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Existence results for Hadamard type fractional functional integro-differential equations with integral boundary conditions

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Abstract

The proposal of tsshis paper, gives the study of existence results that involves Hadamard-type fractional functional integro-differential equation. Our examined results implies some well-defined fixed points theorems. Combined usage of integral boundary conditions also gives the results involving integro-differential equations. The present results has been developed the recent results on this issue.

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1. Introduction

The main objective behind this approach is to investigate the following Hadamard boundary value problem

$$D^q y(t) = f\left(t, y(t), (T_1 y)(t), (T_2 y)(t)\right), \quad 1 < q \leq 2, \quad t \in (1, e), \tag{1.1}$$

$$y(1) = 0, \; \sum_{i=1}^{m} \lambda_i J^{\alpha_i} y(\eta_i) = \sum_{j=1}^{n} \mu_j \left(J^{\beta_j} y(b) - J^{\beta_j} y(\xi_j)\right), \tag{1.2}$$

where D^q denotes the Hadamard fractional derivative of order q, $f : [1, e] \times R \to R$ is a continous function, $\eta_i, \xi_j \in (1, e), \lambda_i, \mu_j \in R$, for all $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, \eta_1 < \eta_2 < \cdots < \eta_m, \xi_1 < \xi_2 < \cdots < \xi_n$, and J^{ϕ} is the Hadamard fractional integral of order $\phi > 0$, $(\phi = \alpha_i, \beta_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$.

Here T_1, T_2 are integral operators given by

$$(T_1 y)(t) = \int_0^t k_1(t, s) y(s) ds,$$

$$(T_2 y)(t) = \int_0^t k_2(t, s) y(s) ds,$$

with

$$\gamma_0 = max \int_0^t k_1(t, s) ds,$$

$$\delta_0 = max \int_0^t k_2(t, s) ds,$$

$$k_1, k_2 \in C([1, e] \times [1, e], R^+).$$

The integral boundary conditions that are specified previously are used in several applications such as population dynamics, blood flow model, chemical engineering, cellular systems, heat transmission, plasma physics, termoelasticity, etc.

The condition (1.2) is a general form of the integral boundary conditions considered in [2] and covers many special cases. For example, if $\alpha_i = \beta_j = 1$, for all $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, then the condition (1.2) reduces to

$$\begin{aligned} y(1) = 0, \lambda_1 \int_1^{\eta_1} y(s) \frac{ds}{s} + \cdots + \lambda \int_1^{\eta_m} y(s) \frac{ds}{s} \\ = \mu_1 \int_{\xi_1}^{e} y(s) \frac{ds}{s} + \cdots + \mu_n \int_{\xi_n}^{e} y(s) \frac{ds}{s}. \end{aligned}$$

Fractional differential eqauations serves as an appropriate phenomenon such that it can even describe the real world problems which is impossible using classical integer order differential equations. Over the past decades, the theory of fractional differential equation deserves more attention, and has obtained a prior position in the field of investigation. Finally, it has with standed strongly due to its advanced applictions in various branches of physics, economics and Engineering sciences [23, 24, 26, 31] and the papers of [27, 33, 34, 17, 29, 30]. A few researches recently contributed ones to the subject can be seen in [7, 1, 8, 9, 2, 25, 3, 16, 15, 6, 28, 10] and references also can be seen there.

It has been observed that major part on the topic covers Riemann-Liouville and Caputo type fractional differential equation mean while there appears another kind of fractional derivative side by side to Riemann-Liouville and Caputo derivative in the literature which is the fractional derivative due to Hadamard introduced in 1892 [19].

The above said concept differs from the previous ones such that the kernal of the integral involve contains logarithmic function of aribtrary exponent. The most preferable properties and ideas of Hadamard fractional derivative and integral can be found in [23, 11, 12, 13, 20, 21]. For some recently examined results on Hadamard boundary value problem we refer to [4, 5] and references therein.

