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Axiomatic Determination Of Elementary Multi-Valued Functions For A Complex Variable

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Abstract

An axiomatic approach to the determination of basic elementary functions for a real variable admits a simple extension in the case of single-valued elementary functions for a complex variable. The case of multi-valued elementary functions is beyond the scope of a general concept. It stands alone and requires a separate independent study. In this article, the author addresses this gap and develops the axiomatics of multivalued elementary functions for a complex variable, based on the concept of a continuous representation of multivalued mapping and the concept of continuous multivalued mapping.

Keywords: multivalued mappings, continuous representations of multi-valued mappings, continuous multi-valued mappings, elementary functions of a complex variable, functional equations of Cauchy, axiomatic determination of elementary functions.

1. INTRODUCTION

The functional equation $f(x+y) = f(x)+f(y)$ was initially considered by Legendre during the proof of a projective geometry fundamental theorem [1] and by Gauss during the study of probability distributions [2]. The solution of this equation and the equations $f(x+y) = f(x)f(y)$, $f(xy) = f(x) + f(y)$, $f(xy) = f(x)f(y)$ within the class of continuous functions were originally obtained by Cauchy [3]. It was found that the continuous solutions of these equations are exhausted by four functions ax, a^x, $\log_a x$ and x^a, respectively. This means that Cauchy equations may be used in the axiomatic determination of basic elementary functions concerning one real variable.

Further studies of the Cauchy equations were associated with the weakening of the continuity condition (local limitation, integrability, measurability, the domination with a measurable function, etc). Finally, Hamel [4] gave the first example of the equation *f(x+y) = f(x)+f(y)* solution, different from *ax*. Cauchy equations were solved in more general functional classes, for example, within the class of multiple variable functions (Abel, [5]). The additional information about the Cauchy equations may be taken from the works [6-8].

In this article, the Cauchy equations are solved within the class of continuous multi-valued functions of a complex variable with discrete images. The concept of continuous multivalued mapping is introduced in the second section. The basic properties of continuous multi-valued mappings and their continuous representations are also considered there. The concept of continuous multi-valued mapping should not be confused with the concept of continuous multivalued mapping inclusion [9-10].

In the case of multi-valued solutions the Cauchy equations are complemented with inverse Cauchy equations and the systems of equations are considered:

$$F(x+y)=F(x)+F(y),\ F(x-y)=F(x)-F(y);$$

$$F(x+y)=F(x)F(y),\ F(x-y)=F(x)/F(y);$$

$$F(xy)=F(x)+F(y),\ F(x/y)=F(x)-F(y);$$

$$F(xy)=F(x)F(y),\ F(x/y)=F(x)/F(y).$$

The solution of these systems (with discrete initial conditions) is carried out in the third section devoted to the axiomatic determination of multi-valued elementary functions for a complex variable.

2. CONTINUOUS REPRESENTATIONS OF MULTI-VALUED MAPPINGS

According to the common grounds the mapping $f: X \to Y$ is defined as a non-empty subset of the Cartesian product, satisfying the term of uniqueness

$$(x,y_1),(x,y_2)\in f \ \Rightarrow\ y_1=y_2.$$

The multivalued mapping $F: X \to Y$ is defined as a non-empty subset of the Cartesian product $X \times Y$. The term of uniqueness is omitted. If $Y \subseteq \overline{\mathbf{C}}$ and $X \subseteq \overline{\mathbf{C}}$ then the multi-valued mapping F is called a multi-valued function.

Let X,Y are topological spaces. The symbol $c[X,Y]$ denotes the set of all continuous mappings $f: X \to Y$. If $f \in c[X,Y]$ and the definition domain $D_f \subseteq X$ of the display f is open, then we write $f \in c(X,Y)$. Let's choose an arbitrary multivalued mapping $F : X \to Y$. Let's call the subset $\mathbf{F} \subseteq c[X,Y]$ a continuous representation of a multivalued mapping F if $F = \bigcup_{f\in\mathbf{F}} f$. The multivalued mapping F is called continuous if it admits a continuous representation $\mathbf{F} \subseteq c(X,Y)$.

Any multi-valued mapping F admits a continuous representation $\mathbf{F}$. One may assume for example that each mapping $f \in \mathbf{F}$ consists of a single point $(x,y) \in F$. Such a continuous representation of the multivalued mapping F is called pointwise. The opposite is also true, every family $\mathbf{F}$ of continuous mappings $f : X \to Y$ defines a multivalued mapping $F := \bigcup_{f \in \mathbf{F}} f$.

