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On Commutative Ternary Semigroups Of Principal Ideals

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Abstract:

The class of ternarysemigroups under the title includes the finitely generated commutative ternarysemigroups and noetherian commutative ternarysemigroups. We develop here some properties of noetherian ternarysemigroups related to primeideal structure.

Key words: Principal ideal, primary ideal, *T*-primary ideal, noetherian ternarysemigroups.

Introduction and preliminaries:

Throughtout this paper all ternarysemigroups under consideration are commutative. An ideal A in a ternarysemigroup T is said to be finitely generated if $A=\bigcup_{i=1}^{n}(x_i \cup x_iTT)=\bigcup_{i=1}^{n} x_iT^1T^1$. It is called a principal ideal or is principally generated if $A=xT^1T^1$ for some $x\in T$. T can be treated as an ideal and every ideal different from T is called a proper. A ternarysemigroup T is called a noetherian ternarysemigroup if every increasing chain of ideals teriminates at a finite state or equivalentely each ideal is finitely generated. T is called finitely generated if there exists $x_1, x_2, ...x_n$ in T such that every element is a product of powers of x_i's. An ideal A is primary (prime) if $xyz\in A$ and $x\notin A, y\notin A$ then for some odd integer n, $z^n\in A$ $(z\in A)$. For any ideal A in a ternarysemigroup T, $\sqrt{A} = \{x \in T : x^n \in A$ for some odd integer $n\}$. If A is a primary ideal, then $\sqrt{A}$ is a primeideal. An ideal A is called T-primary if $\sqrt{A}$ = T. It can be shown that every ideal in a noetherian ternarysemigroup is an intersection of finite number of primaryideals. The ternarysemigroup $T = \{x_i\}_{i \in N}$ with maximum multiplication is a noetherian ternarysemigroup and T is principal ideal. But T is not a finitely genetared ternarysemigroup. So one will be interested in knowing which noetherian ternarysemigroups are finitely genetared. The result in this paper is the supplement the works of Anjaneyulu.A [1] and Satyanarayana.M [6].

Primary ideals in Ternary semigroups

Lemma 1.1. Let H be the collection of all ideals in a ternarysemigroup T, which are not principal. If $H \neq \phi$ then there exists a primeideal which is not prinicipal.

***Proof*:** We shall prove the theorem when no ideal in H is principal. Similar proof can be given for finitely generated case. Let $\{A_\alpha\}$ be a chain of ideals in H. If $\bigcup A_\alpha = xT^1T^1$, then $A_\alpha = xT^1T^1$ for some α, which is not true. So $\bigcup A_\alpha \in H$. Then by the application of Zorn's lemma to H a maximal element P in H is guaranteed. Now the proof is completed by showing that P is a primeideal. Suppose that P is not a primeideal. Then there exist $a, b, c \notin P$ and $abc \in P$. By maximality of $P, P \cup bT^1T^1 = xT^1T^1$, which implies $x \in P$ or $x \in bT^1T^1$. If $x \in P$ then $P = xT^1T^1$ which is not ture. So, if $x \in bT^1T^1$, then $P \subseteq bT^1T^1$. Since $abc \in P$ and $a \notin P$, $P : bT^1T^1 = \{t : bT^1T^1tt \subseteq P\}$ is an ideal containing P properly. Again by the maximality of P, $P : bT^1T^1 = yT^1T^1$. Now we assert $P = byT^1$, which is evidently a contradiction. Clearly $byT^1 \subseteq P$. Now if $t \in P$, $t \in bT^1T^1$ and so t = brs, since $t \neq b$. But $brs \in P$, so that $r \in P : bT^1T^1 = yT^1T^1$. Thus $t \in byT^1$ and hence $P \subseteq byT^1$.

An immediate consequeness of 1.1 is

Corollary 1.2. If every primeideal including T is principal in a ternarysemigroup T, then every ideal in T is principal.

Lemma 1.3: Let T be a ternarysemigroup, which is a union of a finite number of principal ideals. If every proper primeideal is principal, then the following are true.

a) Every ideal is an intersection of a principal ideal and an T-primary ideal.

b) If $T = T^3$ then every proper ideal is principal.