Establishment of various results for the problem (1.1)-(1.2) by using well-defined fixed point theorems are made-successfully. The first result, depends on Banach contraction mapping principle and depicts about the existence and uniqueness results for the solutions of the problem (1.1)-(1.2). The second result, has also proved the existence and uniqueness phenomen through non-linear contractions and a fixed point theorem due to Boyd and Wong. The same results of existence are again proved in the third result, by using Krasnoselskii fixed point theorem.

The following criteria in this paper is as follows. In section 2, we summarize few preliminary concepts that concerns the sequel and preliminary lemma is proved. Section 3, has got the preferable results for the problem $(1.1)-(1.2)$. In section 4, some illustrative examples are explained.

2. Preliminaries

In this section, we introduce some notations and defintions of fractional calculus [23, 24, 26] and present preliminary results needed in our proofs later.

Definition 2.1. The Hadamard derivative of fractional order q for a function $f : [1, \infty) \to R$ is defined as

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(log\frac{t}{s}\right)^{n-q-1} \frac{f(s)}{s} ds, \quad n-1 < q < n, \quad n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q, $log(.) = log_e(.)$ and Γ is the Gamma function.

Definition 2.2. The Hadamard fractional integral $f : [1, \infty) \to R$ is defined by

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_1^t \left(log\frac{t}{s}\right)^{q-1} \frac{f(s)}{s} ds, \quad q > 0.$$

provied the integral exists.

For convenience, we set

$$\Pi = \sum_{i=1}^{m} \lambda_i \frac{\Gamma(q)}{\Gamma(q+\alpha_i)} (log\eta_i)^{q+\alpha_i-1} - \sum_{j=1}^{n} \mu_j \frac{\Gamma(q)}{\Gamma(q+\beta_j)} \left(1-(log\xi_j)^{q+\beta_j-1}\right). \tag{2.1}$$

Proposition 2.3. If $R(\alpha) > 0$, $R(\beta,\ and\ 0 < a < b < \infty$, then

$$\left(J_{a+}^{\alpha}\left(log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(log\frac{x}{a}\right)^{\beta+\alpha-1},$$
$$\left(D_{a+}^{\alpha}\left(log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(log\frac{x}{a}\right)^{\beta-\alpha-1},$$
$$and \left(J_{b-}^{\alpha}\left(log\frac{b}{t}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(log\frac{b}{x}\right)^{\beta+\alpha-1},$$
$$\left(D_{b-}^{\alpha}\left(log\frac{b}{t}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(log\frac{b}{x}\right)^{\beta-\alpha-1}.$$

Lemma 2.4. **[32]** Let $\Pi \neq 0, 1 < q \leq 2, \alpha_i, \beta_j > 0, \eta_i, \xi_j \in (1,e)$ for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ and $h \in C([1,2],R)$. The unique solution of the following fractional differential equation.

$$D^q y(t) = h(t), \quad t \in (1,e),$$

subject to the boundary condition

$$y(1) = 0, \quad \sum_{i=1}^{m} \lambda_i J^{\alpha_i} y(\eta_i) = \sum_{j=1}^{n} \mu_j \left(J^{\beta_j} y(e) - J^{\beta_j} y(\xi_j)\right),$$

is given by the integral equation

$$y(t) = \frac{(logt)^{q-1}}{\Pi} \sum_{j=i}^{n} \mu_j \left(J^{q+\beta_j} h(e) - J^{q+\beta_j} h(\xi_j)\right) - \frac{(logt)^{q-1}}{\Pi} \sum_{i=1}^{m} \lambda_i J^{q+\alpha_i} h(\eta_i) + J^q h(t).$$