Let's define within a power set $c[X,Y]$ the equivalence relation $\mathbf{F}_1 \sim \mathbf{F}_2$ by the following rule: $\mathbf{F}_1 \sim \mathbf{F}_2$ if and only if when $\bigcup_{f \in \mathbf{F}_1} f = \bigcup_{f \in \mathbf{F}_2} f$. All the families of an individual equivalence class represent the same multi-valued mapping. On the other hand, each multi-valued mapping F defines a specific equivalence class. This is the class that contains a pointwise representation of the mapping F. Thus, any multi-valued mapping may be identified with a particular class of equivalent families for continuous mappings $f : X \to Y$. You may set a multivalued mapping F by selecting an arbitrary representative of this class, i.e. by the specification of a continuous representation $\mathbf{F}$ of this mapping. In this case, the elements of the continuous representation $\mathbf{F}$ of the multivalued mapping F are called continuous branches.

Let K is the set endowed with the discrete topology, X is the topological space. The Cartesian product $X \times K$ may be considered as the union $\bigcup_{x \in X} (x)_K$ of $(x)_K := \{(x,k) : k \in K\}$ sets. In this case, the topological product $X \times K$ is called a multiple area and is denoted by $(X)_K$. The sets $(x)_K$ are called multiple elements of the area $(X)_K$. The elements $(x)_k \in (x)_K$ are called the element elevations $x \in X$. The subareas $(X)_k \subseteq (X)_K$ are called the area elevations X. The related subareas of the area $(X)_K$ are called the space paper.

Let F is a multi-valued mapping of the topological space X into a topological space Y, $\mathbf{F} = \{f_k : k \in K\}$ - its continuous representation. We may assume that each continuous branch $f_k \in \mathbf{F}$ is defined on its elevation $(X)_k$ of the space X. In this case, the mapping F may be viewed as a single mapping from the multiple space $(X)_K$ into the space Y. For the point $(x)_k$ it assigns the correspondence in the point $y := f_k(x) \in Y$. In this case, the definition area F of the mapping coincides with the disjoint union $\coprod_{k \in K} D_{f_k} := \{(x)_k : x \in D_{f_k}\} \subseteq (X)_K$.

Let's introduce on $\coprod_{k \in K} D_{f_k}$ the equivalence relation: $(x)_k \sim (x')_{k'}$ if and only if when $x = x' \in D_{f_k}$ $D_{f_{k'}}$ and $f_k(x) = f_{k'}(x)$. The space factor $(X)_{\mathbf{F}}$ of the space $\coprod_{k \in K} D_{f_k}$ by this equivalence relation is called the space factor of continuous mappings family $\mathbf{F}$. The related subspaces of the space $(X)_{\mathbf{F}}$ are called the paper of this space. They say that the space papers $(X)_{\mathbf{F}}$ and the space $(X)_{\mathbf{F}}$ itself is obtained by gluing the papers of the multiple space $(X)_K$ according to equivalent points, i.e. according to the points at which the continuous branches have the same values. The space elements $(X)_{\mathbf{F}}$ are designated as $(x)_k$ and called the element $x \in X$ elevations. At that the symbols $(x)_k$ and $(x)_{k'}$ denote the same element of the space $(X)_{\mathbf{F}}$ if $f_k(x) = f_{k'}(x)$. For any $x \in \coprod_{k \in K} D_{f_k}$

the set $(x)_{\mathbf{F}} := \{(x)_k \;:\; x \in D_{f_k}\}$ is called a multiple element or a layer (above x) of the space $(X)_{\mathbf{F}}$. The space X is called the space projection $(X)_{\mathbf{F}}$, and the mapping $p : (X)_{\mathbf{F}} \to X \big| (x)_k \to x$ is called a projection operator or a projector.

In connection with the study of multi-valued mappings inverse to single mappings the concept of separation is introduced. The reverse mappings in respect to single ones are completely characterized by the fact that the images of different points for such mappings are not overlapped. Let's s is a single circle $\{\zeta \;:|\, \zeta \,|= 1\}$, endowed with the topology induced from $\mathbf{C}$. Let's consider the mapping $t \;:\; \mathbf{R} \to s,$ which assigns the correspondence between the point $\varphi \in \mathbf{R}$ and the point $e^{i\varphi} \in s$. The inverse mapping $T \;:\; s \to \mathbf{R}$ is a separation over the space s and is called a trigonometric separation. From the definition of a complex number argument it follows that $T(\zeta) = \mathrm{Arg}\,\zeta$ for any $\zeta \in s$. Let's suppose that $u_k := (-\pi, \pi) + \pi k,$ $v_k := t(u_k)$ for any integer k. Let's choose as a continuous representation of the trigonometric separation the family $\mathbf{T} := \{T_k \;:\; k \in \mathbf{Z}\}$, where a continuous single-valued function T_k is defined on the arc v_k using the relations

$$T_k(\zeta) = \arg\left(e^{-\pi k i}\right) + \pi k = \arg\left(\cos \pi k\right) + \pi k.$$

