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An Efficient Quadrature Rule For Approximate Solution Of Non Linear Integral Equation Of Hammerstein Type

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ABSTRACT

In this paper we apply an efficient quadrature (i.e mixed quadrature rule) of precision nine to find an approximate numerical solution of nonlinear integral equation of Hammerstein type through discrete adomain decomposition method(DADM). Some examples are given to illustrate the validity of our method taking absolute error for different components.

KEYWORDS- Adomain Decomposition Method, Discrete Adomain Decomposition Method, Hammerstein Integral Equation, Efficient quadrature.

1. INTRODUCTION

Non linear integral equations of Hammerstein type appear very often in many applications. For example it occurs in solving problems arising in economics, engineering and physics. One of the most important frequently investigated non linear integral equation of Hammerstein type. [1], [6], [9], [11], [16].

In this paper we study the problems for approximate solutions for the non linear integral equations of the Hammerstein type.

$$\lambda x(t) = y(t) + \int_a^b k(t,s) F(x(s))\,ds, \lambda \neq 0, a \leq t \leq b \qquad (1.1)$$

Adomain decomposition method (ADM) for solving integral equations has been presented by G. Adomain [7], [8]. In [3], [4], wazwaz extended ADM to solve Volterra integral equations and boundary value problems for higher order integro-differential equations. There are significant interests in applying the adomain decomposition method (ADM) for a wide class of non-linear integral equations. For example, ordinary and partial differential equation. In [14], Behiry etal. introduced a discrete version of the Adomain decomposition method and applied it to (1.1). This method is called a Discrete Adomain Decomposition Method (DADM). We use the advantage of the fact that the Bull's rule of precision five $(R_{B5}(f))$ and Cleanshaw-Curtis rule of precision five $(R_{CC5}(f))$ are linearly combined to form mixed quadrature rule of precision seven $(R_{CC5B5}(f))$. Again we mix the Cleanshaw-Curtis rule of precision seven $(R_{C7}(f))$ with the first mix up rule $(R_{CC5B5}(f))$ of precision seven to form another mixed quadrature rule(i.e so called efficient quadrature) of higher degree of precision nine. The present paper is designed in the following sections. Section II consists of basic idea of the above quadrature rules. Section III deals with DADM with new nodes. Section IV contains the formulation of efficient quadrature rule. The numerical examples are illustrated in Section V .

2. QUADRATURE RULES

In this section, we recall the definitions of two rules, namely Cleanshaw-Cutris quadrature rule and Bull's quadrature rule. Cleanshaw-Cutris quadrature rule is based on expansion of the integrand in terms of Chebyshev polynomials. So we have to know some facts about Chebyshev polynomials. It is worth mentioning that Chebyshev polynomials are everywhere dense in numerical analysis [10]. The Chebyshev polynomials $T_n(x)$ of the first kind is a polynomial in x of degree n denoted by the following relation,

$$T_n(x) = \cos n\theta, where\, x = \cos\theta \tag{2.1}$$

From formula (2.1), the zero for x in $[-1, 1]$ of $T_n(x)$ must correspond to the zeros for θ in $[0, \pi]$ of $\cos n\theta$ so that

$$n\theta = (j-1)\frac{\pi}{2},\ j = 1, 2, 3, \ldots. n \tag{2.2}$$

Hence the zeros of $T_n(x)$ are

$$x_j = \frac{\cos(j-1)\pi}{2n},\ j = 1, 2, 3, \ldots. n \tag{2.3}$$

The internal extrema of $T_n(x)$ corresponds to the external value of $\cos n\theta$, namely the zeros of $\sin n\theta$, since $\left(\frac{d}{dx}\right)T_n(x) = \frac{\sin n\theta}{\sin\theta}$. Hence including those at $x = \pm 1$, the extrema of $T_n(x)$ on $[-1, 1]$ are

$$x_j = \cos\left(\frac{j\pi}{n}\right), \; j = 1, 2, 3, \ldots. n \tag{2.4}$$

The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial in x of degree n, defined by the following relation

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \; where \;\; x = \cos\theta \tag{2.5}$$

The zeroes of $U_n(x)$ are given by

$$x_j = \cos\left(\frac{j\pi}{n+1}\right), \; j = 1, 2, 3, \ldots. n \tag{2.6}$$

2.1. CLEANSHAW-CURTIS QUADRATURE-

Cleanshaw-Curtis quadrature method proposed by Cleanshaw and Curtis [5] amounts to integrating via a change of variable $x = \cos\theta$. The algorithm is normally expressed for integration of a function $f(x)$ over the interval $[-1, 1]$. Any other interval can be obtained by appropriate rescaling. For this integral, we can write

$$\int_{-1}^{1} f(x)\,dx = \int_{0}^{\pi} f(\cos\theta)\sin\theta\, d\theta \tag{2.7}$$