3. Main results

Let $C = ([1,e],R)$ denotes the Banach space of all continuous functions from $[1,e]$ to R endowed with the norm defined by $\|y\| = \sup_{t\in[1,e]} |y(t)|$. As in Lemma 2.1, we define an operator $F : C \to C$ by

$$\begin{aligned}(Fy)(t) = {} & J^q f\,(s, y(s), (T_1 y)(s), (T_2 y)(s))\,(t) \\ & - \frac{(logt)^{q-1}}{\Pi} \sum_{i=1}^{m} \lambda_i J^{\alpha_i+q} f\,(s, y(s), (T_1 y)(s), (T_2 y)(s))\,(\eta_i) \\ & + \frac{(logt)^{q-1}}{\Pi} \sum_{j=1}^{n} \mu_j \Big[J^{\beta_j+q} f\,(s, y(s), (T_1 y)(s), (T_2 y)(s))\,(e) \\ & - J^{\beta_j+q} f(s, y(s), (T_1 y)(s), (T_2 y)(s))(\xi_j) \Big]. \end{aligned} \tag{3.1}$$

with $\Pi \neq 0$. It should be noticed that problem (1.1)-(1.2) has solution if and only if the operator F has Fixed points. For the sake of convenience, we put

$$\varphi = \frac{1}{\Gamma(q+1)} + \frac{1}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| \frac{(log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{1}{\Pi} \sum_{j=1}^{n} |\mu_j| \frac{1+(log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}. \tag{3.2}$$

The first existence and uniqueness result is based on the Banach contraction mapping principle.

Theorem 3.1. Let $f : [1, e] \times R \to R$ be a continuous function satisfying the assumption:

(H1) There exists a constant $N^* > 0$ such that

$$|f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)| \leq N_1 |y_1 - z_1| + N_2 |y_2 - z_2| + N_3 |y_3 - z_3|$$

for each $t \in [1, e]$ and $y_i, z_i \in R, i = 1, 2, 3$.

$$If \quad N^*\varphi < 1, \quad where \quad N^* = N_1 + \gamma_0 N_2 + \delta_0 N_3. \tag{3.3}$$

where φ is given by (3.2), then the boundary value problem (1.1)-(1.2) has a unique solution on $[1, e]$.

Proof. We transform the problem $(1.1) - (1.2)$ into a fixed point problem, $y = Fy$, where the operator F is defined by (3.1). By using the Banach's contraction mapping principle, we shall show that F has a fixed point which is a unique solution of problem (1.1)-(1.2). We set

$$\sup_{t \in [1,e]} |f(t, 0, 0, 0)| = M < \infty.$$

and choose

$$r \geq \frac{M\varphi}{1 - N^*\varphi} \quad \text{where} \quad N^* = N_1 + \gamma_0 N_2 + \delta_3 N_3.$$

Now, we show that $FB_r \subset B_r$, where $B_r = \{y \in C : \|y\| \leq r\}$. For any $y \in B_r$, we have

$$
\begin{aligned}
\|Fy\| \leq \sup_{t\in[1,e]} \Big\{ & J^q \big| f(s, y(s), (T_1y)(s), (T_2y)(s)) \big|(t) \\
& + \frac{(logt)^{q-1}}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| J^{\alpha_i+q} \big| f\,(s, y(s), (T_1y)(s), (T_2y)(s))\, \big|(\eta_i) \\
& + \frac{(logt)^{q-1}}{|\Pi|} \sum_{j=1}^{n} |\mu_j| \Big[J^{\beta_j+q} \big| f\,(s, y(s), (T_1y)(s), (T_2y)(s))\, \big|(e) \\
& + J^{\beta_j+q} \big| f\,(s, y(s), (T_1y)(s), (T_2y)(s))\, \big|(\xi_j) \Big] \Big\} \\
\leq\ & J^q \left(\big| f\,(s, y(s), (T_1y)(s), (T_2y)(s)) - f(s,0,0,0) \big| + \big| f(s,0,0,0) \big| \right)(e) \\
& + \frac{1}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| J^{\alpha_i+q} \left(\big| f(s, y(s), (T_1y)(s), (T_2y)(s)) \right. \\
& \left. - f(s,0,0,0) \big| + \big| f(s,0,0,0) \big| \right)(\eta_i) \\
& + \frac{1}{|\Pi|} \sum_{j=1}^{n} |\mu_j| \Big[J^{\beta_j+q} \Big(\Big| f \Big(s, y(s), (T_1y)(s), (T_2y)(s) \Big) \\
& - f(s,0,0,0) \big| + \big| f(s,0,0,0) \big| \Big)(e) \\
& + J^{\beta_j+q} \left(\big| f\,(s, y(s), (T_1y)(s), (T_2y)(s)) - f(s,0,0,0) \big| + \big| f(s,0,0,0) \big| \right)(\xi_j) \Big] \\
\leq\ & \big[\, (N_1 + \gamma_0 N_2 + \delta_0 N_3)\, r + M \big] \Big\{ \frac{1}{\Gamma(q+1)} + \frac{1}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| \frac{(log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
& + \frac{1}{|\Pi|} \sum_{j=1}^{n} \frac{1+(log\xi_i)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \Big\} \\
\leq\ & \big[\, (N_1 + \gamma_0 N_2 + \delta_0 N_3)\, r + M \big] \varphi \\
\leq\ & \big[N^* r + M \big] \varphi \quad \text{where} \quad N^* = N_1 + \gamma_0 N_2 + \delta_0 N_3 \\
\leq\ & r.
\end{aligned}
$$