It is natural to assume that every function $T_k \in \mathbf{T}$ is a mapping from a separate copy $(s)_k$ of a single circle within the space $\mathbf{R}$. The arcs v_k are the papers of the multiple space $(s)_{\mathbf{Z}}$. When you pass to the space factor $(s)_{\mathbf{T}}$ all the papers of the space $(s)_{\mathbf{Z}}$ are glued together in one paper.

Let $\mathbf{R}_+$ is the set of positive real numbers $\mathbf{R}_+^2$ is the Cartesian product $\mathbf{R}_+ \times \mathbf{R}$. Let's consider single-valued mapping $p \;:\; \mathbf{R}_+^2 \to \mathbf{C}$ that makes a correspondence between the point $(\rho, \theta) \in \mathbf{R}_+^2$ and the point $\rho t(\varphi) = \rho e^{i\varphi} \in \mathbf{C}$. The inverse mapping P is a separation over the space $\mathbf{C}$ and is called the polar separation. From the definition of a complex number argument it follows that $P(z) = (|\, z\,|, \mathrm{Arg}\, z)$ for any $z \in \mathbf{C}^* := \mathbf{C} \setminus \{0\}$. Let $U_k := \mathbf{R}_+ \times u_k$, $V_k := p(U_k)$. Let's choose the family $\mathbf{P} := \{P_k \;:\; k \in \mathbf{Z}\},$ as a continuous representation of the polar separation P where the vector function P_k is defined on the area V_k according to the rule

$$P_k(z) := \left(|\, z\,|, T_k\left(\frac{z}{|z|}\right)\right), T_k \in \mathbf{T}.$$

It is natural to assume that every function $P_k \in \mathbf{P}$ is the mapping of the individual copy $\mathbf{C}^*$. The areas V_k are the multiple papers of space $(\mathbf{C}^*)_{\mathbf{Z}}$. When you pass to the space factor $(\mathbf{C}^*)_{\mathbf{P}}$ of all the space papers $(\mathbf{C}^*)_{\mathbf{Z}}$ are glued together in a single paper.

3. MULTIVALUED ELEMENTARY FUNCTIONS OF COMPLEX VARIABLE

3.1. Multi-valued linear function

Let $h := (h_1,...,h_k)$ is a fixed set of complex numbers. Let's denote the discrete set of points $\{\langle h,\alpha\rangle := h_1\alpha_1 + \ldots + h_k\alpha_k \,:\, \alpha := (\alpha_1,\ldots,\alpha_k) \in \mathbf{Z}^k\}$ in the complex plane by the $\langle h,\mathbf{Z}^k\rangle$ symbol. This set satisfies the equation $\langle h,\mathbf{Z}^k\rangle + \langle h,\mathbf{Z}^k\rangle = \langle h,\mathbf{Z}^k\rangle - \langle h,\mathbf{Z}^k\rangle = \langle h,\mathbf{Z}^k\rangle$..

The linear multi-valued function of a real variable (with a complex coefficient a and the step h) is called a continuous multi-valued function $F:\mathbf{R}\to\mathbf{C}$ defined on the set of all actual numbers and satisfying the axioms:

1) $F(x_1 + x_2) = F(x_1) + F(x_2)$ for all $x_1, x_2 \in \mathbf{R}$;

2) $F(x_1 - x_2) = F(x_1) - F(x_2)$ for any $x_1, x_2 \in \mathbf{R}$;

3) $F(\frac{1}{n}) = \frac{a}{n} + \langle h,\mathbf{Z}^k\rangle$ for any natural number n.

