on M is the largest of numbers of the used judgments, M is not fixed beforehand. The judgment with the zero number expresses the fact of the indistinguishableness of the comparable objects and it is always used because every object is identical to itself.

As the space of the qualitative valuations is discrete, one can use the same judgment by comparisons of objects in different pairs. That's why in general case the equalities will take place:

$0=i_0 < i_1 < \dots < i_J = M,$ (10)

where i_k are numbers of used judgments; and J+1 is their amount;

$i_k \in \{1, \dots, M\}$; $k=1, \dots, J$; $J \le M$. (11)

On the other side, we must always take into account the symmetry about transpositions of objects in pairs. Therefore, the inequality will take place

$J \le N(N-1)/2.$ (12)

Let $B = \{b_{ij}\}$; $i,j = 1, \dots, M$ – is a matrix, its elements are filled by the following rule.

If it is acknowledged that object Ai exceeds the object Aj and the intensity of its superiority expresses by the judgment number k under the order, then

$b_{ij} = k,$ (13)

and

$b_{ij} = -k,$ (14)

when on the contrary object Aj surpasses Ai at the same degree, at that

$b_{ii} = 0, i=1,\dots M.$ (15)

Further on, the condition

$(i,j) \in B_k , i \ne j$ (16)

means that equality (13) takes place, B_k is the set of such pairs of indexes and I_k is the power of this set, k =0, 1,…, M.

It's evident that the distribution of numbers of judgments (ranks) by elements of the matrix B is in general case that a posteriori information, which one can use for the computing of weights of objects. In addition to this information it's necessary to form some statements and principles, which permit to construct mathematical models and computing algorithms that are caused by these models.

The basic thesis consists in that the expert judgment with the number $k \in \{0,1, \dots, M\}$ reflects values of structure functions $f_{ij}(\vec{v})$,such as pairs of indexes of these functions satisfy to the condition (16). The unknown argument $\vec{v} \in W$ is the subject to evaluation by results of pair comparisons.

Let $\vec{x}$ is a vector of unknown numerical gradations of intensity of used judgments

$\vec{x} = (x_1, x_2, \dots, x_M)' \in X$, (17)

where in accordance to hierarchy of the intensity of superiorities, which are expressed with judgments, the set X of admissible vectors must satisfy to the condition

$f_{ii}(\vec{w}) = x_0 < x_1 < x_2 < \dots < x_M$, (18)

Here and further on $f_{ij}(\vec{w})$ are described with the correlations (5) or (6). At that the identicalness of indexes corresponds to the comparison of objects with itselves that is expressed by the equality in the left part of (18). Besides, there are some another limitations that can be applied to the set X.

It's evident that real MPC may be represented in the manner of the matrix of the function

$$C = C(\vec{x}), \qquad (19)$$

if one takes into account that if the condition (16) takes place, then equalities (20) must take place

$$C_{ij} = x_k, \quad C_{ji} = \Theta_f[x_k], \qquad (20)$$

at that

$$C_{ii} = x_0, \; i.j = 1,..., N. \qquad (21)$$

Here and further on the index f means that operations (operators) that are marked with it, are determined by the species of the structure function, and Θ_f is a function that describes the calibration.

We'll call the pair comparisons as ideal, if two vectors $\vec{v} \in W \; u \; \vec{y} \in X$ exist, and the equality (22) executes for these vectors

$$C(\vec{y}) - F(\vec{v}) = 0, \qquad (22)$$

This equality means that the exact equality of the corresponding elements of real and ideal MPCs is reached.

In general case, comparisons will not be ideal. In addition to above mentioned it's expedient to apply the principle of the minimization of differences in elements of matrixes C and F as the basis of the computing procedure of the decision of the inverse task. The mathematical expression of such approach is the requirement of the minimization of some functional of the proximity of the left part of (22) to zero

$$S[C(\vec{x}), F(\vec{w})] = \sum_{i,j=1}^{N} \rho_f [C_{ij}(\vec{x}), f_{ij}(\vec{w})] \to \min, \vec{w} \in W \qquad (23)$$

where ρ_f is a some nonnegative function

$$\rho_f(a, b) \geq 0, \qquad (24)$$

at that the equality of this function to zero is reached only when the condition (25) executes.

$$a = b. \qquad (25)$$

It's evident that the right part of the functional (23) may be used as the measure of the proximity of ideal and real MPCs.