It follows that $FB_r \subset B_r$. For $y, z \in C$ and for each $t \in [1, e]$, we have

$$
\begin{aligned}
|Fy(t) - Fz(t)| \leq J^q \big(\big| & f(s, y(s), (T_1y)(s), (T_2y)(s)) \\
& - f(s, z(s), (T_1z)(s), (T_2z)(s)) \big| \big)(t)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(logt)^{q-1}}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| J^{\alpha_i+q}\big(\big| f(s, y(s), (T_1y)(s), (T_2y)(s)) \\
&- f(s, z(s), (T_1z)(s), (T_2z)(s))\big|\big)(\eta_i) \\
&+ \frac{(logt)^{q-1}}{|\Pi|} \sum_{j=1}^{n} |\mu_j| \Big[J^{\beta_j+q}\big(\big| f\,(s, y(s), (T_1y)(s), (T_2y)(s)) \\
&- f\,(s, z(s), (T_1z)(s), (T_2z)(s))\,\big|\big)(e) \\
&+ J^{\beta_j+q}\,\big(\big| f(s, y(s), (T_1y)(s), (T_2y)(s)\big) \\
&- f\,\big(s, z(s), (T_1z)(s), (T_2z)(s))\big|\big)\,(\xi_j)\Big] \\
\le\ & \big[N_1 + \gamma_0 N_2 + \delta_0 N_3\big]\|y - z\| \Big\{ \frac{1}{\Gamma(q+1)} + \frac{1}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| \frac{(log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i + q + 1)} \\
&+ \frac{1}{|\Pi|} \sum_{j=1}^{n} \frac{1 + (log\xi_i)^{\beta_j+q}}{\Gamma(\beta_j + q + 1)} \Big\} \\
\le\ & \big[N_1 + \gamma_0 N_2 + \delta_0 N_3\big]\varphi \|y - z\| \\
\le\ & N^*\varphi \|y - z\| \quad \text{where} \quad N^* = (N_1 + \gamma_0 N_2 + \delta_0 N_3).
\end{aligned}
$$

The above result implies that $\|Fy - Fz\| \le N^*\varphi \|y - z\|$. As $N^*\varphi < 1$, therefore F is a contraction. Hence, by the Banach contraction mapping principle, we deduce that F has a fixed point which is the unique solution of the problem $(1.1) - (1.2)$. ■

Next, we get the second existence and uniqueness result by using non-linear contractions an a fixed point.

Definition 3.2. Let E be a Banach space and Let $F : E \to E$ be a mapping F is said to be a non-linear contraction if there exists a continuous non decreasing function $\Omega : R^+ \to R^+$ such that $\Omega(0) = 0$ and $\Omega(\theta) < \theta$ for all $\theta > 0$ with the property $\|Fy - Fz\| \le \Omega(\|y - z\|)$ for all $y, z \in E$.