It is easy to check that the multi-valued function $x \to ax + \langle h,\mathbf{Z}^k\rangle$ satisfies the conditions 1)-3). Then let's make sure that there are no two different continuous multi-valued functions defined on the set of all real numbers and satisfying the conditions 1)-3). From the equality $F(0) = F(1) - F(1)$ it follows that $F(0) = \langle h,\mathbf{Z}^k\rangle$. From the equality $F(x) - F(x) = F(0)$ it follows that $F(x) = f(x) + \langle h,\mathbf{Z}^k\rangle$, where f is an arbitrary single-valued branch F, of the function defined everywhere on $\mathbf{R}$. We believe that $f(\frac{1}{n}) = \frac{a}{n}$ for any $n \in \mathbf{N}$. From the equalities $F(-x) = F(0) - F(x) = -F(x)$ it follows that the f function may be considered as an odd one. Moreover, from the obvious equalities $F(\frac{m}{n}) = mf(\frac{1}{n}) + \langle h,\mathbf{Z}^k\rangle = \frac{m}{n}a + \langle h,\mathbf{Z}^k\rangle$ the definition $f(\frac{m}{n}) := \frac{m}{n}a$ eligibility follows for any natural m and n. This means that $F(r) = ar + \langle h,\mathbf{Z}^k\rangle$ for any rational r. Due to the continuity of a multi-valued function F we have

$$F(x) = ax + \langle h,\mathbf{Z}^k\rangle$$

for any rational x. Indeed, let's set the arbitrary $x \in \mathbf{R}$ and $y \in F(x)$. Let $f_{x,y}$ is an arbitrary continuous branch of the function F that takes the value y at the point x. Then for any sequence of rational numbers $r_n \to x$ we have $f_{x,y}(r_n) = ar_n + h_1 m_{1,n} + \ldots + h_k m_{k,n} \to y$. So $y = ax + h_1 m_1 + \ldots + h_k m_k \in ax + \langle h,\mathbf{Z}^k\rangle$, where $m_j := \lim m_{j,n} \in \mathbf{Z}$. Hence $F(x) = y + \langle h,\mathbf{Z}^k\rangle = ax + \langle h,\mathbf{Z}^k\rangle$.

The linear multi-valued function of a complex variable (with a complex coefficient a and the step h) is called a continuous multi-valued function $F:\mathbf{C}\to\mathbf{C}$ defined on the set of all complex numbers satisfying the axioms:

4) $F(z_1+z_2)=F(z_1)+F(z_2)$ for all $z_1,z_2\in\mathbf{C}$;

5) $F(z_1-z_2)=F(z_1)-F(z_2)$ for all $z_1,z_2\in\mathbf{C}$;

6) $F(\frac{1}{n})=\frac{a}{n}+\langle h,\mathbf{Z}^k\rangle$, $F(\frac{1}{n}i)=\frac{a}{n}i+\langle h,\mathbf{Z}^k\rangle$ for any natural number n.

It is easy to check that the multi-valued function $z\to az+\langle h,\mathbf{Z}^k\rangle$ satisfies the conditions 4)-6). Then let's make sure that there are no two different continuous multi-valued functions defined on the set of all complex numbers and satisfying the conditions 4)-6). Indeed, let's suppose that F satisfies the specified axioms $z\in\mathbf{C}$, $x:=\operatorname{Re} z$ and $y:=\operatorname{Im} z$. Then $F(z)=F(x)+F(iy)=F_1(x)+F_2(y)$ where F_1 is multi valued function $x\to F(x)$, and F_2 is the multi-valued function $y\to F(iy)$. The multi-valued functions F_1 and F_2 are continuous multi-valued functions of a real variable. They are defined everywhere on $\mathbf{R}$ and satisfy the axioms 1)-2). At that $F_1(\frac{1}{n})=F(\frac{1}{n})=\frac{a}{n}+\langle h,\mathbf{Z}^k\rangle$ and $F_2(\frac{1}{n})=F(\frac{1}{n}i)=\frac{a}{n}i+\langle h,\mathbf{Z}^k\rangle$ for any natural number n. From the definition of a linear multi-valued function of a real variable it follows that $F_1(x)=ax+\langle h,\mathbf{Z}^k\rangle$ and $F_2(y)=aiy+\langle h,\mathbf{Z}^k\rangle$ for all real numbers x and y. Thus, $F(z)=F_1(x)+F_2(y)=az+\langle h,\mathbf{Z}^k\rangle$ for any complex z.

The natural continuous linear representation of a multi-valued function for a complex variable is the set $\{f_\alpha : \alpha\in\mathbf{Z}^k\}$, where

$$f_\alpha(z):=az+\langle h,\alpha\rangle.$$

We note that the linear multi-valued function of a complex variable is a periodic one. If $a\neq 0$, then its period is any arbitrary complex number of the set $\frac{1}{a}\langle h,\mathbf{Z}^k\rangle$.