***Proof*:** We now prove firstly, every primaryideal $Q \ni \sqrt{Q} \neq T$ is a principal ideal. By hypothesis the proper primeideal $P = \sqrt{Q}$ is of the form aT^1T^1 for some $a \in T$. This implies that $\exists$ a natural number $r \ni a^r \in Q$. Therefore $P^r = a^rT^1T^1 \subseteq Q$. In the case when Q is accommodated in every power of P we have $Q = P^r = a^rT^1T^1$. On the another hand let $\exists$ an odd natural number $m \ni Q \subseteq P^m$ and $Q \nsubseteq P^{m+2}$. Since P^m is a principal ideal, $Q = P^m AB$ for some ideals A, B and $Q \nsubseteq P^{m+2}$ implies that $A \nsubseteq P$, $B \nsubseteq P$. Since Q is P- primary we must have that $P^m \subseteq Q$. So that $Q = P^m$ and hence Q is principal. Now by 1.2, T is noetherian and so any arbitrary ideal A is of the form $Q_1 \cap Q_2 \cap ... \cap Q_n$ where Q_i's are primaryideals $\ni$ $P_i = \sqrt{Q_i} \neq \sqrt{Q_j} = P_j$ for $i \neq j$. We may assume $P_i \neq T$ for some i =1,2,...m and $P_i = T$ for $m+2 \leq i \leq n$. Clearly $\sqrt{(Q_{m+2} \cap ... \cap Q_n)} = T$ and hence $Q_{m+2} \cap ... \cap Q_n$ is a T-primaryideal. Now we claim that $Q_1 \cap Q_2 \cap ..... \cap Q_m = Q_1 Q_2 ..... Q_m$ which proves that $Q_1 \cap Q_2 \cap ... \cap Q_m$ is a principal ideal. Since every one of $Q_1, Q_2, ..., Q_m$ is a principal ideal. This establishes (*a*). For this order these P_i's $1 \leq i \leq m$ so that we can assume without loss of generality that P_1 is large in $\{P_i\}_1^m$, P_2 maximal in $\{P_i\}_2^m$ and so forth. This means no $P_i \subseteq P_j$ for $i \neq j$.

Now suppose for $r < m$, $Q_1 \cap Q_2 \cap ... \cap Q_r = Q_1 Q_2 ... Q_r$.

Then $Q_1 \cap Q_2 \cap ... \cap Q_{r+2} = (Q_1 \cap Q_2 \cap ... \cap Q_r) \cap Q_{r+2} = aT^1T^1 \cap Q_{r+2}$ for some $a \in T$, since every one of $Q_1, Q_2, ... Q_r$ is principal. Let $x = ayz \in Q_{r+2}$. By the choice of P_i's, $a \notin P_{r+2}$. Since $a \in P_{r+2}$ implies $\sqrt{(aT^1T^1)} = \sqrt{Q_1 \cap Q_2 \cap ... \cap Q_r} = P_1 \cap P_2 \cap ... \cap P_r$ and thus $P_i \subseteq P_{r+2}$ for $i < r+2$, which is not true. Hence $y \in Q_{r+2}$. Since Q_{r+2} is a primaryideal such that $\sqrt{Q_{r+2}} = P_r$. Thus $aT^1T^1 \cap Q_{r+2} \subseteq aT^1T^1 \cdot Q_{r+2}$, which implies $aT^1T^1 \cap Q_{r+2} = aT^1T^1 \cdot Q_{r+2}$. Therefore by induction, $Q_1 \cap Q_2 \cap ... \cap Q_m = Q_1 Q_2 ... Q_m$ where m is an odd natural number. To show (*b*), it suffices to prove that there are no proper T-primaryideals by justice of (*a*). We can write $T = \bigcup_{i=1}^{n} x_i T^1 T^1$ where $x_i \notin x_j T^1 T^1$ for $i \neq j$. Then the condition $T = T^3$ implies that $x_i \in x_i^3 T^1 T^1$ for every i and so $x_i T^1 T^1 = e_i T^1 T^1$ where e_i is an idempotent. Thus $T = \bigcup_{i=1}^{n} e_i TT$. Now, if A is a proper ideal such that $\sqrt{A} = T$, then $e_i^{n_i} \in A$ for some n_i, so that $A = T$ which is not true.

Lemma 1.4: Let T be a ternary semigroup in which $T \neq T^3$ and every maximalideal is principal. Then *T* has atmost two maximalideals and for any proper prime ideal *P*, either *P* is a principal ideal or *P* = *xyP* for some $x, y \in T$.