After the corresponding transformations followed presentations for relative values of functionals of proximities. For representations (5) and (6) were obtained

$$SR(\vec{x}) = S_0(\vec{x}) \Big/ \sum_{i,j=1}^{N} C_{ij}^{\;2}(\vec{x}) = 1 - 2\sum_{i=1}^{N}\left(\sum_{j=1}^{N} C_{ij}\right)^2 \Big/ N \sum_{i,j=1}^{N} C_{ij}^{\;2}, \qquad (26)$$

$$SP(\vec{x}) = S_2(\vec{x}) \Big/ \sum_{i,j=1}^{N} \ln^2 C_{ij}(\vec{x}) = 1 - 2\sum_{i=1}^{N}\left(\sum_{j=1}^{N} \ln C_{ij}\right)^2 \Big/ N \sum_{i,j=1}^{N} \ln^2 C_{ij}, \qquad (27)$$

where, in accordance with (20),

$$C_{ij} = C_{ij}(\vec{x}). \qquad (28)$$

The natural basis of the computing of the vector $\vec{x}$ is the minimization of right parts of (26) or (27) with the presence of some limitations on the class of solutions. Correlations (29) and (30), (31) are the mathematical expression of this principle:

$$SR(\vec{y}) = \min SR(\vec{x}), \forall \vec{x} \in X, \quad (29)$$

$$SP(\vec{y}) = \min SP(\vec{x}), \forall \vec{x} \in X \quad (30)$$

$$\vec{y} \in X \quad (31)$$

Additional limitations must contain standardizing conditions, because the solution $\vec{y}$ will be received with the exactness to arbitrary multiplier.

In separate cases, proceeding from some suggestions, for example, on the basis of authentic measurements, the expert can indicate for some pair of indexes (m,n) the value of the structure function that is the right part of the correlation

$$f_{mn} = f_0. \quad (32)$$

It's naturally that the solution that is received, that is $F(\vec{v})$ in (22), must satisfy to this condition. The adaptation to a priori information of the indicated species reaches by correlation (32).

It's easy to understand, that structure functions of the species (5) are dimensional, because the RIs itselves can be measured in some digits.

That's why it's naturally to lay

$$y_1 = 1, \quad (33)$$

for the decision of the task (29) as some standardized digit of dimension.

2. ABOUT DECISION OF VARIATIONAL TASKS (29) AND (30)

We can offer such presentations of sought variables and limitations on the elements of the set X, that permit to simplify the elaboration of the corresponding algorithms.

It's useful to use presentations (34) or (35) accordingly, for the decision of tasks (29) and (30) with the account of conditions (19)

$$x_k = (\vec{p}, \vec{e}_k), \quad (34)$$

Or

$$\ln x_k = (\vec{p}, \vec{e}_k), \quad (35)$$

where

$$\vec{p} \in P_M = \{\vec{z} | z_i > 0, i = 1,...M\}; \quad (36)$$

$$\vec{e}_k = (1,..., 1, 0,..., 0)'; \quad (37)$$

the symbol $(.,.)$ means the scalar product of vectors of the same dimension. Vectors $\vec{e}_k$ have M components. First k of these components are equal to one, other are equal to zero.

It's evident that it's easier to control the execution of the condition (36) then conditions in the form (18).

It's not difficult to receive presentations of elements of MPCs for structure functions (5) and (6) accordingly, meaning the rule (20).

$$C_{ij} = (\vec{g}_{ij}, \vec{p}), \quad (38)$$

$$C_{ij} = \exp(\vec{g}_{ij}, \vec{p}), \quad (39)$$

where, when the condition executes $(i, j) \in B_{|k|}, k = 0, \pm 1, ..., \pm M$, this correlation takes place

$$\vec{g}_{ij} = sign(b_{ij})\vec{e}_{|k|}; \; sign(z) = \begin{cases} 1, z > 0; \\ 0, z = 0; \\ -1, z < 0. \end{cases} \quad (40)$$

At that, presentations for estimations of RIs accept the species (41) and (42) accordingly

$$v_i = (\vec{g}_i, \vec{p}) + A; \quad (41)$$

$$v_i = \exp((\vec{g}_i, \vec{p})), \quad (42)$$

Where

$$\vec{g}_i = \sum_{j=1}^{N} \vec{g}_{ij} / N. \quad (43)$$

It's evident, that the requirement (43) may be executed by corresponding standardization of the vector $\vec{p}$.

The substitution of (38), (41) or (39),(42) accordingly, into (26) or (27) results (leads) to the same form of the functionals of proximities to ideal comparisons

$$S(\vec{p}) = \vec{p}'(H - G)\vec{p} / \vec{p}' H \vec{p}$$

where

$$H = \sum_{i,j=1}^{N} \vec{g}_{ij} \vec{g}_{ij}'; \quad (45)$$

$$G = 2N \sum_{i=1}^{N} \vec{g}_i \vec{g}_i'. \quad (46)$$

Thus, the task leads to the search of the vector $\vec{p}$ with positive components $(\vec{p} > 0)$, that minimizes the right part of (44).

It seems to be expedient to lay additional requirements to the set of vectors, where the solution of this variational task is searched, besides the condition (36) of positiveness of components.