3.2. Axiomatic definition of argument.

The axiomatic definition of a multi-valued linear function allows to perform the axiomatic approach to the definition of a complex number argument. That is the argument of a complex number is called a continuous multi-valued function $F:\mathbf{C}\to\mathbf{R}$ defined on the set $\mathbf{C}^*$ and satisfying the axioms:

1) $F(z_1z_2)=F(z_1)+F(z_2)$ for all $z_1,z_2\in\mathbf{C}^*$;

2) $F\left(\frac{z_1}{z_2}\right)=F(z_1)-F(z_2)$ for any $z_1,z_2\in\mathbf{C}^*$;

3) $F(e^{\frac{1}{n}})=2\pi\mathbf{Z}$, $F(e^{\frac{\pi}{n}i})=\frac{\pi}{n}+2\pi\mathbf{Z}$ for any natural number n.

Let's be sure that the conditions 1)-3) define the argument of a complex number in a definite way. Suppose that a continuous multi-valued function F is defined on $\mathbf{C}^*$ and satisfies the axioms 1)-3). Let $F_1(x):=F(e^x)$ and $F_2(x):=F(e^{\pi x i})$ for any real x. Then $F_1(\frac{1}{n}) = F(e^{\frac{1}{n}}) = 2\pi\mathbf{Z}$ and $F_2(\frac{1}{n}) = F(e^{\frac{\pi}{n}i}) = \frac{\pi}{n} + 2\pi\mathbf{Z}$ for any natural number n. The multi-valued functions F_1 and F_2 are defined on $\mathbf{R}$, they are the continuous ones and satisfy the axioms of a linear multi-valued function for a real variable (make it sure). Therefore $F_1(x) = 2\pi\mathbf{Z}$, $F_2(x) = \pi x + 2\pi\mathbf{Z}$ for any $x \in \mathbf{R}$. As the following presentation is possible for $z \neq 0$

$$z = |z|\frac{z}{|z|} = e^{\ln|z|}(\cos\varphi + i\sin\varphi) = e^{\ln|z|}e^{i\varphi},$$

where φ is an arbitrary solution of the equation system

$$\cos\varphi = \frac{\operatorname{Re} z}{|z|}, \quad \sin\varphi = \frac{\operatorname{Im} z}{|z|}$$

then $F(z) = F(e^{\ln|z|}) + F(e^{i\varphi}) = F_1(\ln|z|) + F_2(\frac{\varphi}{\pi}) = \varphi + 2\pi\mathbf{Z}$. This means that $\operatorname{Arg} z = \varphi + 2\pi\mathbf{Z}$.

According to the definition of polar separation

$$\operatorname{Arg} z = \operatorname{Pr}_2 P(z),$$

where Pr_2 is the projection operator $\mathbf{R}^2 \to \mathbf{R}$ on the second component. This means that the argument of a complex number may be considered as a continuous single-valued function defined on the Riemann surface of a polar separation $(\dot{\mathbf{C}})_{\mathbf{P}}$. At that the point $(z)_k$ corresponds to a real number $T_{2k}\left(\frac{z}{|z|}\right) = \arg z + 2\pi k$. In different points of $(z)_{\mathbf{P}}$ layer this function takes different values. Therefore, the space factor of a continuous representation of the argument $\{(\operatorname{Arg} z)_k : k \in \mathbf{Z}\}$, where $(\operatorname{Arg} z)_k := T_k\left(\frac{z}{|z|}\right) = \arg((-1)^k\zeta) + \pi k$ coincides with the space factor of the polar separation $(\mathbf{C}^*)_{\mathbf{P}}$.

3.3. Logarithmic function of complex variable

The logarithmic function of a complex variable is defined as a continuous multi-valued function F, defined within the set $\mathbf{C}^*$ and satisfying the following conditions:

1) $F(z_1 z_2) = F(z_1) + F(z_2)$ for all $z_1, z_2 \in \mathbf{C}^*$;

2) $F\left(\frac{z_1}{z_2}\right) = F(z_1) - F(z_2)$ for any $z_1, z_2 \in \mathbf{C}^*$;

3) $F(e^{\frac{1}{n}}) = \frac{1}{n} + 2\pi i\mathbf{Z}$ and $F(e^{\frac{\pi}{n}i}) = \frac{\pi}{n}i + 2\pi i\mathbf{Z}$ for any natural number n.