That is we have to transform the problem from integrating $f(\cos\theta)\sin\theta$. This can be performed if we know the cosine series for $f(\cos\theta)$. The reason is that this is connected to Chebyshev polynomials $T_j(x)$, by (2.1), $T_j(\cos\theta) = \cos(j\theta)$, and so the cosine series is really an approximation of $f(x)$ by Chebyshev polynomials.

$$f(x) = \frac{a_0}{2} T_0(x) + \sum_{j=1}^{n} a_j T_j(x), \qquad x \in [-1, 1] \tag{2.8}$$

and thus we are integrating $f(x)$ by integrating its approximate expansion in terms of Chebyshev polynomials. The evaluation points $x_j = \cos\left(\frac{j\theta}{n}\right)$ correspond to the extrema of Chebyshev polynomials $T_n(x)$; see [6]. The fact that such Chebyshev approximation is just a cosine series under a change of variables is responsible for the rapid convergence of the approximation as more terms $T_j(x)$ are included. A cosine series converges very rapidly for functions that are even, periodic and sufficiently smooth. This is true here since $f(\cos\theta)$ is even and periodic in θ by construction, and is j times differentiable everywhere if $f(x)$ is j times differentiable on $[-1, 1]$.

3. DISCRETE ADOMAIN DECOMPOSITION METHOD

The solution $x(t)$ of (1.1) when applying ADM is expressed in a series form define by,

$$x(t) = \sum_{m=0}^{\infty} x_m(t) \tag{3.1}$$

Where the component $x_m(t), m \geq 0$ can be computed as will be shown next. The non linear term $F(x(t))$ of the equation (1.1) should be represented, using a distinct scheme [13], by the so called Adomain polynomials $A_m(t)$ as

$$F(x(t)) = \sum_{m=0}^{\infty} A_m(x_0(t), x_1(t), \ldots\ldots x_m(t)) \tag{3.2}$$

Where $A_m(t)$ can be evaluated by the following formula, [13]

$$A_m(x_0(t), x_1(t), \ldots\ldots x_m(t)) = \frac{1}{m!}\frac{d^m}{d\alpha^m}\left[F\left(\sum_{m=0}^{\infty}\alpha^m x_m\right)\right]_{\alpha=0} \tag{3.3}$$

The Adomain polynomials are arranged into the form

$$A_0 = F(x_0) \tag{3.4}$$

$$A_1 = x_1 F^{i}(x_0) \tag{3.5}$$

$$A_2 = x_2 F^{i}(x_0) + \frac{1}{2!}x_1^{2} F^{ii}(x_0) \tag{3.6}$$

$$A_3 = x_3 F^{i}(x_0) + x_1 x_2 F^{ii}(x_0) + \frac{1}{3!}x_1^{3} F^{iii}(x_0) \tag{3.7}$$

$$A_4 = x_4 F^{i}(x_0) + \left(\frac{1}{2!}x_1^{2} + x_1 x_3\right)F^{ii}(x_0) + \frac{1}{2!}x_1^{2} x_2 F^{iii}(x_0) + \frac{1}{4!}x_1^{4} F^{iv}(x_0) \tag{3.8}$$

Substituting (3.1) and (3.2) into (1.1), we obtain

$$\sum_{m=0}^{\infty} x_m(s) = \frac{1}{\lambda}\left[y(s) + \sum_{m=0}^{\infty} \int_a^b K(s,t) A_m(t)\, ds \right] \tag{3.9}$$

The components $x_m(s), m \geq 0$ are to be computed using the following recursive relation [14];

$$x_0(s) = \frac{1}{\lambda}[y(s)] \tag{3.10}$$

$$x_{m+1}(s) = \frac{1}{\lambda}\int_a^b K(s,t) A_m(t)\, ds, m \geq 0 \tag{3.11}$$

It is noticed that computation of each component $x_m(s), m \geq 0$ requires the computation of an interval in (3.11). If the evaluation of integrals are analytically possible, the ADM can be applied in a simple manner. In the cases where the evaluation of integral (3.11) is analytically impossible, the ADM cannot be applied. In order to use numerical integration for integral in (3.11), we transform the interval $[a, b]$ to $[-1, 1]$ by using the transformation,

$$x = \frac{1}{2}[(a+b) + (b-a)t] \tag{3.12}$$

Now we will make use of the above two quadrature rules.