Lemma 3.3. (Boyd and Wong [14]) Let E be a Banach space and Let $F : E \to E$ be a non-linear contraction. The F has a unique fixed point in E.

Theorem 3.4. Let $f : [1, e] \times R \to R$ be a continuous function satisfying the assumption

(H2)

$$
\begin{aligned}
&|f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)| \\
&\quad \le h(t) \Big[\frac{|y_1 - z_1|}{G^* + |y_1 - z_1|} + \frac{|y_2 - z_2|}{G^* + |y_2 - z_2|} + \frac{|y_3 - z_3|}{G^* + |y_3 - z_3|} \Big]
\end{aligned}
$$

$t \in [1, e]$, $y_i, z_i \geq 0$ where $h : [1, e] \to R^+$ is continuous and a constant G^* defined by

$$G^* = J^q h(e) + \frac{1}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| J^{\alpha_i+q} h(\eta_i) + \frac{1}{|\Pi|} \sum_{j=1}^{n} |\mu_j| \left(J^{\beta_j+q} h(e) + J^{\beta_j+q} h(\xi_j) \right). \tag{3.4}$$

Then the boundary value problem (1.1) $-$ (1.2) has a unique solution.

Proof. We define the operator $F : C \to C$ as (3.1) and a continuous non-decreasing function $\Omega : R^+ \to R^+$ by

$$\Omega(\theta) = \frac{G^*\theta}{G^* + \theta}, \quad \text{for all} \quad \theta \geq 0 \quad \text{where} \quad \Omega = \Omega + \gamma_0 \Omega + \delta_0 \Omega.$$

Note that the function Ω satisfies $\Omega(0) = 0$ and $\Omega(\theta) < \theta$ for all $\theta > 0$.

For any $y, z \in C$ and for each $t \in [1, e]$, we have

$$\begin{aligned}
|Fy(t) - Fz(t)| \leq\ & J^q \Big(\big| f(s, y(s), (T_1 y)(s), (T_2 y)(s)) \\
& - f(s, z(s), (T_1 z)(s), (T_2 z)(s)) \big| \Big)(t) \\
& + \frac{(logt)^{q-1}}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| J^{\alpha_i+q} \Big(\big| f(s, y(s), (T_1 y)(s), (T_2 y)(s)) \\
& - f(s, z(s), (T_1 z)(s), (T_2 z)(s)) \big| \Big)(\eta_i) \\
& + \frac{(logt)^{q-1}}{|\Pi|} \sum_{j=1}^{n} |\mu_j| \Big[J^{\beta_j+q} \Big(\big| f(s, y(s), (T_1 y)(s), (T_2 y)(s)) \\
& - f(s, z(s), (T_1 z)(s), (T_2 z)(s)) \big| \Big)(e) \\
& + J^{\beta_j+q} \Big(\big| f(s, y(s), (T_1 y)(s), (T_2 y)(s)) \\
& - f(s, z(s), (T_1 z)(s), (T_2 z)(s)) \big| \Big)(\xi_j) \Big] \\
\leq\ & J^q \Bigg[h(s) \Bigg(\frac{|y(s) - z(s)|}{G^* + |y(s) - z(s)|} + \gamma_0 \frac{|y(s) - z(s)|}{G^* + |y(s) - z(s)|} \\
& + \delta_0 \frac{|y(s) - z(s)|}{G^* + |y(s) - z(s)|} \Bigg) \Bigg](e) + \frac{1}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| J^{\alpha_i+q} \Bigg[h(s) \Bigg(\frac{|y(s) - z(s)|}{G^* + |y(s) - z(s)|} \\
& + \gamma_0 \frac{|y(s) - z(s)|}{G^* + |y(s) - z(s)|} + \delta_0 \frac{|y(s) - z(s)|}{G^* + |y(s) - z(s)|} \Bigg) \Bigg](\eta_i)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Pi|}\sum_{j=1}^{n}|\mu_j|\Bigg\{J^{\beta_j+q}\Bigg[h(s)\Bigg(\frac{|y(s)-z(s)|}{G^*+|y(s)-z(s)|}\\
&+\gamma_0\frac{|y(s)-z(s)|}{G^*+|y(s)-z(s)|}+\delta_0\frac{|y(s)-z(s)|}{G^*+|y(s)-z(s)|}\Bigg)\Bigg](e)\\
&+J^{\beta_j+q}\Bigg[h(s)\Bigg(\frac{|y(s)-z(s)|}{G^*+|y(s)-z(s)|}\\
&+\gamma_0\frac{|y(s)-z(s)|}{G^*+|y(s)-z(s)|}+\delta_0\frac{|y(s)-z(s)|}{G^*+|y(s)-z(s)|}\Bigg)\Bigg](\xi_j)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{(\Omega+\gamma_0\Omega+\delta_0\Omega)}{G^*}(\|y-z\|)\Big[J^q h(e)+\frac{1}{|\Pi|}\sum_{i=1}^{m}|\lambda_i|J^{\alpha_i+q}h(\eta_i)\\
&\quad+\frac{1}{|\Pi|}\sum_{j=1}^{n}|\mu_j|\left(J^{\beta_j+q}h(e)+J^{\beta_j+q}h(\xi_j)\right)\Big]\\
&=[\Omega+\gamma_0\Omega+\delta_0\Omega](\|y-z\|)\\
&=\Omega(\|y-z\|)\quad\text{where}\quad \Omega=(\Omega+\gamma_0\Omega+\delta_0\Omega).
\end{aligned}
$$