Let us make sure that there are no two different continuous multi-valued functions defined on $\mathbf{C}^*$ and satisfying the conditions 1)-3). Let's suppose that a continuous multi-valued function F is defined on $\mathbf{C}^*$ and satisfies the axioms 1)-3). For any real r and φ we have $F(re^{i\varphi}) = F(r) + F(e^{i\varphi}) = F_1(\ln r) + F_2\left(\frac{\varphi}{\pi}\right)$ where $F_1(x) := F(e^x)$, $F_2(x) := F(e^{\pi x i})$. At that, $F_1(\frac{1}{n}) = F(e^{\frac{1}{n}}) = \frac{1}{n} + 2\pi i\mathbf{Z}$ and $F_2(\frac{1}{n}) = F(e^{\frac{\pi}{n}i}) = \frac{\pi}{n}i + 2\pi i\mathbf{Z}$ for any natural number n. The complex multi-valued functions F_1 and F_2 are defined everywhere on $\mathbf{R}$ and are continuous ones and satisfy the axioms of linear multi-valued function for a real variable. Therefore $F_1(x) = x + 2\pi i\mathbf{Z}$ and $F_2(x) = \pi x i + 2\pi i\mathbf{Z}$ for any real x. Hence, $F(re^{i\varphi}) = F_1(\ln r) + F_2\left(\frac{\varphi}{\pi}\right) = \ln r + i\varphi + 2\pi i\mathbf{Z}$. If $r := |z| \neq 0$ and $\varphi \in \operatorname{Arg} z$, then $F(z) = \ln|z| + i\operatorname{Arg} z$. Thus, $F(re^{i\varphi}) = F_1(\ln r) + F_2\left(\frac{\varphi}{\pi}\right) = \ln r + i\varphi + 2\pi i\mathbf{Z}$ the conditions 1) - 3) define the multi-valued function F quite clearly. According to the conventional notes $\operatorname{Ln} z = \ln|z| + i\operatorname{Arg} z$ for any $z \in \mathbf{C}^*$.

According to the definition of a polar separation for any $z \in \mathbf{C}^*$

$$\operatorname{Ln} z = \ln \operatorname{Pr}_1 P(z) + i \operatorname{Pr}_2 P(z),$$

where $\operatorname{Pr}_1, \operatorname{Pr}_2$ are the projection operators $\mathbf{R}^2 \to \mathbf{R}$ for the first and second components, respectively. Consequently the logarithmic function of a complex variable may be viewed as a continuous single-valued function defined on the space factor of a polar separation $(\mathbf{C}^*)_{\mathbf{P}}$. At that the point $(z)_k$ corresponds to a complex number $\ln|z| + i(\operatorname{Arg} z)_k$. In different parts of the layer $(z)_{\mathbf{P}}$, this function has different values. Therefore, the space factor of continuous representation $\{\ln|z| + i(\operatorname{Arg} z)_k : k \in \mathbf{Z}\}$ (the Riemann surface of the logarithmic function) coincides with $(\mathbf{C}^*)_{\mathbf{P}}$.

3.4. Power function of complex variable.

The power function of a complex variable (with the value $\alpha \in \mathbf{C}^*$) is defined as the continuous multi-valued function F defined on the set $\mathbf{C}^*$ of all nonzero complex numbers and satisfying the following conditions:

1) $F(z_1 z_2) = F(z_1)F(z_2)$ for all $z_1, z_2 \in \mathbf{C}^*$;

2) $F\left(\frac{z_1}{z_2}\right) = \frac{F(z_1)}{F(z_2)}$ for any $z_1, z_2 \in \mathbf{C}^*$;

3) $F(e^{\frac{1}{n}}) = e^{\frac{\alpha}{n} + 2\pi\alpha i\mathbf{Z}}$ and $F(e^{\frac{i}{n}}) = e^{\frac{\alpha}{n}i + 2\pi\alpha i\mathbf{Z}}$ for any natural number n.

Let us show that there are not two different continuous multi-valued functions defined on $\mathbf{C}^*$ and satisfying the conditions 1)-3). Suppose that a continuous multi-valued function is defined on and satisfies the axioms 1) -3). From the axiom 2) it follows that $0 \notin F(z)$, i.e. $F(z) \subseteq \mathbf{C}^*$ for any $z \in \mathbf{C}^*$.. Let $\Phi(z) := \frac{1}{\alpha}\operatorname{Ln} F(e^z)$. It is easy to see that the multi-valued function Φ satisfies the axioms of a linear multi-valued function for a complex variable and at that $\Phi(\frac{1}{n}) = \frac{1}{n} + 2\pi i\mathbf{Z} + \frac{1}{\alpha}2\pi i\mathbf{Z}$ and $\Phi(\frac{i}{n}) = \frac{i}{n} + 2\pi i\mathbf{Z} + \frac{1}{\alpha}2\pi i\mathbf{Z}$ for any natural number n. Therefore, $\Phi(z) = z + 2\pi i\mathbf{Z} + \frac{1}{\alpha}2\pi i\mathbf{Z}$ for any complex z. The logarithmic function of a complex variable is the inverse multi-valued function for the exponential function of a complex variable, then $F(e^z) = e^{\alpha\Phi(z)} = e^{\alpha z + 2\pi\alpha i\mathbf{Z}}$ for any $z \in \mathbf{C}$ and for any $F(z) = F(e^{\operatorname{Ln} z}) = e^{\alpha \operatorname{Ln} z}$. Thus, the conditions 1)-3) define a multi-valued function F quite clearly. The image of the point $z \neq 0$ at power function presentation may be designated as $\operatorname{Deg}^{\alpha} z$ and called the complex number degree z with the complex value $\alpha \neq 0$.