4. FORMULATION OF AN EFFICIENT QUADRATURE RULE $(R_{CC5B5CC7}(f))$

All the rules are implemented for the approximate evaluation of

$$I(f) = \int_{-1}^{1} f(x)\, dx \tag{4.1}$$

Keeping in the mind [15], here the two rules namely Cleanshaw-Curtis $(R_{CC5}(f))$ and Bull's $(R_{B5}(f))$ rule each of precision five are mixed to form a rule of precision seven. We consider the Cleanshaw-Curtis rule of degree of precision five

$$R_{CC5}(f) = \frac{1}{15}\left[f(-1) + 8f\left(-\frac{1}{\sqrt{2}}\right) + 12f(0) + 8f\left(\frac{1}{\sqrt{2}}\right) + f(1) \right] \tag{4.2}$$

and Bull's rule of degree of precision five

$$R_{B5}(f) = \frac{1}{45}\left[7f(-1) + 32f\left(-\frac{1}{2}\right) + 12f(0) + 32f\left(\frac{1}{2}\right) + 7f(1)\right] \tag{4.3}$$

Taking the convex combination of (4.2) and (4.3) we obtain the mixed quadrature rule of precision seven for the approximate evaluation of (4.1) namely,

$$R_{CC5B5}(f) = \frac{1}{7}\left[R_{CC5}(f) + 2R_{B5}(f)\right] \tag{4.4}$$

Equation (4.4) can be written in analytic form

$$R_{CC5B5}(f) = \frac{1}{315}\left[\begin{array}{l} 29\{f(-1) + f(1)\} + 120\left\{f\left(-\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{2}}\right)\right\} \\ + 64\left\{f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right)\right\} + 204f(0) \end{array}\right] \tag{4.5}$$

The mixed quadrature rule (i.e. the so called efficient quadrature rule $R_{CC5B5CC7}(f)$) of degree of precision nine is formulated by linear combination of (4.5) and (4.6) is of degree of precision seven for the approximate evaluation of (4.1). The Cleanshaw-Curtis rule of precision seven is

$$R_{CC7}(f) = \frac{1}{315}\left[\begin{array}{l} 9\{f(-1) + f(1)\} + 80\left\{f\left(-\frac{\sqrt{3}}{2}\right) + f\left(\frac{\sqrt{3}}{2}\right)\right\} \\ + 144\left\{f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right)\right\} + 164f(0) \end{array}\right] \tag{4.6}$$

We obtain an efficient quadrature rule i.e.

$$R_{CC5B5CC7}(f) = \frac{1}{15}\left[R_{CC5B5}(f) + 14R_{CC7}(f)\right] \tag{4.7}$$

Equation (4.7) can also be written in analytic form as,

$$R_{CC5B5CC7}(f) = \frac{1}{4725}\left[\begin{array}{l} 155\{f(-1)+f(1)\}+1120\left\{f\left(-\frac{\sqrt{3}}{2}\right)+f\left(\frac{\sqrt{3}}{2}\right)\right\} \\ +2080\left\{f\left(-\frac{1}{2}\right)+f\left(\frac{1}{2}\right)\right\}+ \\ 120\left\{f\left(-\frac{1}{\sqrt{2}}\right)+f\left(\frac{1}{\sqrt{2}}\right)\right\}+2500 f(0) \end{array}\right] \tag{4.8}$$

5. NUMERICAL EXAMPLES

In this section we are applying our methods to some non linear integral equations of the Hammerstein type (1.1). These examples show the efficiency and accuracy of our method. The tables show computed absolute error for different m.

$$|e_m(t)| = |X_{exact}(t) - X_{approx}(t)| \tag{5.1}$$

Where m is the number of components $x_1, x_2, x_3, ..., x_m$. The computation associated with different examples is numerically evaluated using Matlab Format Long.

EXAMPLE1:

Consider the non linear integral equation of Hammerstein type

$$10\,x(t) = 10t - \frac{1}{4}(e-1)\exp(t^4) + \int_0^1 \exp(s^4 + t^4)(x(s))^3\, ds \tag{5.2}$$

Here

$\lambda = 10$, $y(t) = 10t - \frac{1}{4}(e-1)\exp(t^4)$

$K(t,s) = \exp(s^4 + t^4)$ and $F(s, x(s)) = (x(s))^3$

Equation (5.2) has an exact solution $x_e(t) = t$, [2]

Let $x_0(t) = \frac{y(t)}{\lambda} = t - \frac{1}{40}(e-1)\exp(t^4)$

Table (1) and table (2) display the numerical solution of example-1.