Therefore F is a non-linear contraction. Hence, by Lemma 3.2 the operator F has a fixed point. which is the unique solution of the problem (1.1)-(1.2).

Next, we give an existence result by using Krasnoselskii's fixed point theorem. ■

Lemma 3.5. (Krasnoselskii's fixed point theorem) [22] Let M be a closed, bounded, convex and non-empty subset of a Banach space X. Let A, B be the operators such that $(a) Ax + By \in M$ whenever $x, y \in M$;$(b) A$ is compact and continuous $(c) B$ is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3.6. Assume that $f : [1, e] \times R \to R$ is a continuous function satisfying the assumption $(H1)$. In addition we suppose that:

(H3) $|f(t, y_1, y_2, y_3)| \le \tau(t)$ for all $(t, y) \in [1, e] \times R$ and $\tau = C\left([1, e], R^+\right)$

$$
\frac{N^*}{\Gamma(q+1)} < 1, \tag{3.5}
$$

then the boundary value problem(1.1)-(1.2) has at least one solution on $[1, e]$.

Proof. We define $\sup_{t\in[1,e]} |\tau(t)| = \|\tau\|$ and choose a suitable constant $\overline{r}$ as $\overline{r} \ge \|\tau\|\varphi^*$ put $\varphi^* = (1 + \gamma_0 + \delta_0)\varphi$ where φ is denoted by (3.2) Furthere more, we define the operator

P and Q on $B_{\overline{r}} = \{y \in C : \|y\| \leq \overline{r}\}$ as

$$\begin{aligned}(Py)(t) = & \frac{(logt)^{q-1}}{\Pi} \sum_{j=1}^{n} \mu_j (J^{\beta_j+q} f(s, y(s), (T_1y)(s), (T_2y)(s))(e) \\ & - J^{\beta_j+q} f(s, y(s), (T_1y)(s), (T_2y)(s))(\xi_j)) \\ & - \frac{(logt)^{q-1}}{\Pi} \sum_{i=1}^{m} \lambda_i J^{\alpha_i+q} f\,(s, y(s), (T_1y)(s), (T_2y)(s))\,(\eta_i), \quad t \in [1, e] \\ (Qy)(t) = & J^q f\,(s, y(s), (T_1y)(s), (T_2y)(s))\,(t), \quad t \in [1, e].\end{aligned}$$