According to the definition of a polar separation for any $z \in \mathbf{C}^*$

$$\mathrm{Deg}^{\alpha} z = \exp(\alpha \ln \mathrm{Pr}_1 P(z) + i\alpha \mathrm{Pr}_2 P(z))$$

where $\mathrm{Pr}_1, \mathrm{Pr}_2$ is the projection operators $\mathbf{R}^2 \to \mathbf{R}$ for the first and second components, respectively. Therefore the power function of a complex variable α with the complex index may be regarded as a continuous single-valued function defined on the space factor of polar separation $(\mathbf{C}^*)_{\mathbf{P}}$. At that the number $\exp(\alpha \ln|z| + i\alpha(\mathrm{Arg}\, z)_{2k})$ corresponds with the point $(z)_k$. At $\alpha \in \mathbf{C} \setminus \mathbf{Q}$ in various points of the layer $(z)_{\mathbf{P}}$, this function takes different values. Therefore, the space factor of continuous representation $\{\exp(\alpha \ln|z| + i\alpha(\mathrm{Arg}\, z)_k) : k \in \mathbf{Z}\}$ (the Riemann surface of the power function) in this case coincides with the space factor of the polar separation.

Let $\alpha \in \mathbf{Q}$. We may assume that $\alpha = \frac{m}{n} \neq 0$, where $m \in \mathbf{Z}$, $n \in \mathbf{N}$ and NOD $(m,n) = 1$. The value of the power function at the point $z \in \mathbf{C}^*$ in this case is usually denoted as $z^{\frac{m}{n}}$. As we already mentioned, the function $z \to z^{\frac{m}{n}}$ may be viewed as a continuous single-valued function defined on the space factor of the polar separation. It is associated with the complex number $\exp(\frac{m}{n} \ln|z| + i\frac{m}{n}(\mathrm{Arg}\, z)_{2k})$ at the point $(z)_k$. This function takes different values in the points $(z)_0, \ldots (z)_{n-1}$ of the layer $(z)_{\mathbf{P}}$ and takes the same values at the points $(z)_k, (z)_{k+n} \in (z)_{\mathbf{P}}$ at any integer k. This means that at $n = 1$ the power function is a unique one and coincides with the rational function $z \to z^m$ restriction on the plane with the punctured point $\mathbf{C}^*$. If $n \neq 1$, then the space factor of the power function is obtained from the space factor of the polar separation with the topological identification (gluing) of points $(z)_k, (z)_{k+n} \in (z)_{\mathbf{P}}$ at any $z \in \mathbf{C}^*$ and every integer k.

3.5. Exponential function of complex variable.

Suppose that a continuous-valued function $F : \mathbf{C} \to \mathbf{C}$ is defined on $\mathbf{C}$, and satisfies the following conditions:

1) $F(z_1 + z_2) = F(z_1)F(z_2)$ for all $z_1, z_2 \in \mathbf{C}$;

2) $F(z_1 - z_2) = \frac{F(z_1)}{F(z_2)}$ for any $z_1, z_2 \in \mathbf{C}$;

3) $F(1) = e$, $F(\frac{\pi}{2} i) \subseteq e^{\frac{\pi}{2} i \mathrm{Ln}\, e}$ and $\mathrm{Re}\, F(yi) > 0$ for all $y \in (0, \frac{\pi}{2})$.