Table 1- $(R_{CC5B5}(f))$ *The effect of* m *in the absolute error at* $n=7$

t	$\|e_3(t)\|$	$\|e_4(t)\|$	$\|e_5(t)\|$
0.00000	1.24198×10^{-3}	4.05460×10^{-4}	7.87349×10^{-5}
0.06699	1.24200×10^{-3}	4.05493×10^{-4}	7.87364×10^{-5}
0.25000	1.24684×10^{-3}	4.07047×10^{-4}	7.90430×10^{-5}
0.50000	1.28422×10^{-3}	4.31610×10^{-4}	8.38128×10^{-5}
0.75000	1.69207×10^{-3}	5.56367×10^{-4}	1.08039×10^{-4}
0.93301	2.64982×10^{-3}	8.65066×10^{-4}	1.67984×10^{-4}
1.00000	3.37606×10^{-3}	1.10215×10^{-3}	2.14023×10^{-4}

Table 2- $(R_{CC5B5CC7}(f))$ *The effect of* m *in the absolute error at* $n=9$

t	$\|e_3(t)\|$	$\|e_4(t)\|$	$\|e_5(t)\|$
0.00000	1.34640×10^{-3}	5.36810×10^{-4}	2.25173×10^{-4}
0.06699	1.34643×10^{-3}	5.36823×10^{-4}	2.25177×10^{-4}
0.14645	1.34702×10^{-3}	5.37057×10^{-4}	2.52276×10^{-4}
0.25000	1.35167×10^{-3}	5.38911×10^{-4}	2.26054×10^{-4}
0.50000	1.43671×10^{-3}	5.71432×10^{-4}	2.39695×10^{-4}
0.75000	1.84670×10^{-3}	7.36664×10^{-4}	3.08979×10^{-4}
0.85355	2.30191×10^{-3}	9.12718×10^{-4}	3.82853×10^{-4}
0.93301	2.87260×10^{-3}	1.14530×10^{-3}	4.80415×10^{-4}
1.00000	3.68552×10^{-3}	1.45920×10^{-3}	6.12083×10^{-4}

EXAMPLE 2:

Consider the non linear integral equation of Hammerstein type

$$20x(t) = 20t + \cos(e+t) - \cos(1+t) + \int_0^1 \exp(x(s))\sin(t+e^s)\,ds \qquad (5.3)$$

Here

$\lambda = 20$, $y(t) = 20t + \cos(e+t) - \cos(1+t)$

$K(t,s) = \sin(t+e^s)$ and $F(x(s)) = \exp(x(s))$

Equation (5.3) has an exact solution $x_e(t) = t$, [2]

Let $x_0(t) = \dfrac{y(t)}{\lambda} = t + \dfrac{1}{20}\left(\cos(e+t) - \cos(1+t)\right)$

Table (3) and table (4) display the numerical solution of example-2.

Table 3- $(R_{CC5B5}(f))$ *The effect of* m *in the absolute error at* $n = 7$

t	$\|e_1(t)\|$	$\|e_2(t)\|$	$\|e_3(t)\|$
0.00000	3.47613×10^{-3}	2.73309×10^{-4}	5.22558×10^{-5}
0.06699	3.43977×10^{-3}	2.70622×10^{-4}	5.25980×10^{-5}
0.25000	3.26251×10^{-3}	2.57146×10^{-4}	5.23295×10^{-5}
0.50000	2.84605×10^{-3}	2.24994×10^{-4}	4.91496×10^{-5}
0.75000	2.25264×10^{-3}	1.78853×10^{-4}	4.29139×10^{-5}
0.93301	1.72699×10^{-3}	1.37835×10^{-4}	3.66288×10^{-5}
1.00000	1.51916×10^{-3}	1.21592×10^{-4}	3.40099×10^{-5}

Table 4- $(R_{CC5B5CC7}(f))$ *The effect of* m *in the absolute error at* $n = 9$

t	$\|e_1(t)\|$	$\|e_2(t)\|$	$\|e_3(t)\|$
0.00000	3.47781×10^{-3}	2.75000×10^{-4}	5.05684×10^{-5}
0.06699	3.44143×10^{-3}	2.72284×10^{-4}	5.09396×10^{-5}
0.14645	3.37828×10^{-3}	2.67479×10^{-4}	5.10833×10^{-5}
0.25000	3.26408×10^{-3}	2.58690×10^{-4}	5.11491×10^{-5}
0.50000	2.84740×10^{-3}	2.26296×10^{-4}	4.78481×10^{-5}
0.75000	2.25369×10^{-3}	1.79832×10^{-4}	4.19342×10^{-5}
0.85355	1.96429×10^{-3}	1.57120×10^{-4}	3.86941×10^{-5}
0.93301	1.72778×10^{-3}	1.38537×10^{-4}	3.59246×10^{-5}
1.00000	1.51985×10^{-3}	1.22187×10^{-4}	3.34130×10^{-5}

6. CONCLUSION

We have solved the non linear integral equation of Hammerstein type by an efficient quadrature $(R_{CC5B5CC7}(f))$ using DADM. The approximate results of integral equation obtained by our proposed method indicate that our method is remarkably successful numerical technique for solving non linear integral equation of Hammerstein type.

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