For $y, z \in B_{\overline{r}}$, we have

$$\begin{aligned}\|Py + Qy\| \; & \leq \|\tau\|(1 + \gamma_0 + \delta_0)\Big[\frac{1}{\Gamma(q+1)} + \frac{1}{|\Pi|}\sum_{i=1}^{m}|\lambda_i|\frac{(log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\ & + \frac{1}{|\Pi|}\sum_{j=1}^{n}|\mu_j|\frac{1+(log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\Big] \\ & \leq \|\tau\|(1+\gamma_0+\delta_0)\varphi \\ & \leq \|\tau\|\varphi^* \quad put \quad \varphi^* = (1+\gamma_0+\delta_0) \\ & \leq \overline{r}.\end{aligned}$$

This shows that $Py + Qz \in B_{\overline{r}}$. It follows from the assumption $(H1)$ together with (3.5) that Q is a contraction mapping. Since the function f is continuous, we have that the operator P is continuous. It is easy to verify that

$$\|Py\| \; \leq \|\tau\|(1+\gamma_0+\delta_0)\Big[\frac{1}{|\Pi|}\sum_{i=1}^{m}|\lambda_i|\frac{(log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{1}{|\Pi|}\sum_{j=1}^{n}|\mu_j|\frac{1+(log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\Big].$$

Therefore, P is uniformly bounded on $B_{\overline{r}}$. Next, we prove the compactness of the operator P.

Let us set

$$\sup_{(t,y)\in[1,e]\times B_{\overline{r}}} |f(t, y, T_1y, T_2y| = \overline{f} < \infty,$$

consequently we get

$$
\begin{aligned}
&|(Py)(t_1) - (Py)(t_2)| \\
&= \Bigg| \frac{(logt_1)^{q-1}}{\Pi} \sum_{j=1}^{n} \mu_j \left(J^{\beta_j+q} f(s, y(s), (T_1y)(s), (T_2y)(s)) \right) (e) \\
&\quad - J^{\beta_j+q} f \left(s, y(s), (T_1y)(s), (T_2y)(s))(\xi_j)\right) \\
&\quad - \frac{(logt_1)^{q-1}}{\Pi} \sum_{j=1}^{m} \lambda_i J^{\alpha_i+q} f\,(s, y(s), (T_1y)(s), (T_2y)(s))\,(\eta_i) \\
&\quad - \frac{(logt_2)^{q-1}}{\Pi} \sum_{j=1}^{n} \mu_j \left(J^{\beta_j+q} f(s, y(s), (T_1y)(s), (T_2y)(s)) \right) (e) \\
&\quad - J^{\beta_j+q} f \left(s, y(s), (T_1y)(s), (T_2y)(s))(\xi_j)\right) \\
&\quad + \frac{(logt_2)^{q-1}}{\Pi} \sum_{i=1}^{m} \lambda_i J^{\alpha_i+q} f\,(s, y(s), (T_1y)(s), (T_2y)(s))\,(\eta_i) \Bigg| \\
&\leq \overline{f}(1+\gamma_0+\delta_0) \frac{|(logt_2)^{q-1} - (logt_1)^{q-1}|}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| \frac{(log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\quad + \overline{f}(1+\gamma_0+\delta_0) \frac{|(logt_2)^{q-1} - (logt_1)^{q-1}|}{|\Pi|} \sum_{j=1}^{m} |\mu_j| \frac{1-(log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)},
\end{aligned}
$$

which is independent of y and tends to zero as $t_2 \to t_1$. Thus, ρ is equicontinuous. So P is relatively compact on $B_{\overline{r}}$. Hence, by the Arzela-Ascoli theorem, P is compact on $B_{\overline{r}}$. Thus all the assumptions of lemma 3.3 are satisfied. So the boundary value problem (1.1)-(1.2) has at least one solution on $[1, e]$. The proof is completed. ■

Remark 3.7. In the above theorem we can interchange the roles of the operators P and Q to obtain a second result replacing (3.5) by the following condition.

$$
\frac{N^*}{|\Pi|} \sum_{i=1}^{m} |\lambda_i| \frac{(log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{N^*}{|\Pi|} \sum_{j=1}^{n} |\mu_j| \frac{1+(log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} < 1.
$$

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