From the conditions 2) and 3) it follows that $F(0) = F(1-1) = \frac{F(1)}{F(1)} = 1$. Therefore, $\frac{F(z)}{F(z)} = F(z - z) = F(0) = 1$This means that the function F is a single-valued one. Let $F(\frac{\pi}{2} i) = ie^{-\pi^2 k}$. Let's consider the unique on $\mathbf{C}$ function

$$f(z) := e^{\mathrm{Ln} F(z) - z 2\pi i k}.$$

It is easy to verify that it satisfies the following conditions:

$f(z_1+z_2)=f(z_1)f(z_2)$ for all $z_1,z_2\in\mathbf{C}$, $f(1)=e^{\mathrm{Ln}e}=e$, $f(\frac{\pi}{2}i)=e^{\ln\left(ie^{-\pi^2k}\right)+\pi^2k}=i$ and $\mathrm{Re}\,f(yi)>0$ for all $y\in\left(0,\frac{\pi}{2}\right)$. According to the definition of a complex variable exponential function $f(z)=e^z$. Consequently, $F(z)e^{-z2\pi ik}=e^z$ and $F(z)=e^ze^{2\pi ikz}$. Thus, the conditions 1) -3) are determined on the family $\mathbf{C}$ of single-valued continuous functions $z\to\left(\mathrm{Exp}\,z\right)_k$, where

$$\left(\mathrm{Exp}\,z\right)_k:=e^{z\left(1+2\pi ik\right)},\ k\in\mathbf{Z}.$$

This means that the conditions 1) to 3) define a multi-valued function $z\to\mathrm{Exp}\,z$ where

$$\mathrm{Exp}\,z:=\left\{\left(\mathrm{Exp}\,z\right)_k:\ k\in\mathbf{Z}\right\}=e^{z\mathrm{Ln}e}=\mathrm{Deg}^z\,e,$$

which is commonly referred as a multi-valued exponential function of a complex variable with the base e. Suppose that a continuous multi-valued function $F:\mathbf{C}\to\mathbf{C}$ satisfies the conditions 1) and 2) and the condition

4) $F(1)=a$, $F(\frac{\pi}{2}i)\subseteq e^{\frac{\pi}{2}i\mathrm{Ln}a}$ and $\mathrm{Re}\exp\frac{\ln F(iy)}{\ln a}>0$ for all $y\in\left(0,\frac{\pi}{2}\right)$.

Here $a\in\mathbf{C}\setminus\{0;1\}$. It is easy to see that the conditions 1)-2) and 4) are determined on C family of continuous single-valued functions $z\to\left(\mathrm{Exp}_a\,z\right)_k$ where

$$\left(\mathrm{Exp}_a\,z\right)_k:=e^{z\left(\ln a+2\pi ik\right)},\ k\in\mathbf{Z}.$$

This means that the conditions 1)-2) and 4) define a continuous multi-valued function $z\to\left(\mathrm{Exp}_a\,z\right)_k$ on $\mathbf{C}$ where

$$\mathrm{Exp}_a\,z:=\left\{\left(\mathrm{Exp}_a\,z\right)_k:\ k\in\mathbf{Z}\right\}=e^{z\mathrm{Ln}a}=\mathrm{Deg}^z\,a,$$

which is commonly referred to as a multi-valued exponential function of a complex variable with the base a. The space factor of this function is obtained from multiple space $(\mathbf{C})_{\mathbf{Z}}$ with the topological identification (gluing) of $(z)_k$ and $(z)_n$ points and satisfying the condition $z\left(k-n\right)\in\mathbf{Z}$.

REFERENCES

[1] A. M. Legendre. Éléments de géométrie. Paris. 1794.

[2] G. F. Gauss, Theoria motus corporum coelestium, Hamburg, 1809, § 175, Werke 7, 240–244.

[3] A. L. Cauchy, Cours d'analyse de l'Ecole Polyt., 1. Analyse algebrique, Paris, 1821, 103; Oevres (2), 3, 98–105", 220.

[4] G. Hamel, Eine Basis aller Zahlen und die unstetigen Losungen der FunktionaL gleichung *f*(*x*+*y*) = *f*(*x*)+*f*(*y*), Math. Ann. 60 (1905), 459–462.

[5] N. H. Abel, Untersuchungen über die Reihe 1+mx/1+ m(m+1)x/2+..., Journ. reine und angew. Math. 1 (1826), 311–339; Oeuvres 1, 219–250.

[6] Kuczma, Marek. An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality. Basel: Birkhäuser, 2009.
[7] Introduction to Functional Equations: Theory and problem-solving strategies for mathematical competi-tions and beyond
Costas Efthimiou, University of Central Florida, Or-lando, FL, 2011.
[8] Henrik Stetkær, Functional Equations on Groups, first edition, World Scientific Publishing, 2013
[9] K. Kuratowski. Topology. Vol. I. – New York, 1966.
[10] J. L. Kelley. General topology. – New York, 1955.