***Proof*:** Let $a \in T \setminus T^3$. Then $T \setminus a$ is a maximalideal and so by hypothesis $T \setminus a = bT^1T^1$. Clearly $b \neq a$. Let $b \in T^3$. Thus $T \setminus a = T^3$. If $M = cT^1T^1$ is any maximalideal and if $c \in T^3$, then $M \subseteq T^3$ and $M = T^3 = T \setminus a$. Now if $c \notin T^3$, then $c \notin T \setminus a$ so that *c* = *a*. Thus $M = aT^1T^1$. Hence in this case when $b \in T^3, T$ can have atmost two maximalideals, namely $T \setminus a$ and aT^1T^1. Let $b \notin T^3$. Then $T = a \cup bT^1T^1 = a \cup b \cup T^3$. We claim that $T \setminus a$ and $T \setminus b$ are the only two maximalideals. If $M = cT^1T^1$ is a maximalideals, then consider the case when $c \notin T^3$. This implies *c* = *a* or *b*, so that $M = T \setminus a$ or $T \setminus b$. The case that $c \in T^3$ is in admissible, since otherwise $M = T^3$, which implies that the maximalideal T^3 is contained properly in the maximal ideal $T \setminus a$.

To prove the second part consider any proper primeideal *P*. If $a \notin P$, then $P \subseteq T \setminus a = bT^1T^1$. This implies that $P = bT^1T^1$ if $b \in P$ and *P* = *bPP* if $b \notin P$ since *P* is a prime ideal. Let $a \in P$. If $b \in P$ also, then *P* = *T*. If $b \notin P$ then $P \subseteq T \setminus b$. In the first part we have proved $T \setminus b$ is a maximalideal and so $T \setminus b = xT^1T^1$ for some *x*. Then as before $P = xT^1T^1$ or *P* = *xPP*.

Theorem 1.5: Let every maximalideal in a ternarysemigroup *T* be principal. If $T \neq T^3$ and $\bigcap_{n=1}^{\infty} x^n TT = \phi$ for every $n \in T$ then *T* is a union of three principal ideals and every ideal is an intersection of a principal ideal and an *T*-primary ideal.

***Proof*:** By Lemma 1.4, every proper primeideal is principal. If $a \in T \setminus T^3$, then by hypothesis, the maximalideal $T \setminus a$ is of the form bT^1T^1 for some $b \in T$. Therefore $T = a \cup bT^1T^1 = aT^1T^1 \cup bT^1T^1$. Now the conclusion is evident from 1.3.

Theorem 1.6: Let *T* be a noetherian ternarysemigroup such that $T = \bigcup_{i=1}^{n} x_i T^1T^1$. Suppose $a \notin x_i aT^1$ for all $a \in T$, which is not a product of powers of x_i's. Then *T* is finitely generated. In particular if *T* is a noetherian cancellative ternarysemigroup without identity, then *T* is finitely generated.

***Proof*:** Suppose there exists an element a such that a is not a product of x_i 's. Then $a = x_i t_1 t_1$, where $a \neq t_1$ and t_1 is not a product of powers of x_i 's. Hence $t_1 = x_j t_2 t_2$. If $t_2 \in t_1 TT$ or $t_2 = t_1$, then we have $t_1 = x_j t_1 T^1$ which is not true by hypothesis. Thus $t_1 T^1 T^1$ is properly accommodated in $t_2 T^1 T^1$. Proceeding in this manner, we have a non-terminating chain of ideals, $t_1 T^1 T^1 \subset t_2 T^1 T^1 \subset ......$ this is impossible by the noetherian condition. The second assertion follows now immediately by noting that in cancellative ternarysemigroups, the condition $a = aab$ implies that b is an identity.

Proposition 1.7: Let T be a ternarysemigroup which is a union of finite number of ideals. Then T contains idempotents if $T = T^3$. If T is cancellative, then T contains an identity if and only if $T = T^3$.

Proof: Let $T = \bigcup_{i=1}^{n} x_i T^1 T^1$ with $x_i \notin x_j \cup x_j TT$ for $i \neq j$. Since $T = T^3$, $\bigcup_{i=1}^{n} x_i T^1 T^1$ $= \bigcup_{i=1}^{n} (x_i^3 \cup x_i x_j T)$ which implies $x_i = x_i^3$ or $x_i = x_i^3 tt$ for every *i*. Thus T contains idempotents. If T is cancellative, T can have atmost one idempotent, which is the identity itself. Hence the second part is evident.

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