The common characteristic property of vectors p_k of type is the unpositiveness of the first differences of components

$$d_k = p_{k+1} - p_k \le 0, k = 1, .., M - 1, \quad (47)$$

and the nonnegativity of the second differences

$$r_k = d_{k+1} - d_k \ge 0, k = 1, .., M - 2. \quad (48)$$

After determination of the index of the severely single component of the vector $\vec{y} = (y_1, .., y_M)'$ that satisfies to the condition

$$(hr_{mm} - gr_{mm}) / hr_{mm} = \min(hr_{ii} - gr_{ii}) / hr_{ii}, \forall i \in \{1, ..., M\}, \quad (49)$$

it's useful to use the approach of the by-coordinate descent [7] for the computing of values of other components. At that the iteration is in the successive computing of the value of the right part of the functional (50) at the increase of each component of the sought vector (besides the single) on the indicated supplement to define what component's increase leads to the most decrease of the value of the functional. It's natural that components at this method increase at first, and after calculations in accordance with (50) components return to their previous values, and only the component that gives the maximal decrease to the functional, is modified for the realization of the following iteration.

After receiving the solution of the task (50)

$$S(\vec{y}) = \vec{y}'(HR - GR)\vec{y} / \vec{y}'HR\vec{y} = \min \quad (50)$$

by the indicated approach we determine components of the sought vector. Then these founded components must be rationed.

For the askew symmetrical calibration of the MPC (correlations (5)) it's naturally to use the requirement (33), whereas for the degree calibration (look correlations (6)) one can use the variational principle

$$\sum_{k=2}^{M-1}((\ln((k+1)/k) - u * p_k)^2 * ((k+1)/k)^M) = \min, \forall u > 0 \ . \quad (51)$$

The use of the ratio of species $(k+1)/k$ permits to proceed from ranks to real numbers, and the weight succession $((k+1)/k)^M$ reflects the fact of lesser confidence to estimations with larger rank (with the account of its maximal value).

Thus, the vector that was calculated in accordance with (52),

$$\vec{p} = T * TR(1, \vec{z}')' , \quad (52)$$

where T and TR' - are the lower square triangular matrixes with the dimension of M that is the second column of the second matrix is negative and the elements of the first column, besides the first element of this column, are equal to zero. Must be multiplied on the coefficient $u > 0$, that satisfies to the condition (51).

3. COMPUTING EXPERIMENTS

Computing experiments by elaboration of the data of tables 3.3, 3.4 and 3.5 from work [2] were realized to illustrate the efficiency of the offered procedure of adaptation.

The presentation of species (53) is used below for the calculation of estimations of RI.

$$\tilde{w}_k = v_k \Big/ \sum_{i=1}^{N} v_i \ , . \quad (53)$$

where v_i satisfy to (39), and vector $\vec{p}$ is found from the minimization of the right part of (50) and the use of correlations (52) with further setting of norms that satisfies to the condition (51).

The proximity of received estimations and initial RIs of objects w_k is determined by the measure (54), as in work [2]

$$\rho = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(w_k - \tilde{w}_k)^2} \ . \quad (54)$$

Results of computing experiments are reduced in tables 1 and 2, which correspond to tables 3.2, 3.4 in work [2]. Results of calculations of RIs of Saati, when the basic scale of AHP is used (Number 4 of tables), are reduced in tables 1 and 2 for the useful comparisons. The Saati's RIs have index 'c'.

Table 1

k	1	2	3	4	The measure of proximity
w	0,608	0,219	0,111	0,062	
$\tilde{w}$	0,610	0,202	0,116	0,072	ρ=0,009
$\tilde{w}_c$	0,617	0,224	0,097	0,062	ρ_c=0,008

Table 2

k	1	2	3	4	5	6	The measure of proximity
w	0,279	0,381	0,032	0,132	0,177	0,019	
$\tilde{w}$	0,255	0,367	0,035	0,143	0,168	0,032	ρ=0,013
$\tilde{w}_c$	0,262	0,397	0,033	0,116	0,164	0,028	ρ_c=0,014

The data of these tables show that the offered approach to calculation of relative importances of alternatives in indicated examples on the basis of real expert estimations by the approach of pair comparisons permits to receive quite adequate results that are comparable with initial values.

It's evident that the data of last three tables confirm the good efficiency of the approach that is offered in this work, to the receipt and elaboration of expert judgments in the approach of pair comparisons at askew symmetrical and degree calibrations of MPCs.

CONCLUSION

It's evident, that the purpose of the work that was formulated in the introduction is reached. Results of the computing experiment show the efficiency of the offered approach of the determination of the relative importances of several objects (alternatives) in the quantitative species on the basis of the realization of pair qualitative comparisons without a priori elaboration of scales of expert judgments and quantitative values of degrees of the supremacy that are expressed by the scales.

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